A Note on Strongly Invo-Clean Rings

Santosh Kumar Pandey

Dept. of Mathematics, SPUP,

Vigyan Nagar-342037, Jodhpur, India.

E-mail: skpandey12@gmail.com

ABSTRACT

In this note, some important observations have been reported on recent works related to strongly invo-clean rings [1-3].

Key-words: invo-clean ring, strongly invo-clean ring, weakly tripotent ring .

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1. Introduction

In this paper R is a unital and associative ring. Let $U(R)$ and $Id(R)$ denote the set of all units and idempotents in R respectively. Then R is called a strongly invo-clean ring if [1-3]

i) for each $a \in R$, there exist $b \in U(R)$ with $b^2 = 1$ and $e \in Id(R)$ such that $a = b + e$ and (ii) $be = eb$.

A ring R is called a strongly nil-clean if for each $a \in R$ we have $a = n + e$ such that $ne = en$ for some $n \in N(R)$ and $e \in Id(R)$ [1-3]. Here $N(R)$ is the set of all nilpotent elements of R.

In [1] an important result on strongly invo clean rings is given which states that a ring R is a strongly invo-clean ring iff $R \cong R_1 \times R_2$. Here R_1 is a strongly invo-clean ring of characteristic ≤ 8 which is strongly nil-clean. R_2 is either zero or a commutative semiprimitive (reduced) invo-clean ring of characteristic three which can be embedded as an isomorphic copy of the direct product of copies of Z_3 (please refer corollary 2.17 of [1]). We provide a counterexample for this result. Please see example 1 for the same.

Further we consider the following result on strongly invo-clean ring [2-3] and provide counterexamples.

Let R is a strongly invo-clean ring, then $R \cong R_1 \times R_2$, here $R_1 = \{0\}$ or $\frac{R_1}{J(R_1)}$ is Boolean with nil $J(R_1)$ of index of nilpotence at most three, and $R_2 = \{0\}$ or R_2 is a subdirect product of a family of copies of the field of order three (please refer [2], page 22 and [3], page 74].

In the next section we provide important observations for the above results of [1-3] and we substantially improve these results of [1-3].

2. Some Observations on Strongly Invo-Clean Rings

In this section we provide our observations in the form of the following examples. Example 1 serves as a counterexample for the above result of [1] and the remaining examples serve as counterexamples for the above result of [2-3].

Example 1. Let $R = Z_2 \times Z_4$. We have $R_1 = Z_2$ and $R_2 = Z_4$. One may verify that $R_1 = Z_2$ is a strongly invo-clean ring of characteristic two. Also, $R_1 = Z_2$ is a strongly nil-clean ring as desired. We note that R is strongly invo-clean ring however $R_2 = Z_4$ is not a reduced ring. Also its characteristic is not three. Therefore this is a counterexample for the above of [1].

Below we shall provide some counterexamples for the above result of [2-3].

Example 2. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Clearly R is a unital ring under addition and multiplication of matrices modulo four. One may easily verify that $x^3 = x$ or $(1-x)^3 = 1-x$ for each $x \in R$. Therefore R is a weakly tripotent ring [6].

It may be noted that each weakly tripotent ring is a strongly invo-clean ring [1-5]. Thus R is a strongly invo-clean ring. We note that $R \cong R_1 \times R_2$. Here $R_1 = \{0\}$ and $R_2 = Z_4$.

One may note that R_2 is not a subdirect product of a family of copies of the field of order three even though R is a strongly invo-clean ring.

Example 3. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix} \right\}$.

Here R is a unital ring under addition and multiplication of matrices modulo eight. Since R is weakly tripotent ring and therefore it is a strongly invo-clean ring. Clearly $R \cong R_1 \times R_2$. Here $R_1 = \{0\}$ and $R_2 = Z_8$. However R_2 is not a subdirect product of a family of copies of the field of order three even though R is a strongly invo-clean ring.

Example 4. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$.

Here R is a unital ring under addition and multiplication of matrices modulo two. Clearly R is weakly tripotent and hence a strongly invo-clean ring. Let us take $R_1 = \{0\}$ and $R_2 = R$. Then $R \cong R_1 \times R_2$. However R_2 is not a subdirect product of a family of copies of the field of order three even though R is a strongly invo-clean ring.

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