Proof of Beal's conjecture
Solving of generalized Fermat equation $x^p + y^q = z^w$

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Abstract

The difference between of the Fermat's generalized equation and Fermat's regular equation is the different exponents of the variables and the method of solution. As we will show, for the proof of this equation to be complete, Fermat's theorem will be must hold,as we know it has been proven. There are only 10(+4) known solutions and all of them appear with exponent 2. This very fact is proved here using a uniform and understandable method. Therefore, the solution is feasible under the above conditions, as long as it we accepts that there is no solution if any other case, occurs for which all exponents values are greater than 2. The truth of this premise is proved in Theorem 6, based on the results of Theorem 4 and 5. The primary purpose for solving the equation is to see what happens in solving of Diophantine equation $a \cdot x + b \cdot y = c \cdot z$ which refers to Pythagorean triples of degree 1. This is the generator of the theorems and programs that follow.

I.1. Theorem 1 (Pythagorean triples 1st degree)

Let $\mathrm{P_1}$ be the set of Pythagorean triples and defined as $\mathrm{P_1} = \{(\mathrm{x,y,z}) \mid \mathrm{a,b,c,x,y,z} \in \mathrm{Z} - \{0\}$ and $\mathrm{a \cdot x + b \cdot y = c \cdot z}, \{x,y,z\}$ are pairwise relatively primes$\}$. Let $\mathrm{G_1}$ be the set defined as: $\mathrm{G_1} = \{(\mathrm{x} = \mathrm{k \cdot (c \cdot \lambda - b), y = k \cdot (a - c), z = k \cdot (a \cdot \lambda - b)}), (\mathrm{x = k \cdot (b - c),\ y = k \cdot (c \cdot \lambda - a), z = k \cdot (b \cdot \lambda - a)}), (\mathrm{x} = \mathrm{k \cdot (c + b \cdot \lambda), y = k \cdot (c - a \cdot \lambda), z = k \cdot (\alpha + b)}) \mid \mathrm{k}, \lambda \in \mathrm{Z}^+\}$. We need to prove that the sets $\mathrm{P_1} = \mathrm{G_1}$.

Proof.

Given a triad (a,b,c) such that $abc \neq 0$ and are these positive integers, if we divide by $y \neq 0$, we get according to the set $\mathrm{P_1}$ then apply $\mathrm{a} \cdot (\mathrm{x/y}) + \mathrm{b} = \mathrm{c} \cdot (\mathrm{z/y})$ and we call $\mathrm{X = x/y}$ and $\mathrm{Z = z/y}$. We declare now the sets:

$$\mathrm{F_1} = \{(\mathrm{X,Z})\} \in \mathrm{Q}^2 - \{0\} \mid a \cdot \mathrm{X} + \mathrm{b} = \mathrm{Z} \cdot \mathrm{c},\ \text{where } \mathrm{a,b,c} \in \mathrm{Z} - \{0\},\ \text{and where } \mathrm{X,Z} \in \mathrm{Q} - \{0\}\}$$

and

$$\mathrm{S_1} = \left\{(\mathrm{X,Z}) \in \mathrm{Q}^2 - \{0\} \mid \mathrm{X} = \mathrm{m} - \lambda \wedge \mathrm{Z} = \mathrm{m},\ \text{where } \mathrm{m}, \lambda \in \mathrm{Q} - \{0\}\right\}$$

The set $F_1 \cap S_1$ has 3 points as a function of parameters m, λ and we have solutions for the corresponding final equations,

$$F_1 \cap S_1 = \left\langle \begin{array}{c} a \cdot (m - \lambda) + b = m \cdot c \Leftrightarrow m = \frac{a \cdot \lambda - b}{a - c}, a - c \neq 0 \\ m - \lambda = \frac{c \cdot \lambda - b}{a - c}, a - c \neq 0, y = k \cdot (a - c), k \in Z^+ \\ x = \frac{c \cdot \lambda - b}{a - c} \cdot y \wedge z = \frac{c \cdot \lambda - b}{a - c} \cdot y, a - c \neq 0 \\ x = (c \cdot \lambda - b) \cdot k, y = k \cdot (a - c), z = k \cdot (a \cdot \lambda - b), k \in Z^+, a - c \neq 0 \end{array} \right\rangle$$

Therefore

$$F_1 \cap S_1 = \langle x = (c \cdot \lambda - b) \cdot k, y = k \cdot (a - c), z = k \cdot (a \cdot \lambda - b), k \in Z^+, a - c \neq 0\rangle \ (I)$$

Dividing respectively by $\mathrm{x} \neq 0$ we get the set and the relations we call $\mathrm{Y} = \mathrm{y}/\mathrm{x}$ and $\mathrm{Z} = \mathrm{z}/\mathrm{x}$

$$\mathrm{F}_2 = \left\{(\mathrm{Y}, \mathrm{Z}) \in \mathrm{Q}^2 - \{0\} \mid \mathrm{a} + b \cdot (\mathrm{y}/\mathrm{x}) = c \cdot (\mathrm{z}/x),\ \text{where } \mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{Z} - \{0\},\ \text{and where } \mathrm{Y}, \mathrm{Z} \in \mathrm{Q} - \{0\}\right\}$$

and

$$\mathrm{S}_2 = \left\{(\mathrm{Y}, \mathrm{Z}) \in \mathrm{Q}^2 - \{0\} \mid \mathrm{Y} = \mathrm{m} - \lambda \wedge \mathrm{Z} = \mathrm{m},\ \text{where } \mathrm{m}, \lambda \in \mathrm{Q} - \{0\}\right\}$$

Then as the type (I) we get the result

$$F_2 \cap S_2 = \langle x = (b - c) \cdot k, y = k \cdot (c \cdot \lambda - a), z = k \cdot (b \cdot \lambda - a), k \in Z^+, b - c \neq 0\rangle \ \text{(II)}$$

and finally dividing by $\mathrm{z} \neq 0$ similarly as before we call $\mathrm{X} = \mathrm{x}/\mathrm{z}$ and $\mathrm{Y} = \mathrm{y}/\mathrm{z}$

$$\mathrm{F}_3 = \left\{(\mathrm{X}, \mathrm{Y}) \in \mathrm{Q}^2 - \{0\} \mid \mathrm{a} \cdot (\mathrm{x}/\mathrm{z}) + b \cdot (\mathrm{y}/\mathrm{z}) = c,\ \text{where } \mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{Z} - \{0\},\ \text{and where } \mathrm{X}, \mathrm{Y} \in \mathrm{Q} - \{0\}\right\}$$

and

$$\mathrm{S}_3 = \left\{(\mathrm{X}, \mathrm{Y}) \in \mathrm{Q}^2 - \{0\} \mid \mathrm{X} = \mathrm{m} - \lambda \wedge \mathrm{Y} = \mathrm{m},\ \text{where } \mathrm{m}, \lambda \in \mathrm{Q} - \{0\}\right\}.$$

$$F_3 \cap S_3 = \langle x = (c + b \cdot \lambda) \cdot k, y = k \cdot (c - a \cdot \lambda), z = k \cdot (a + b), k \in Z^+, a + b \neq 0\} \ \text{(III)}$$

As a complement we can state that the parameter λ can be equal with $\lambda = \mathrm{p}/\mathrm{q}$, where p and q relatively primes. Therefore $\mathrm{P}_1 = \mathrm{G}_1$ and the proof is complete.

General Solving $x^p + y^q = z^w$

1.2 Theorem 2 [4,5,6]

To be proven that the equation $x^q + y^p = z^w, \{x, y, z, q, p, w\} \in Z^+_{>1}$can be equivalently transformed into 3 linear forms of equations according to the theorem of primary Pythagorean triads 1st degree.

Proof

To we prove initially must accept the general solution of the equation $\mathrm{ax} + \mathrm{by} = \mathrm{cz}$ (1) and $\mathrm{a}, \mathrm{b}, \mathrm{c} \in Z$, which, as we know from Theorem 1 has solution

$$\begin{aligned} \mathrm{x} &= (\mathrm{c} \cdot \lambda - \mathrm{b}) \cdot \mathrm{k} \\ \mathrm{y} &= (\mathrm{a} - \mathrm{c}) \cdot \mathrm{k} \qquad \text{with } \mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{k} \in Z, \lambda \in Q \\ \mathrm{z} &= (\mathrm{a} \cdot \lambda - \mathrm{b}) \cdot \mathrm{k} \end{aligned}$$

From the definition of the function we take

$$f(x,y,z)=\left\{\exists x,y,z\in C^{3}:A(x)=x^{q-1},B(y)=y^{p-1},C(z)=z^{w-1}\wedge A(x)x+B(y)y=C(z)z\right\}\subseteq C^{3}$$

with $q,p,w\in Z^{+}\wedge\{q>1,p>1,w>1\}$.

Therefore we have the correspondence $A(x)=x^{q-1},B(y)=y^{p-1},C(z)=z^{w-1}$ on the primary equation (1) and then we have the equivalence of the systems,

$$\begin{array}{l}\mathrm{x}=(\mathrm{C(z)}\cdot\lambda-\mathrm{B(y)})\cdot\mathrm{k}\\ \mathrm{y}=(\mathrm{A(x)}-\mathrm{C(z)})\cdot\mathrm{k}\\ \mathrm{z}=(\mathrm{A(x)}\cdot\lambda-\mathrm{B(y)})\cdot\mathrm{k}\end{array}\Leftrightarrow\begin{array}{l}\mathrm{x}=\left(z^{w-1}\cdot\lambda-y^{p-1}\right)\cdot\mathrm{k}\\ \mathrm{y}=\left(x^{q-1}-z^{w-1}\right)\cdot\mathrm{k}\\ \mathrm{z}=\left(x^{q-1}\cdot\lambda-y^{p-1}\right)\cdot\mathrm{k}\end{array}\quad\text{with } \mathrm{x,y,z,k}\in Z^{+},\lambda\in Q^{+}\tag{2}$$

Substituting $\mathrm{x,y,z}$ to solve the equation $x^{q}+y^{p}=z^{w}$ resulting

$$A(x)x+B(y)y=C(z)z\Rightarrow x^{q-1}\cdot\left(z^{w-1}\cdot\lambda-y^{p-1}\right)\cdot\mathrm{k}+y^{p-1}\cdot\left(x^{q-1}-z^{w-1}\right)\cdot\mathrm{k}-z^{w-1}\cdot\left(x^{q-1}\cdot\lambda-y^{p-1}\right)\cdot\mathrm{k}=0$$

Which seeing as applicable for each $q,p,w\in Z^{+}_{>1}$. Of course this extends to $x,y,z\in R$ or C. The general solution of

$$m\cdot x^{q}+n\cdot y^{p}=d\cdot z^{w}\quad m,n,d\in Z\wedge\{q>1,p>1,w>1\}$$

of course if we apply the same method we developed, F is given by:

$$\mathrm{F(x,y,z)}=\left(\begin{array}{l}\mathrm{x}=(\mathrm{d}\cdot\mathrm{C(z)}\cdot\lambda-n\cdot\mathrm{B(y)})\cdot\mathrm{k}\\ \mathrm{y}=(\mathrm{m}\cdot\mathrm{A(x)}-d\cdot\mathrm{C(z)})\cdot\mathrm{k}\\ \mathrm{z}=(\mathrm{m}\cdot\mathrm{A(x)}\cdot\lambda-n\cdot\mathrm{B(y)})\cdot\mathrm{k}\end{array}\right)\quad\text{with } \mathrm{x,y,z,m,n,d,k}\in Z^{+},\lambda\in Q^{+}\tag{3}$$

It remains to see that finally produced the solutions of the equation with more great from unit powers of unity with 3 equivalent forms

$$\begin{array}{l}\mathrm{x}=\left(z^{w-1}\cdot\lambda-y^{p-1}\right)\cdot\mathrm{k}\\ \mathrm{y}=\left(x^{q-1}-z^{w-1}\right)\cdot\mathrm{k}\\ \mathrm{z}=\left(x^{q-1}\cdot\lambda-y^{p-1}\right)\cdot\mathrm{k}\end{array}\quad\text{or}\quad\begin{array}{l}\mathrm{x}=\left(\mathrm{y}^{\mathrm{p}-1}-\mathrm{z}^{\mathrm{w}-1}\right)\cdot\mathrm{k}\\ \mathrm{y}=\left(\mathrm{z}^{\mathrm{w}-1}\cdot\lambda-\mathrm{x}^{\mathrm{q}-1}\right)\cdot\mathrm{k}\\ \mathrm{z}=\left(\mathrm{y}^{\mathrm{p}-1}\cdot\lambda-\mathrm{x}^{\mathrm{q}-1}\right)\cdot\mathrm{k}\end{array}\quad\text{or}\quad\begin{array}{l}\mathrm{x}=\left(\mathrm{z}^{\mathrm{w}-1}+\lambda\cdot\mathrm{y}^{\mathrm{p}-1}\right)\cdot\mathrm{k}\\ \mathrm{y}=\left(\mathrm{z}^{\mathrm{w}-1}-\lambda\cdot\mathrm{x}^{\mathrm{q}-1}\right)\cdot\mathrm{k}\\ z=\left(\mathrm{x}^{\mathrm{q}-1}+\lambda\cdot\mathrm{y}^{\mathrm{p}-1}\right)\cdot\mathrm{k}\end{array}\tag{4}$$

In the analysis of this special case $x^{q}+y^{p}=z^{w}$ we assume $x,y,z,p,q,w\in Z_{>=2}$, $\lambda\in Q^{+}$ where $\lambda=u/s$, $\{u,s$ co-primes) or $\lambda\in Z^{+}$.

It should be mentioned that for the solution of the simple equation, $x^{q}+y^{p}=z^{w}$, which is reduced to a pseudo-linear system as we have shown, we use the newer Gröbner basis method proposed in mathematica. In the next chapter we will discuss some of the elements of the method and how it is applied.

2.1 Method Gröbner [8]

For systems with equational constraints generating a zero-dimensional ideal, Mathematica uses a variant of the CAD algorithm that finds projection polynomials using Gröbner basis methods. If the lexicographic order Gröbner basis of contains linear polynomials with constant coefficients in every variable but the last one (which is true "generically"), then all coordinates of solutions are easily represented as polynomials in the last coordinate. Otherwise the coordinates are given as Root objects representing algebraic numbers defined by triangular systems of equations. Setting to True causes Mathematica to represent each coordinate as a single numeric Root object defined by a minimal polynomial and a root number. Computing this reduced representation often takes much longer than solving the system.

Correctness of the Algorithm

The correctness of the algorithm is based on the following "Main Theorem of Gröbner Bases Theory:

F is a Gröbner basis $\Leftrightarrow \underset{f_1, f_2 \in F}{\forall} RF\left[F, S- \text{ polynomial } \left[f_1, f_2\right]\right] = 0$

2.2 Examples

A Simple Set of Equations

We now show how Gröbner bases can be applied to solving systems of polynomial equations. Let us, first, consider again the example:

$$\begin{aligned} f_1 &= xy - 2y \\ f_2 &= 2y^2 - x^2 \\ F &= \{f_1, f_2\} \end{aligned}$$

The Gröbner basis G of F is

$$\mathrm{G} := \left\{-2x^2 + x^3, -2y + xy, -x^2 + 2y^2\right\}$$

By the fact that F and G generate the same ideal, F and G have the same solutions. The elimination property of Gröbner bases guarantees that, in case G has only finitely many solutions, G contains a univariate polynomial in x. (Note that, here, we use the lexicographic order that ranks y higher than x. If we used the lexicographic order that ranks x higher than y then, correspondingly, the Gröbner basis would contain a univariate polynomial in y.) In fact, the above Gröbner basis is "reduced", i.e. all polynomials in the basis are reduced modulo the other polynomial in the basis. It can be shown that reduced Gröbner bases (with finitely many solutions) contain exactly one univariate polynomial in the lowest indeterminate.

If we look at this example using the mathematica, and the method Groebner Basis summarized in the following program and for two variables $\{x, y\}$

polys $= \{x^*y - 2y, 2y^\wedge 2 - x^\wedge 2\}$;

gb=GroebnerBasis[polys, $\{x, y, z\}$, Method="Buchberger"]

soly1=NSolve[gb[[1]]=0,{y}]

soly2=NSolve[gb[[2]]=0,{x}]

Reduce[polys=0,{y,x,z},Backsubstitution=True]//N

NSolve[polys $= 0, \{y, x\}$]

Results.

$\left\{-2y + y^3, -2y + xy, x^2 - 2y^2\right\}$

$\{\{y = -1.41421\}, \{y = 0.\}, \{y = 1.41421\}\}$

$\{\{x = 2.\}\}$

$(y = 0.\&\&x = 0.)||(y = -1.41421\&\&x = 2.)||(y = 1.41421\&\&x = 2.)$

$\{\{y = -1.41421, x = 2.\}, \{y = 1.41421, x = 2.\}, \{y = 0., x = 0.\}, \{y = 0., x = 0.\}\}$

Then we have to solve the generalized system generated from the equation $x^q + y^p = z^w$ and which written before as

$$\begin{aligned} &\mathrm{x} = \left(z^{w-1} \cdot \lambda - y^{p-1}\right) \cdot \mathrm{k} \\ &\mathrm{y} = \left(x^{q-1} - z^{w-1}\right) \cdot \mathrm{k} \qquad \text{with k} \in Z+, \lambda \in Q+ \\ &\mathrm{Z} = \left(x^{q-1} \cdot \lambda - y^{p-1}\right) \cdot \mathrm{k} \end{aligned}$$

In language mathematica can be written as a partial but total solutions for triplets $\{x, y, z\}$ as one of the sets of solutions $x^3 + y^3 = z^3$ and for the values $k = r = 1$, written

q:=3;p:=3;w:=3;λ:=1;k=1;

eqs1={x-k* $z^{w-1}\lambda + k^*y^{p-1}, y - k^*x^{q-1} + k^*z^{w-1}, z - k^*x^{q-1} * \lambda + k^*y^{p-1}$};

gb=GroebnerBasis[eqs1,{x,y,z},Method-"Buchberger"]

soly1=NSolve $[gb[[1]] == 0, \{z\}, Integers]$

soly2=NSolve $[gb[[2]] == 0, \{x\}, Integers]//N$

soly3=NSolve $[gb[[3]] == 0, \{y\}, Integers]//N$

NSolve[eqs1 $== 0, \{x, y, z\}$]

Results.

$$\{-z+z^3, z+2yz-z^2, -y+y^2+z-z^2, x+y-z\}$$

$$\{\{z\to -1.\},\{z\to 0\},\{z\to 1.\}\}$$

$$\{\{y\to 1.-1.z\},\{y\to z\}\}$$

$$\{\{x\to 1., y\to 0., z\to 1.\},\{x\to -1, y\to 1, z\to 0.\},$$

$$\{x\to 0., y\to -1, z\to -1\},\{x\to 0., y\to 0., z\to 0.\}\}$$

This analysis does not reflect all solutions but an elementary part of them with $r=k=1$. For the case $g=p=w=3$ we obtain complex or real roots with at least one root equal to zero. Of course we can take many different powers for $\{q,p,w\}$ with $q=p=w>3$ and we will see that the program follows the same logic as for the values $q=p=w=3$. This is the computational procedure of the proof of Fermat's theorem. Of course we can take different values for $\{q,p,w\}$ and we will notice that only in some cases where one of them is at least equal to 2, it has integer solutions.

3. Theorem 3 [2,3,6]

In the general solution of $x^q+y^p=z^w$, where $x,y,z\in Z_{>=2}$ and $p,q,w\in Z_{>=2}$ but and x,y,z co-primes numbers after final simplification by the common factor of GCD[x, y, z] will eventually obtain 4 variables $\{\mathrm{x}',\mathrm{y}',\mathrm{z}'\}$ which will be the primes among then, and variables related with one of the 4 general linear relations, $x'+y'\cdot\lambda=z'$, or $\lambda\cdot x'+y'=z'$ or $\lambda\cdot z'+y'=x'$ or $\lambda\cdot z'+x'=y'$ independent of the exhibitors and which are equivalent forms of powers equivalent to the general form.

Proof

Initially each Diophantine equation of the form $x^q+y^p=z^w$ is equivalent to the form $x\cdot x^{q-1}+y\cdot y^{p-1}=z\cdot z^{k-1}\Leftrightarrow a\cdot x+b\cdot y=c\cdot z, a\equiv x^{q-1}\wedge b\equiv y^{p-q}\wedge c\equiv z^{w-1}$.

To prove this we must first accept the general solution according to theorem (1) of the equation a·x+b·y = c·z (1) with $a,b,c\in Z^+$.

As we know from Theorem 1 If you divide the variables $\{y,z\}$ with $x,\{x,z\}$ with y and $\{x,y\}$ with z and we get:

$$\begin{array}{lllllll} x=(c\cdot\lambda-b)\cdot k & & y=(c\cdot\lambda-a)\cdot k & & x=(c+b\cdot\lambda)\cdot k & & \\ y=(a-c)\cdot k & \text{or} & x=(b-c)\cdot k & \text{or} & z=(a+b)\cdot k & & \text{with } k\in Z,\lambda\in Q \\ z=(a\cdot\lambda-b)\cdot k & & z=(b\cdot\lambda-a)\cdot k & & y=(c-a\cdot\lambda)\cdot k & & \end{array}$$

In conjunction with Theorem 2 we will apply the same method and construct a relevant function that will link all these variables.

So we have 4 cases: i) For the first case in general form for $x\neq 0$ the function will has relationship

$$f(x,y,z) = \begin{pmatrix} x = (C(z)\cdot\lambda - B(y))\cdot k \\ y = (A(x) - C(z))\cdot k \\ z = (A(x)\cdot\lambda - B(y))\cdot k \end{pmatrix} \quad \text{with } k \in Z^{+}, \lambda \in Q^{+} \tag{2}$$

therefore dividing by k resulting

$$\begin{pmatrix} \mathrm{x/k = (C(z)\lambda - B(y))} \\ \mathrm{y/k = (A(x) - C(z))} \\ \mathrm{z/k = (A(x)\cdot\lambda - B(y))} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \mathrm{x/k = (C(z)\lambda - B(y))} \\ \mathrm{y/k = (A(x) - C(z))} \\ \mathrm{x/k - z/k = -(A(x)\cdot\lambda - \lambda\cdot C(z))} \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} \mathrm{x/k = (C(z)\lambda - B(y))} \\ \mathrm{y/k = (A(x) - C(z))} \\ \mathrm{x/k - z/k} = -\lambda\cdot y/k \end{pmatrix} \Leftrightarrow x + \lambda\cdot y = z \tag{3}$$

if we divide the triad $\{\mathrm{x,y,z}\}$ with the greatest common divisor $\varepsilon = \mathrm{GCD[x,y,z]}$, then it is clearly arises the final triad $\{\mathrm{x' = x/\varepsilon, y' = y/\varepsilon, z' = z/\varepsilon}\}$ and the relationship. $x' + \lambda\cdot y' = z'$.

ii) If we divide the variables $\{x,z\}$ with $y, y \neq 0$ and therefore will occur in accordance with the basic relationship and certainly using of Theorem 1 that the relationship meets these data are:

$$\begin{aligned} &\mathrm{y = (c\cdot\lambda - a)\cdot k} \\ &\mathrm{x = (b - c)\cdot k} \qquad \text{with } \mathrm{x} \in Z, \lambda \in Q \\ &\mathrm{z = (b\cdot\lambda - a)\cdot k} \end{aligned}$$

Of course we will apply the same method and the function f will be given by the relationship in general form:

$$\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{y = (C(z)\cdot\lambda - A(y))\cdot k} \\ \mathrm{x = (B(x) - C(z))\cdot k} \\ \mathrm{z = (B(x)\cdot\lambda - A(y))\cdot k} \end{pmatrix} \quad \text{with more generally } \mathrm{k \in Z+, \lambda \in Q+}$$

by the sequence

$$\begin{pmatrix} \mathrm{y/k = (C(z)\cdot\lambda - A(y))} \\ \mathrm{x/k = (B(x) - C(z))} \\ \mathrm{z/k = (B(x) - \lambda - A(y))} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \mathrm{y/k = (C(\{}z\mathrm{)\cdot\lambda - A(y))} \\ \mathrm{x/k = (B(x) - C(z))} \\ \mathrm{y/k - z/k = -(B(x)\cdot\lambda - \lambda\cdot C(y))} \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} \mathrm{x/k = (C(z)\cdot\lambda - B(y))} \\ \mathrm{y/k = (A(x) - C(z))} \\ \mathrm{y/k - z/k} = -\lambda\cdot x/k \end{pmatrix} \Leftrightarrow y + \lambda\cdot x = z \tag{4}$$

Similarly If we divide the triad $\{x, y, z\}$ by the greatest common divisor $\varepsilon = GCD[x, y, z]$, then it is clearly that the final triad $\{x' = x/\varepsilon, y' = y/\varepsilon, z' = z/\varepsilon\}$ and relationship is obtained $\lambda \cdot x' + y' = z'$.

iii) Continuing, in the same way and dividing by z variables $\{x, y\}$ with z which $z \neq 0$ will occur in accordance with the basic relationship and certainly using Theorem 1 that the relationship meets these data will be:

$$\begin{aligned} &\mathrm{x} = (\mathrm{c} + \mathrm{b} \cdot \lambda) \cdot \mathrm{k} \\ &\mathrm{z} = (\mathrm{a} + \mathrm{b}) \cdot \mathrm{k} \quad \kappa, \lambda \in Q \\ &\mathrm{y} = (\mathrm{c} - \mathrm{a} \cdot \lambda) \cdot \mathrm{k} \end{aligned}$$

By the same method and to the display function f which is given by the general form:

$$\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{x} = (\mathrm{C(z)} + \lambda \cdot \mathrm{B(y)}) \cdot \mathrm{k} \\ \mathrm{z} = (\mathrm{A(x)} + \mathrm{B(y)}) \cdot \mathrm{k} \\ \mathrm{y} = (\mathrm{C(z)} - \lambda \cdot \mathrm{A(x)}) \cdot \mathrm{k} \end{pmatrix} \quad \begin{array}{l} \text{more generally with } \mathrm{k} \in Z^+, \lambda \in Q^+, \\ \text{where } \lambda = u/s, (u, s \text{ co-primes) or } \lambda \in Z^+ \end{array}$$

by the sequence

$$\begin{pmatrix} \mathrm{x/k} = (\mathrm{C(z)} + \lambda \cdot \mathrm{B(y)}) \\ \mathrm{z/k} = (\mathrm{A(x)} + B(\mathrm{y})) \\ \mathrm{y/k} = (\mathrm{C(z)} - \lambda \cdot \mathrm{A(y)}) \end{pmatrix} \Leftrightarrow \begin{pmatrix} \mathrm{x/k} = (\mathrm{C(z)} \cdot \lambda - \mathrm{B(y)}) \\ \mathrm{z/k} = (\mathrm{A(x)} + \mathrm{B(y)}) \\ \mathrm{x/k} - \mathrm{y/k} = (\mathrm{B(y)} \cdot \lambda + \lambda \cdot A(\mathrm{x})) \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} \mathrm{x/k} = (\mathrm{C(z)} \cdot \lambda - \mathrm{B(y)}) \\ \mathrm{z/k} = (\mathrm{A(x)} + \mathrm{B(y)}) \\ \mathrm{x/k} - \mathrm{y/k} = \lambda \cdot z/k \end{pmatrix} \Leftrightarrow y + \lambda \cdot z = x \tag{5}$$

These resulted from the initial $a \cdot x + b \cdot y = z$ assuming that $\frac{x}{z} = m$ and $\frac{y}{z} = m - \lambda$:

iv) Finally, if we assume that $\frac{x}{z} = m - \lambda$ and $\frac{y}{z} = m$ resulting system:

$$\begin{aligned} &\mathrm{x} = (\mathrm{c} - \mathrm{a} \cdot \lambda) \cdot \mathrm{k} \\ &\mathrm{z} = (\mathrm{a} + b) \cdot \mathrm{k} \quad \text{ with k} \in Z, \lambda \in Q \\ &\mathrm{y} = (\mathrm{c} + \mathrm{b} \cdot \lambda) \cdot \mathrm{k} \end{aligned}$$

By the same method and the function f which is given by the general form

$$\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{x} = (\mathrm{C(z)} - \lambda \cdot \mathrm{A(x)}) \cdot \mathrm{k} \\ \mathrm{z} = (\mathrm{A(x)} + \mathrm{B(y)}) \cdot \mathrm{k} \\ \mathrm{y} = (\mathrm{C(z)} + \lambda \cdot \mathrm{B(y)}) \cdot \mathrm{k} \end{pmatrix} \quad \begin{array}{l} \text{more generally with } \mathrm{k} \in Z^+, \lambda \in Q^+, \\ \text{where } \lambda = u/s, (u, s \text{ co-primes) or } \lambda \in Z^+ \end{array} \tag{6}$$

by sequence follows

$$\begin{pmatrix} \text{x/k} = (\text{C(z)} - \lambda \cdot \text{A(x)}) \\ \text{z/k} = (\text{A(x)} + B\text{(y)}) \\ \text{y/k} = (\text{C(z)} + \lambda \cdot \text{B(y)}) \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{x/k} = (\text{C(z)} \cdot \lambda - \text{A(x)}) \\ \text{z/k} = (\text{A(x)} + \text{B(y)}) \\ \text{y/k} - \text{x/k} = (\text{C(z)} \cdot \lambda + \lambda \cdot A\text{(x)}) \Leftrightarrow \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \text{x/k} = (\text{C(z)} \cdot \lambda - A\text{(x)} \\ \text{z/k} = (\text{A(x)} + \text{B(y)}) \\ \text{y/k} - \text{x/k} = \lambda \cdot z/k \end{pmatrix} \Leftrightarrow x + \lambda \cdot z = y \tag{7}$$

We will therefore obtain table (9) which is directly related to the variables x, y, z

$$\begin{pmatrix} \text{x} + \text{y} \cdot \lambda = \text{z} \\ \text{y} + x \cdot \lambda = \text{z} \\ \text{x} = \text{y} + \text{z} \cdot \lambda \\ \text{y} = \text{x} + \text{z} \cdot \lambda \end{pmatrix} \lambda \in Q^{+}, \text{ where } \lambda = u/s, (u, s \text{ co-primes }) \text{ or } \lambda \in Z^{+} \tag{8}$$

and also the simplified table (9) which is obtained by dividing each variable x, y, z by $GCD[x, y, z]$ and therefore the variables x', y', z' will be obtained.

$$\begin{pmatrix} x' + y' \cdot \lambda = z' \\ y' + x' \cdot \lambda = z' \\ x' = y' + z' \cdot \lambda \\ y' = x' + z' \cdot \lambda \end{pmatrix} \lambda \in Q^{+}, \text{ where } \lambda = u/s, (u, s \text{ co-primes }) \text{ or } \lambda \in Z^{+} \tag{9}$$

By simple extension of the theorem we can prove that these relations (8, 9) and apply the generalized case Diophantine equation on the form $a'x^q + b'y^p = c'z^w$ which is equivalent to the form $a'x \cdot x^{q-1} + b'y \cdot y^{p-1} = c'z \cdot z^{w-1} \Leftrightarrow a \cdot x + b \cdot y = c \cdot z, a = a'x^{q-1} \wedge b = b' \cdot y^{p-q} \wedge c = c' \cdot z^{w-1}$ and the same applies for the proof, since we accept the general solution of the equation a·x+b·y = c·z as originally defined, ie $a, b, c \in Z^{+}$.

Examples according to the theorem 3

1.6^3 + 5^4 = 29^2 ⇒ 4*6 + 5 = 29
2.7^2 + 2^5 = 3^4 ⇒ 3 + 2 * 2 = 7
3.7^4 + 15^3 = 76^2 => 7 + (23/5)*15 = 76
4.2^7 + 17^3 = 71 ⇒ 2 * 27 + 17 = 71
5.3^5 + 11^4 = 122^2 => 3 * 37 + 11 = 122
6.1414^3 + 2213459^2 = 21063928^2 => 1414 + (442409/13)*65 = 2213459
7.17^7 + 726271^3 = 21063928 => 17 + (21063911/76271) * 76271 = 21063928

Note that λ will be either an integer or explicit in the form mentioned above i.e. where λ = u/s (u, s co-primes) or $\lambda \in Z^{+}$. This property generalizes everywhere to any triad that is a solution of the equation $x^q + y^p = z^r$ where where $x, y, z \in Z^{+}_{>1}$ and final $p, q, w \in Z^{+}_{>=2}$.

4. The Generalized Fermat Equation [9]

We now return to the generalized Fermat equation $x^q + y^p = z^r$ (1) $x, y, z, p, q, r \in Z^+ \wedge \{q >= 2, p >= 2, r >= 2\}$ where x, y and z are integers, and the exponents p, q and r are (potentially distinct) positive integers. We restrict our attention to primitive solutions, i.e. those with $\gcd(x, y, z) = 1$, since, without such a restriction, it is easy to concoct uninteresting solutions in a fairly trivial fashion. Indeed, if we assume, say, that p, q and r are fixed positive integer, then we can choose integers u, v and w such that

$$uqr \equiv -1(\bmod p), vpr \equiv -1(\bmod q), wpq \equiv -1(\bmod r)$$

In the case where they are given a, b, c with $a + b = c$ multiplying this equation by $a^{\mathrm{uqr}}\ b^{\mathrm{vpr}}\ c^{\mathrm{wpq}}$ we have

$$\left(a^{(uqr+1)/p} b^{vr} c^{wq}\right)^p + \left(a^{(ur)} b^{(vpr+1)/q} c^{wp}\right)^q = \left(a^{(uq)} b^{(vp)} c^{(wpq+1)/r}\right)^r$$

We call (p, q, r) the signature of equation (1). The behaviour of primitive solutions depends fundamentally upon the size of the quantity

$$\sigma(p, q, r) = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \tag{2}$$

in particular, whether $\sigma(p, q, r) > 1, \sigma(p, q, r) = 1$ or $\sigma(p, q, r) < 1$. If we set $\chi = \sigma(p, q, r) - 1$, then χ is the Euler characteristic of a certain stack associated to equation (1). It is for this reason that the cases $\sigma(p, q, r) > 1$, $\sigma(p, q, r) = 1$ or $\sigma(p, q, r) < 1$ are respectively termed spherical, parabolic and hyperbolic.

4.1 The spherical case $\boldsymbol{\sigma}(\mathbf{p}, \mathbf{q}, \mathbf{r}) > 1$

In this case, we can assume that (p, q, r) is one of $(2, 2, r), (2, q, 2), (2, 3, 3), (2, 3, 4), (2, 4, 3)$ or $(2, 3, 5)$. In each of these cases, the (infinite) relatively prime integer solutions of (1) belong to finite families of two parameters; in the (more complicated) case $(2, 3, 5)$, there are exactly 27 such families, as Johnny Edwards [11] proved in 2004 via an elegant application of classical invariant theory. In the case $(p, q, r) = (2, 4, 3)$, for example, that the solutions x, y and z satisfy one of the following four parameterizations

$$\begin{cases} x = 4ts\left(s^2 - 3t^2\right)\left(s^4 + 6t^2s^2 + 81t^4\right)\left(3s^4 + 2t^2s^2 + 3t^4\right) \\ y = \pm\left(s^2 + 3t^2\right)\left(s^4 - 18t^2s^2 + 9t^4\right) \\ z = \left(s^4 - 2t^2s^2 + 9t^4\right)\left(s^4 + 30t^2s^2 + 9t^4\right) \end{cases}$$

$\mathrm{s} \neq \mathrm{t} \cdot \mathrm{mod}(2)$ and 3 s

where

$$\begin{cases} x = \pm\left(4s^4 + 3t^4\right)\left(16s^8 - 408t^4s^4 + 9t^8\right), \\ y = 6ts\left(4s^4 - 3t^4\right), \\ z = 16s^8 + 168t^4s^4 + 9t^8, \end{cases}$$

r is odd and 3**s**

where

$$\begin{cases} x = \pm\left(s^4 + 12t^4\right)\left(s^8 - 408t^4s^4 + 144t^8\right), \\ y = 6ts\left(s^4 - 12t^4\right), \\ z = s^8 + 168t^4s^4 + 144t^8, \end{cases}$$

$\mathrm{s} = \pm 1$ mod (6) or

where

$$\begin{cases} x = 2\left(s^4 + 2ts^3 + 6t^2s^2 + 2t^3s + t^4\right)\left(23s^8 - 16ts^7 - 172t^2s^6 - 112t^3s^5\right. \\ \quad \left. -22t^4s^4 - 112t^5s^3 - 172t^6s^2 - 16t^7s + 23t^8\right) \\ y = 3(s-t)(s+t)\left(s^4 + 8ts^3 + 6t^2s^2 + 8t^3s + t^4\right) \\ z = 13s^8 + 16ts^7 + 28t^2s^6 + 112t^3s^5 + 238t^4s^4 \\ \quad +112t^5s^3 + 28t^6s^2 + 16t^7s + 13t^8 \end{cases}$$

$$\text{s} \neq \text{tmod}(2) \text{ and s} \neq \text{tmod}(3)$$

Here, s and t are relatively prime integers. Details on these parametrizations (and much more besides) can be found in Cohen's exhaustive work [10].

4.2 The parabolic case $\sigma(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 1$

If we have $s(p, q, r) = 1$, then, up to reordering, $(p, q, r) = (2, 6, 3), (2, 4, 4), (4, 4, 2), (3, 3, 3)$ or $(2, 3, 6)$. As in Examples 1 and 2, each equation now corresponds to an elliptic curve of rank 0 over Q; the only primitive non-trivial solution comes from the signature $(\mathrm{p}, \mathrm{q}, \mathrm{r}) = (2, 3, 6)$, corresponding to the Catalan solution $3^2 - 2^3 = 1$.

4.3 The hyperbolic case $\sigma(\mathbf{p}, \mathbf{q}, \mathbf{r}) < 1$

It is the hyperbolic case, with $\sigma(\mathrm{p}, \mathrm{q}, \mathrm{r}) < 1$, where most of our interest lies. Here, we are now once again considering the equation and hypotheses [9]. As mentioned previously, it is expected that the only solutions are with (x,y, z, p, q, r) corresponding to the identity $1^p + 2^3 = 3^2$, for $p >= 6$, or to

$$2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2, \quad 3^5 + 11^4 = 122^2,$$
$$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7, 9262^3 + 15312283^2 = 113^7,$$
$$43^8 + 96222^3 = 30042907^2 \quad \text{ and } 33^8 + 1549034^2 = 15613^3$$

A less ambitious conjecture would be that (4) has at most finitely many solutions (where we agree to count those coming from $1^p + 2^3 = 3^2$ only once). In the rest of this section, we will discuss our current knowledge about this equation.

4.4 The Theorem of Darmon and Granville [1,4,9]

What we know for sure in the hyperbolic case, is that, for a fixed signature (p, q, r), the number of solutions to equation (1) is at most finite:

Theorem 4. (Darmon and Granville [9]).

If A, B, C, p, q and r are fixed positive integers of equation $Ax^q + By^p = Cz^r$ and p, q and w are fixed positive integers numbers, with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

then the equation has at most finitely many solutions in coprime non-zero integers x, y and z.

Proof.

The proof by Darmon and Granville is extremely elegant and we cannot resist giving a brief sketch. The hypothesis $1/p + 1/q + 1/r < 1$ is used to show the existence of a cover $\phi : D \to P^1$ that is ramified only above $0, 1, \infty$, where the curve D has genus ≥ 2. Moreover, this cover has the property that the ramification degrees above 0 are all divisors of p, above 1 are all divisors of q, and above ∞ are all divisors of r. Now let (x, y, z) be a non-trivial primitive solution to the equation $Ax^q + By^p = Cz^r$. The above properties of the cover ϕ imply that the points belonging to the fiber $\phi^{-1}(Ax^p/Cz^r)$ are defined over a number field K that is unramified away from the primes dividing 2ABCpqr. It follows from a classical theorem of Hermite that there are only finitely many such number fields K. Moreover, by Faltings' theorem, for each possible K there are only finitely many K-points on D. It follows that the equation $Ax^q + By^p = Cz^r$ has only finitely many primitive solutions. It is worth noting that the argument used in the proof is ineffective, due to its dependence upon Faltings' theorem; it is not currently known whether or not there exists an algorithm for finding all rational points on an arbitrary curve of genus ≥ 2.

4.5 Summary tables of what we know.

What we would really like to do is rather more distant than what Darmon's theorem and Granville tells us. And as we will see in the next chapters we will determine which cases are solvable. In the tables below, we list all known (as of 2015) cases where Eq.(1) has been fully solved. For references to the original papers we recommend the exhaustive search [10,11]. The 2 tables bring together all known infinite families treated to date:

(p, q, r)	reference(s)
(n, n, n)	Wiles, Taylor-Wiles
$(n, n, k), k \in \{2, 3\}$	Darmon-Merel, Poonen
$(2n, 2n, 5)$	Bennett
$(2, 4, n)$	Ellenberg, Bennett-Ellenberg-Ng, Bruin
$(2, 6, n)$	Bennett-Chen, Bruin
$(2, n, 4)$	Bennett-Skinner, Bruin
$(2, n, 6)$	Bennett-Chen-Dahmen-Yazdani
$(3j, 3k, n), j, k \geq 2$	immediate from Kraus
$(3, 3, 2n)$	Bennett-Chen-Dahmen-Yazdani
$(3, 6, n)$	Bennett-Chen-Dahmen-Yazdani
	Bennett-Chen-Dahmen-Yazdani
$(2, 2n, k), k \in \{9, 10, 15\}$	Bennett-Chen-Dahmen-Yazdani
$(4, 2n, 3)$	Anni-Siksek
$(2j, 2k, n), j, k \geq 5$ prime, $n \in \{3, 5, 7, 11, 13\}$	And

Our second table lists "sporadic" triples where the solutions to (1) have been determined, and infinite families of exponent triples where the (p, q, r) satisfy certain additional local conditions.

(p, q, r)	reference(s)
$(3, 3, n)^*$	Chen-Siksek, Kraus, Bruin, Dahmen
$(2, 2n, 3)^*$	Chen, Dahmen, Siksek
$(2, 2n, 5)^*$	Chen
$(2m, 2n, 3)^*$	Bennett-Chen-Dahmen-Yazdani
$(2, 4n, 3)^*$	Bennett-Chen-Dahmen-Yazdani
$(3, 3n, 2)^*$	Bennett-Chen-Dahmen-Yazdani
$(2, 3, n), n \in \{6, 7, 8, 9, 10, 15\}$	Poonen-Schaefer-Stoll, Bruin, Zureick-Brown, Siksek, Siksek-Stoll
$(3, 4, 5)$	Siksek-Stoll
$(5, 5, 7), (7, 7, 5)$	Dahmen-Siksek

The asterisk here refers to conditional results. There are detailed tables for each case in total.

5. Formations of the diophantine equation

5.1 Lemma 1.

The number of forms of $x^q + y^p = z^w, x, y, z, p, q, w \in Z^+ \wedge \{q > 2, p > 2, w > 2\}$ and after simplifying the variables $\{x, y, z\}$ with GCD$[x, y, z] = 1$ is limited to exactly 6 cases and for the exhibitors (p, q, w), we accept from 4 only 3 cases of equality between them.

Proof

Depending on the ascending order of exponents {p, q, w} of original Diophantine $x^p + y^q = z^w, x, y, z, p, q, w \in Z \wedge \{q > 1, p > 1, w > 1\}$. The key point of the equation is if analyzed each variable x, y and z must have a common prime factor. After simplifying the terms of the number $\varepsilon = \mathrm{GCD[x, y, z]}$, where $\mathrm{x} = \varepsilon \cdot \lambda$, $\mathrm{y} = \varepsilon \cdot \mu$, $\mathrm{z} = \varepsilon \cdot \sigma$, $\{\varepsilon, \lambda, \mu, \sigma \in \mathrm{Z}^+\}$ (2). Let's analyze case and the other cases will be similar, according to relations (1 and 2), the following equivalences are obtained:

$$x^p + y^q = z^w \Leftrightarrow \lambda^p \cdot \varepsilon^p + \varepsilon^q \cdot \mu^q = \varepsilon^w \cdot \sigma^w \Leftrightarrow \lambda^p \cdot \varepsilon^{p-q} + \mu^q = \varepsilon^{w-q} \cdot \sigma^w, w > p > q > 1 \in Z+ \tag{3}$$

According to equation (3) the detailed order of all cases is as follows and as we can see there are only 6 :

1. $\lambda^p \cdot \varepsilon^{p-q} + \mu^q = \varepsilon^{w-q} \cdot \sigma^w, \{w, p, q \in \mathbb{Z}^+_{>1}\}$
2. $\lambda^p + \varepsilon^{q-p}\mu^q = \varepsilon^{w-p} \cdot \sigma^w, \{w, p, q \in \mathbb{Z}^+_{>1}\}$
3. $\lambda^p \cdot \varepsilon^{p-q} + \mu^q = \varepsilon^{w-q} \cdot \sigma^w, \{p, w, q \in \mathbb{Z}^+_{>1}\}$
4. $\lambda^p \cdot \varepsilon^{p-w} + \varepsilon^{q-w} \cdot \mu^q = \sigma^w, \{p, q, w \in \mathbb{Z}^+_{>1}\}$
5. $\lambda^p \cdot \varepsilon^{p-w} + \varepsilon^{q-w}\mu^q = \sigma^w, \{q, p, w \in \mathbb{Z}^+_{>1}\}$
6. $\lambda^p + \varepsilon^{q-p}\mu^q = \varepsilon^{w-p} \cdot \sigma^w, \{q, w, p \in \mathbb{Z}^+_{>1}\}$

Let us now analyse the cases that the exponents p, q, w can take only then we can know which cases we can accept that there are

I) $p = q \wedge w = q \Leftrightarrow p = q = w$

II) $p = q \wedge w \neq q$

III) $p \neq q \wedge w = q$

IV) $p \neq q \wedge w \neq q$

Relation (I) cannot be valid because leads to Fermat's equation(If $p, q, w > 2$ integers) and is therefore impossible. All the others can be valid by assumption and as we will see below at least one of p, q and w must be equal to 2 , but not a Pythagorean triad. In our analysis in Lemma 2 , however, we will accept only IV. But the technique that will allow us to analyze each case can only be given by an algorithm and this is given in Lemma 2. There we analyze 2 forms one is related to trying to find what the variables x, y, z are for a range of values up to 100 but being the first among them and the second if is given the variables x, y, z in which set of bases and exponents they are related to Lemma 1.

5.2 Lemma 2.

Prove that there is a relation connecting the bases and exponents to form a simplified algorithm to get the fastest calculation of equation $x^p + y^q = z^w$.

Proof

I) Dividing each variable $\{x, y, z\}$ with $GCD[x, y, z]$ and simplifying exhibitors, removing the minimum of exhibitors from every exhibitor shows a simplified form as final variables, according to the shape.

$x1 = (\mathrm{GCD}\,[\mathrm{x,y,z}])^\wedge\,(\mathrm{p} - \mathrm{Min[p,q,w]})^*\,(\mathrm{x/GCD}\,[\mathrm{x,y,z}])^\wedge\,\mathrm{p};$

$x2 = (\mathrm{GCD}\,[\mathrm{x,y,z}])^\wedge\,(\mathrm{q} - \mathrm{Min}\,[\mathrm{p,q,w}])^*\,(\mathrm{y/GCD}\,[\mathrm{x,y,z}])^\wedge\,\mathrm{q};$

$x3 = (\mathrm{GCD}[x, y, z])^\wedge(w - \mathrm{Min}[\mathrm{p}, q, w])^*(z/\,\mathrm{GCD}[x, y, z])^\wedge\mathrm{w};$

Therefore, we can write a simple but quick program with mathematica, which gives us the value of let's say up to 100 for the variables, with additional conditions for each variable with the others being prime numbers.Surely this could be feasible if we get many instances by constructing the corresponding program, which would be grouped by basis and exponents.

```
Clear[x, y, z, p, q, w]
Do [If [(x)^p + (y)^q == (z)^w&&GCD[x, y, z] = 1,
  x1 = (GCD[x, y, z])^(p − Min[p, q, w])*(x/GCD[x, y, z])^p;
  x2 = (GCD[x, y, z])^(q − Min[p, q, w])*(y/GCD[x, y, z])^q;
  x3 = (GCD[x, y, z])^(w − Min[p, q, w])*(z/GCD[x, y, z])^w
  ff = GCD[x1, x2, x3];
  d1 = FactorInteger [x1/ff]
  d2 = FactorInteger [x2/ff];
  d3 = FactorInteger [x3/ff];
Print["(",x,"∧",p,",",y,"∧",q,",",z,"∧",w,")",",",x1,",",x2,",",x3,",",d1,",",,d2,",",d3]],
{x, 1, 100}, {y, 1, 100}, {z, 1, 100}, {p, 3, 10}, {q, 2, 10}, {w, 3, 10}]
```

Additionally the function FactorInteger [ ] automatically configures all analyzes powers of primes of each variable. In the whole process we accept p > 2 and q >= 2 and w > 2 and furthermore ($xmax = ymax = zmax = 100$) and then we see that at least one exponent is equal to 2 . We also call $\sigma(\mathrm{p,q,w}) = 1/\mathrm{p} + 1/\mathrm{q} + 1/\mathrm{w}$ and compare each case, a condition we will discuss in chapter 4.

Results:

$(2^5, 7^2, 3^4), 32, 49, 81, \{\{2, 5\}\}, \{\{7, 2\}\}, \{\{3, 4\}\}, \sigma(\mathrm{p,q,w}) < 1$

$(3^5, 10^2, 7^3), 243, 100, 343, \{\{3, 5\}\}, \{\{2, 2\}, \{5, 2\}\}, \{\{7, 3\}\}, \sigma(\mathrm{p,q,w}) > 1$

$(3^4, 46^2, 13^3), 81, 2116, 2197, \{\{3, 4\}\}, \{\{2, 2\}, \{23, 2\}\}, \{\{13, 3\}\}, \sigma(\mathrm{p,q,w}) > 1$

$(7^3, 13^2, 2^9), 343, 169, 512, \{\{7, 3\}\}, \{\{13, 2\}\}, \{\{2, 9\}\}, \sigma(\mathrm{p,q,w}) < 1$

If we choose to have p > 2 and q > 2 and w >= 2 then we will get the Printout according to the command

```
Print["(",x,"^",p,",",y,"^",q,",",z,"^",w,")",",",x1,",",x2,",",x3,",",d1,",",,d2,",",d3]],
{x, 1, 100}, {y, 1, 100}, {z, 1, 100}, {p, 3, 10}, {q, 3, 10}, {w, 2, 10}]
```

Results:

$\left(2^7, 17^3, 71^2\right), 128, 4913, 5041, \{\{2,7\}\}, \{\{17,3\}\}, \{\{71,2\}\}, \sigma(\mathrm{p,q,w}) < 1$
$\left(5^4, 6^3, 29^2\right), 625, 216, 841, \{\{5,4\}\}, \{\{2,3\},\{3,3\}\}, \{\{29,2\}\}, \sigma(\mathrm{p,q,w}) > 1$
$\left(6^3, 5^4, 29^2\right), 216, 625, 841, \{\{2,3\},\{3,3\}\}, \{\{5,4\}\}, \{\{29,2\}\}, \sigma(\mathrm{p,q,w}) > 1$
$(7^4, 15^3, 76^2), 2401, 3375, 5776, \{\{7,4\}\}, \{\{3,3\},\{5,3\}\}, \{\{2,4\},\{19,2\}\}\ \sigma(\mathrm{p,q,w}) > 1$

In total we have 4 more cases than the already known ones we will see in chapter 4. As we observe, however, we also have cases with $\sigma > 1$ which result from the computational analysis.

II. To do in the given case especially this analysis knowing the values of x, y, z we can construct the corresponding program

```
Clear[x, y, z]
x := 34; y := 51; z := 85;
Do [If [(x)^p + (y)^q == (z)^w&&GCD[x, y, z] = 1,
  x1 = (GCD[x, y, z])^(p − Min[p, q, w])*(x/GCD[x, y, z])^p;
  x2 = (GCD[x, y, z])^(q − Min[p, q, w])*(y/GCD[x, y, z])^q;
  x3 = (GCD[x, y, z])^(w − Min[p, q, w])*(z/GCD[x, y, z])^w
  ff = GCD[x1, x2, x3];
  d1 = FactorInteger [x1/ff]
  d2 = FactorInteger [x2/ff];
  d3 = FactorInteger [x3/ff];
Print["(",x,"^",p,",",y,"^",q,",",z,"^",w,")",",",x1,",",x2,",",x3,",",d1,",",,d2,",",d3]],
{p, 2, 10}, {q, 2, 10}, {w, 2, 10}]
```

Results

(34^5, 51^4, 85^4) , 544, 81, 625, ({{2, 5}, {17, 1}}, {3, 4}, {5, 4})

To give a typical example, we get the equation $34^5 + 51^4 = 85^4$. Notice that it is analyzed as {2^5*17^1 +3^4 = 5^4}. Of course this case is not acceptable as it is easy to see that the truth of Theorem 3 is valid. So the results of the first programme are the triads we are mainly interested in. In chapter 7 we have some examples that we examine to see if they agree with these 2 Lemmas.

6. Theorems that prove the Proposal 1(Of Beal's)

Proposal 1. The equation $x^p + y^q = z^w$ has no solution in positive integers x, y, z, p, q, w, when apply $(p, q, w > 2)$.

6.1 Theorem 5

Any equation form $x^p + y^q = z^w$ with positive integers x, y, z, p, q, w where $p, q, w > 1$, is transformed into a final Diophantine equation with GCD$(x, y, z) = 1$ then and only then, when at least one exponent equals 2. This equation will belong to a class of equations with exponents that be consistent with the criteria $\sigma(p, q, r) > 1, \sigma(p, q, r) = 1$ or $\sigma(p, q, r) < 1$ with a limited number equations, in accordance with chapter 4.

Proof

The number of the forms of $x^q + y^p = z^w, x, y, z, p, q, w \in Z^+ \wedge \{q >= 2, p >= 2, w >= 2\}$ after simplifying the terms of the $GCD[x, y, z]$, Lemma 1, Lemma 2 limited to 6. Depending on the ascending order of exponents $\{p, q, w\}$ of original Diophantine equation $x^p+y^q = z^w, x, y, z, p, q, w \in Z^+ \wedge \{q >= 2, p >= 2, w >= 2\}$ and after simplifying the terms with the number $\varepsilon = GCD[x, y, z]$, we receive a total of 6 cases where any stemming detail has as follows

1. $\lambda^p \cdot \varepsilon^{p-q} + \mu^q = \varepsilon^{w-q} \cdot \sigma^w, w > p > q >= 2 \in Z+$
2. $\lambda^p + \varepsilon^{q-p}\mu^q = \varepsilon^{w-p} \cdot \sigma^w, w > q > p >= 2 \in Z+$
3. $\lambda^p \cdot \varepsilon^{p-q} + \mu^q = \varepsilon^{w-q} \cdot \sigma^w, p > w > q > 2 \in Z+$
4. $\lambda^p \cdot \varepsilon^{p-w} + \varepsilon^{q-w} \cdot \mu^q = \sigma^w, p > q > w >= 2 \in Z+$
5. $\lambda^p \cdot \varepsilon^{p-w} + \varepsilon^{q-w}\mu^q = \sigma^w, q > p > w >= 2 \in Z+$
6. $\lambda^p + \varepsilon^{q-p}\mu^q = \varepsilon^{w-p} \cdot \sigma^w, q > w > p >= 2 \in Z+$

But these exhibitors must comply with the Fermat-Catalan criteria, but here we will analyse them in general terms, distinguishing 3 general cases:

if we accept that p,q and w are fixed positive integers and that these exponents must satisfy the criteria of chapter 4, and after first accepting $p, q, w >= 2$, we will prove that at least one exponent equals 2 using these criteria alone. So according to this logic the following 3 cases will apply:

Case 1$^{\text{rd}}$

$0 < 1/p + 1/q + 1/w < 1$

In order to we calculate the exhibitors present in the open interval $(0, 1)$ solve the inequality as z and we get

$$1/w < 1 - \frac{p+q}{p \cdot q} \Rightarrow w > \frac{p \cdot q}{p(q-1) - q}$$

The inequality has integer solutions which arise only in accordance with the 3 equations:

$$\begin{aligned}
&(1).\ p \cdot (q-1) - q = 1 \\
&(2).\ q = \varphi \cdot (p \cdot (q-1) - q) \\
&(3).\ p = \varepsilon \cdot (p \cdot (q-1) - q) \\
&\varepsilon, \varphi \in Z
\end{aligned}$$

1.From the first equation it follows that

$p \cdot (q-1) = q+1 \Rightarrow p = \frac{1+q}{q-1} = 1 + \frac{2}{q-1}$ which implies 2 prerequisites:

i) $q-1 = 1 \Rightarrow q = 2 \wedge p = 3$

ii) $q-1 = 2 \Rightarrow q = 3 \wedge p = 2$

because should the $(q-1)$ must divide 2

And for 2 exhibitor cases we get $\boxed{w > 6 \Rightarrow w \geq 7}$

Therefore Thus arise the two triads $\boxed{p = 3, q = 2, w \geq 7}$ and $\boxed{p = 2, q = 3, w \geq 7}$

2. Similarly from the second equation $q = \phi \cdot (p \cdot (q-1) - q)$ we get:

$$q = \phi \cdot p \cdot (q-1) - q \cdot \varphi \Rightarrow p = \frac{q \cdot (1+\phi)}{\phi \cdot (q-1)}$$

$$i)\phi(q-1) = 1 \Rightarrow \phi = \frac{1}{q-1} = 1 \wedge q-1 = 1 \Rightarrow q = 2$$

$$p = \frac{q \cdot (1+\phi)}{\phi \cdot (q-1)} = \frac{2 \cdot 2}{1} = 4$$

$$w > \frac{p \cdot q}{p(q-1) - q} = \frac{4 \cdot 2}{4 \cdot 1 - 2} = 4, w \geq 5$$

Hence the triad

$\boxed{p = 4, q = 2, w \geq 5}$

$$ii)q = \sigma(q-1) \wedge (1+\phi) = \lambda \cdot \phi$$

$$a)\phi = \frac{1}{\lambda - 1} \Rightarrow \lambda - 1 = 1 \Rightarrow (\lambda = 2 \wedge \phi = 1)$$

$$\beta)q \cdot (\sigma - 1) = \sigma \Rightarrow q = \frac{\sigma}{\sigma - 1} = 1 + \frac{1}{\sigma - 1} == 2 \wedge \sigma - 1 = 1 \Rightarrow (\sigma = 2 \wedge q = 2)$$

$$p = \frac{q \cdot (1+\phi)}{\phi \cdot (q-1)} = \frac{2 \cdot 2}{1 \cdot 1} = 4, w > \frac{p \cdot q}{p(q-1) - q} = \frac{4 \cdot 2}{4 \cdot (2-1) - 2} = 4, w \geq 5$$

Therefore resulting triad

$\boxed{|p = 4, q = 2, w \geq 5|}$

$iii) q = \sigma \cdot \phi \wedge (1+\phi) = \lambda \cdot (y-1)$

$a) \lambda = \dfrac{1+\phi}{y-1} = \dfrac{1}{q-1} + \dfrac{\phi}{q-1} \wedge q-1 = 1 \Rightarrow (q=2 \wedge \lambda = 3)$

$\sigma \cdot \phi = 2 \Rightarrow (\sigma = 1 \wedge \phi = 2), (\sigma = 2 \wedge \phi = 1)$

$q = 2 \wedge \phi = 1 \Rightarrow p = \dfrac{q \cdot (1+\phi)}{\phi \cdot (q-1)} = \dfrac{2}{1}\dfrac{2}{1} = 4, w > \dfrac{p \cdot q}{p(q-1)-q} = \dfrac{4 \cdot 2}{4 \cdot 1 - 2} = 4$

$q = 2 \wedge \phi = 2 \Rightarrow p = \dfrac{q \cdot (1+\phi)}{\phi \cdot (q-1)} = \dfrac{2}{2}\dfrac{3}{1} = 3, w > \dfrac{p \cdot q}{p(q-1)-q} = \dfrac{3 \cdot 2}{3 \cdot 1 - 2} = 6$

Thus arise the two triads

$\boxed{p = 4, q = 2, w \geq 5}$ and $\boxed{p = 3, q = 2, w \geq 7}$

3.Similarly from equation $p = \varepsilon \cdot (p \cdot (q-1) - q)$ take that:

$p = \varepsilon \cdot p \cdot (q-1) - q \cdot \varepsilon \Rightarrow q = \dfrac{p \cdot (1+\varepsilon)}{\varepsilon \cdot (p-1)}$

$i) \varepsilon(p-1) = 1 \Rightarrow \varepsilon = \dfrac{1}{p-1} = 1 \wedge p-1 = 1 \Rightarrow p = 2$

$q = \dfrac{p \cdot (1+\varepsilon)}{\varepsilon \cdot (p-1)} = \dfrac{2 \cdot 2}{1} = 4$

$w > \dfrac{p \cdot q}{p(q-1)-q} = \dfrac{4 \cdot 2}{2 \cdot 3 - 4} = 4, w \geq 5$

Therefore shows the triad

$\boxed{q = 4, p = 2, w \geq 5}$

$ii) p = \varepsilon(p-1) \wedge (1+\varepsilon) = \lambda \cdot \varepsilon$

$a) \varepsilon = \dfrac{1}{\lambda - 1} \Rightarrow \lambda - 1 = 1 \Rightarrow (\lambda = 2 \wedge \varepsilon = 1)$

$b) p \cdot (\varepsilon - 1) = \varepsilon \Rightarrow p = \dfrac{\varepsilon}{\varepsilon - 1} = 1 + \dfrac{1}{\varepsilon - 1} == 2 \wedge \varepsilon - 1 = 1 \Rightarrow (\varepsilon = 2 \wedge p = 2)$

$q = \dfrac{p \cdot (1+\varepsilon)}{\varepsilon \cdot (p-1)} = \dfrac{2 \cdot 2}{1 \cdot 1} = 4, w > \dfrac{p \cdot q}{p(q-1)-q} = \dfrac{4 \cdot 2}{2 \cdot (4-1) - 4} = 4, w \geq 5$

Hence the triad

$\boxed{p = 2, q = 4, w \geq 5}$

$iii) p = \varepsilon \cdot \phi \wedge (1+\varepsilon) = \lambda \cdot (p-1)$

$a) \lambda = \dfrac{1+\varepsilon}{p-1} = \dfrac{1}{p-1} + \dfrac{\varepsilon}{p-1} \wedge p-1 = 1 \Rightarrow (p=2)$

$\varepsilon \cdot \phi = 2 \Rightarrow (\varepsilon = 1 \wedge \phi = 2), (\varepsilon = 2 \wedge \phi = 1)$

$p = 2 \wedge \phi = 1 \Rightarrow q = \dfrac{p \cdot (1+\phi)}{\phi \cdot (p-1)} = \dfrac{2}{1}\dfrac{2}{1} = 4, w > \dfrac{p \cdot q}{p(q-1)-q} = \dfrac{4 \cdot 2}{2 \cdot 3 - 4} = 4$

$p = 2 \wedge \phi = 2 \Rightarrow q = \dfrac{p \cdot (1+\phi)}{\phi \cdot (p-1)} = \dfrac{2}{2}\dfrac{3}{1} = 3, w > \dfrac{p \cdot q}{p(q-1)-q} = \dfrac{3 \cdot 2}{2 \cdot 1 - 2} = 6$

Thus arise the two triads

$\boxed{q = 4, p = 2, w \geq 5}$ and $\boxed{q = 3, p = 2, w \geq 7}$

Total we have 12 cases for exhibitors and cyclically we will have

(i)

$$\boxed{\begin{array}{l} p = 3, q = 2, w \geq 7 \wedge p = 2, q = 3, w \geq 7 \\ w = 3, p = 2, q \geq 7 \wedge w = 2, p = 3, q \geq 7 \\ w = 3, q = 2, p \geq 7 \wedge w = 2, q = 3, p \geq 7 \\ q = 4, p = 2, w \geq 5 \wedge q = 2, p = 4, w \geq 5 \\ w = 4, p = 2, q \geq 5 \wedge w = 2, p = 4, q \geq 5 \\ w = 4, q = 2, p \geq 5 \wedge w = 2, q = 4, p \geq 5 \end{array}}$$

Which in relation to equations take the form

(ii)

$$\boxed{\begin{array}{l} x^3 + y^2 = z^w, w \geq 7 \\ x^2 + y^3 = z^w, w \geq 7 \\ x^2 + y^q = z^3, q \geq 7 \\ x^3 + y^q = z^2, q \geq 7 \\ x^p + y^2 = z^3, p \geq 7 \\ x^p + y^3 = z^2, p \geq 7 \end{array}} \qquad \boxed{\begin{array}{l} x^2 + y^q = z^4, q \geq 5 \\ x^4 + y^q = z^2, q \geq 5 \\ x^2 + y^4 = z^w, w \geq 5 \\ x^4 + y^2 = z^w, w \geq 5 \\ x^p + y^2 = z^4, p \geq 5 \\ x^p + y^4 = z^2, p \geq 5 \end{array}}$$

Characteristics mention the work of Jamel Ghanouchi "A new approach of Fermat-Catalan conjecture" that achieves the same result.

The generalized Fermat conjecture (Darmon and Granville, 1995; Darmon, 1997), also known as the Tijdeman-Zagier conjecture and as the Beal conjecture (Beukers, 2012), is concerned with the case $\chi < 1$.

It states that the only non-trivial primitive solutions to $x^q + y^p = z^w$ with $\sigma(\mathrm{p,g,r}) < 1$ are

$$2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2, \quad 3^5 + 11^4 = 122^2,$$
$$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7, 9262^3 + 15312283^2 = 113^7,$$
$$43^8 + 96222^3 = 30042907^2 \text{ and } 33^8 + 1549034^2 = 15613^3.$$

The generalized Fermat conjecture has been documented for many signatures (p, q, r), including many infinite families of signatures, starting with Fermat's last theorem (p, p, p) by Wiles (1995). The remaining cases are reported in Chapter 4.

Case 2rd. $1/p + 1/q + 1/w = 1$

i) From case 1 shows that overall we have 12 cases for exhibitors and and we roundly take:

$$\underset{\text{(i)}}{\boxed{\begin{array}{l} p=3,q=2,w\geq 7 \wedge p=2,q=3,w\geq 7 \\ w=3,p=2,q\geq 7 \wedge w=2,p=3,q\geq 7 \\ w=3,q=2,p\geq 7 \wedge w=2,q=3,p\geq 7 \\ q=4,p=2,w\geq 5 \wedge q=2,p=4,w\geq 5 \\ w=4,p=2,q\geq 5 \wedge w=2,p=4,q\geq 5 \\ w=4,q=2,p\geq 5 \wedge w=2,q=4,p\geq 5 \end{array}}} \quad \Leftrightarrow \quad \underset{\text{(ii)}}{\boxed{\begin{array}{l} p=3,q=2,w>6 \wedge p=2,q=3,w>6 \\ w=3,p=2,q>6 \wedge w=2,p=3,q>6 \\ w=3,q=2,p>6 \wedge w=2,q=3,p>6 \\ q=4,p=2,w>4 \wedge q=2,p=4,w>4 \\ w=4,p=2,q>4 \wedge w=2,p=4,q>4 \\ w=4,q=2,p>4 \wedge w=2,q=4,p>4 \end{array}}}$$

But the inequality (ii), for example, $p = 3, q = 2, w > 6$ as well as the inequality $q = 4, p = 2, w > 4$ which is characteristic of the group of exhibitors according to the criterion $0 < 1/p + 1/q + 1/w < 1$, so for the exponent group to have equality, 12 relations will apply cyclically as follows:

(iii)

$$\boxed{\begin{array}{l} p=3,q=2,w=6 \wedge p=2,q=3,w=6 \\ w=3,p=2,q=6 \wedge w=2,p=3,q=6 \\ w=3,q=2,p=6 \wedge w=2,q=3,p=6 \\ q=4,p=2,w=4 \wedge q=2,p=4,w=4 \\ w=4,p=2,q=4 \wedge w=2,p=4,q=4 \\ w=4,q=2,p=4 \wedge w=2,q=4,p=4 \end{array}}$$

ii) Pending from only the case $3/p = 1 \Rightarrow p = 3$ which implies $p = q = w = 3$. But this case according to the proof of Fermat's theorem does not accept solutions with exponents greater than 2.

Case 3rd. $1/p + 1/q + 1/w > 1$

Originally accept that $p >= 2, q >= 2$ and $w >= 2$. We examine three cases:

i) $p = q = w = 2$ which is true

ii) $p = q = 2 \Rightarrow w > 2$ which is true we cyclically for the other exhibitors that $p = w = 2 \Rightarrow q > 2$ and $q = w = 2 => p > 2$.

iii) For all other cases will apply in accordance with the relation (iii) the second case, because now would force the inequality < 6, i.e total of 12 relations for all exhibitors.

$$\boxed{\begin{array}{l} p = 3, q = 2, \{2 <= w <= 5\} \wedge p = 2, q = 3, \{2 <= w <= 5\} \\ w = 3, p = 2, \{2 <= q <= 5\} \wedge w = 2, p = 3, \{2 <= q <= 5\} \\ w = 3, q = 2, \{2 <= p <= 5\} \wedge w = 2, q = 3, \{2 <= p <= 5\} \\ q = 4, p = 2, \{2 <= w <= 3\} \wedge q = 2, p = 4, \{2 <= w <= 3\} \\ w = 4, p = 2, \{2 <= q <= 3\} \wedge w = 2, p = 4, \{2 <= q <= 3\} \\ w = 4, q = 2, \{2 <= p <= 3\} \wedge w = 2, q = 4, \{2 <= p <= 3\} \end{array}}$$

(iv)

6.2 Theorem 6

The equation $x^p + y^q = z^w$ with positive integers x, y, z and extra $(p, q, w >= 2)$ and p, q and w are fixed positive integers is solved if and only if apply the conditions of Theorem 5, $(1, 2, 3)$ cases for exponents p, q, w with extra $(x, y, z) = 1$, and at least one of them equal 2. Therefore Beal's Conjecture is true with the above conditions, because accepts that there is no solution under the condition that all values of the exponents greater of 2.

Proof

For the equation $x^p + y^q = z^w$ with positive integers $x, y, z, (p, q, w >= 2)$ demonstrated that solved if and only if apply the conditions of Theorem 5 (i, ii, iii) for the exponents p, q, w with extra $(x, y, z) = 1$, so we have analytical

i) $1/p + 1/q + 1/w < 1$

According to Theorem 5, and 1 case, there is a solution to obtain values for the group of exhibitors $\{p, q, w\}$ as follows:

$$\boxed{\begin{array}{l} p = 3, q = 2, w \geq 7 \wedge p = 2, q = 3, w \geq 7 \\ w = 3, p = 2, q \geq 7 \wedge w = 2, p = 3, q \geq 7 \\ w = 3, q = 2, p \geq 7 \wedge w = 2, q = 3, p \geq 7 \\ q = 4, p = 2, w \geq 5 \wedge q = 2, p = 4, w \geq 5 \\ w = 4, p = 2, q \geq 5 \wedge w = 2, p = 4, q \geq 5 \\ w = 4, q = 2, p \geq 5 \wedge w = 2, q = 4, p \geq 5 \end{array}}$$

Which clearly shows that $p = 2$ or $q = 2$ or $w = 2$. Therefore least one exponent $= 2$.

ii) $1/p + 1/q + 1/w = 1$

It happens the second case, Theorem 5, for exist solution will arrive at values for the group of exhibitors

$\{p, q,$ w$\}$ as follows:

$$\boxed{\begin{array}{l} p=3, q=2, w=6 \wedge p=2, q=3, w=6 \\ w=3, p=2, q=6 \wedge w=2, p=3, q=6 \\ w=3, q=2, p=6 \wedge w=2, q=3, p=6 \\ q=4, p=2, w=4 \wedge q=2, p=4, w=4 \\ w=4, p=2, q=4 \wedge w=2, p=4, q=4 \\ w=4, q=2, p=4 \wedge w=2, q=4, p=4 \end{array}}$$

Which also seems that $p = 2$ or $q = 2$ or $w = 2$. Therefore least one exponent equal 2.

iii) $1/p + 1/q + 1/w > 1$

For the third case, the Theorem 5, to obtain a solution we will arrive at values for the group of exhibitors $\{p, q, w\}$ as follows:

$$\boxed{\begin{array}{l} p=3, q=2, \{2 <= w <= 5\} \wedge p=2, q=3, \{2 <= w <= 5\} \\ w=3, p=2, \{2 <= q <= 5\} \wedge w=2, p=3, \{2 <= q <= 5\} \\ w=3, q=2, \{2 <= p <= 5\} \wedge w=2, q=3, \{2 <= p <= 5\} \\ q=4, p=2, \{2 <= w <= 3\} \wedge q=2, p=4, \{2 <= w <= 3\} \\ w=4, p=2, \{2 <= q <= 3\} \wedge w=2, p=4, \{2 <= q <= 3\} \\ w=4, q=2, \{2 <= p <= 3\} \wedge w=2, q=4, \{2 <= p <= 3\} \end{array}}$$

in which at least appear that one of the $p = 2$ or $q = 2$ or $w = 2$.

Therefore at least one exponent equals 2 to have a solution and hence play Beal's Conjecture is true, because it recognizes that there is no solution if all values of the exponents greater 2.

6.3 Lemma 3

According to the known identity that satisfies the raising integer power will apply:

$$a^{\phi} = \left(\begin{array}{l} 1.\phi = 0 \\ 2.\alpha^{*}m, (\alpha, m, \phi \in Z^{+}) \end{array} \right)$$

Proof

If $a, \phi \in \mathrm{Z}^{+}$ then will apply to the exhibitor of a:

i) $a^{0} = 1$ applicable

ii) $a^{\phi} = \alpha \cdot \alpha^{\phi-1} = \alpha \cdot m, m \in Z^{+} \wedge m = a^{\phi-1}$ which this applies.

6.4 Formulations of the equation so that the exponents can be calculated by algorithm

According to Theorem 2 that the Diophantine equation $x^p + y^q = z^w$ where we apply that $q, p, w \in Z^+$ we define the F(x, y, z) function variables, x, y, z in particular integers as follows

$$F(x, y, z) = \left\{\exists x, y, z \in Z^3 : A(x) = x^{q-1}, B(y) = y^{p-1}, C(z) = z^{w-1} \wedge A(x)x + B(y)y = C(z)z\right\}$$

with additional constraints $q, p, w \in Z \wedge \{q > 1, p > 1, w > 1\}$.

For (kerf= 0) which means to solve the system of equations $F(x, y, z) = 0$ we must be true $\mathrm{kerF} = \left\{\exists(\mathrm{x,y,z}) \in \mathrm{Z}^3 : \mathrm{f(x,y,z)} = 0\right\} \subseteq \mathrm{Z}^3$ and finally after replacing the f becomes

i) If we first divide initially with x and $x \neq 0$ the variables {y, z}, all the terms of the equation $x^d + y^p = z^w$, we get with the existing conditions and we have:

$$A(x) = x^{q-1}, B(y) = y^{p-1}, C(z) = z^{w-1}. \wedge A(x)x + B(y)y = C(z) \cdot z.$$

Thus according to Theorem 1 will apply to:

$$\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{x} = (\mathrm{C(z)} \cdot \lambda - \mathrm{B(y)}) \cdot \mathrm{k} \\ \mathrm{y} = (\mathrm{A(x)} - \mathrm{C(z)}) \cdot \mathrm{k} \\ \mathrm{z} = (\mathrm{A(x)} \cdot \lambda - \mathrm{B(y)}) \cdot \mathrm{k} \end{pmatrix} (1) \text{ with more generally } \mathrm{k} \in Z^+, \lambda \in Q^+.$$

But in accordance with Lemma 3 will have the equivalences:

$$A(x)^{x-1} = A(x) \cdot m_1 \wedge B(y)^{y-1} = B(y) \cdot m_2 \wedge C(z)^{z-1} = C(z) \cdot m_3 (2)$$

From (1) and (2) implies that

$$\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{A(x)} \cdot \mathrm{m_1} = (\mathrm{C(z)} \cdot \lambda - \mathrm{B(y)}) \cdot \mathrm{k} \\ \mathrm{k} = \dfrac{\mathrm{B(y)^{y-1}}}{\mathrm{A(x)} - \mathrm{C(z)}} \\ \mathrm{C(z)} \cdot \mathrm{m_3} = (\mathrm{A(x)} \cdot \lambda - \mathrm{B(y)}) \cdot \mathrm{k} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} (\mathrm{A}(x))(\mathrm{A(x)} - \mathrm{C(z)}) \cdot \mathrm{A(x)} \cdot \mathrm{m_1} = \mathrm{A(x)} \cdot (\mathrm{C(z)} \cdot \lambda - \mathrm{B(y)}) \cdot \mathrm{B(y)^{y-1}} \\ \mathrm{k} = \dfrac{\mathrm{B(y)^{y-1}}}{\mathrm{A(x)} - \mathrm{C(z)}} \\ (-C(z)) \cdot (\mathrm{A(x)} - \mathrm{C(z)}) \cdot \mathrm{C(z)} \cdot \mathrm{m_3} = (-C(z)) \cdot (\mathrm{A(x)} \cdot \lambda - \mathrm{B(y)}) \cdot \mathrm{B(y)} \end{pmatrix}$$

and after replacing the relations and simplifying the system we will arrive at the final

$$(\mathrm{A(x)} - \mathrm{C(z)}) \cdot \left\{\mathrm{A(x)^2} \cdot \mathrm{m_1} - \mathrm{C(z)^2} \cdot \mathrm{m_3}\right\} = (\mathrm{C(x)} - \mathrm{A(x)}) \cdot \mathrm{B(y)^y} \Leftrightarrow$$

relationship:

$$\boxed{\mathrm{C(z)^2} \cdot \mathrm{m_3} - \mathrm{A(x)^2} \cdot \mathrm{m_1} = \mathrm{B(y)^y}} \tag{3}$$

ii) Also if we divide by y if $y \neq 0$ the variables $\{x, z\}$, we will certainly force the same method on the function F, given will by the general form:

$$\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{y = (C(z) \cdot \lambda - A(y)) \cdot k} \\ \mathrm{x = (B(x) - C(z)) \cdot k} \\ \mathrm{z = (B(x) \cdot \lambda - A(y)) \cdot k} \end{pmatrix} \tag{4}$$

With extra generally $\kappa, \lambda \in Q^+$. But according to Lemma 3 and relation (3), we have the equivalences:

$$\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{k} = \dfrac{\mathrm{A(y)^{x-1}}}{\mathrm{B(y) \cdot C(z)}} \\ \mathrm{B(y) \cdot m_2 = (C(z) \cdot \lambda - A(x)) \cdot k} \\ \mathrm{C(z) \cdot m_3 = (B(y) \cdot \lambda - A(x)) \cdot k} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \mathrm{k} = \dfrac{\mathrm{A(y)^{x-1}}}{\mathrm{B(y) - C(z)}} \\ (B(y))\mathrm{(B(y) - C(z)) \cdot B(y) \cdot m_1 = B(y) \cdot (C(z) \cdot \lambda - A(x)) \cdot A(x)^{x-1}} \\ (-C(z)) \cdot \mathrm{(B(y) - C(z)) \cdot C(z) \cdot m_3} = (-C(z)) \cdot \mathrm{(B(y) \cdot \lambda - A(x)) \cdot A(x)^{x-1}} \end{pmatrix}$$

and after replacing the relations and simplifying the system we arrive at the final relationship

$$\begin{aligned} &\mathrm{(B(y) - C(z)) \cdot \left\{B(y)^2 \cdot m_2 - C(z)^2 \cdot m_3\right\} = (C(x) - B(y)) \cdot A(x)^x} \\ &\Leftrightarrow \boxed{\mathrm{C(z)^2 \cdot m_3 - B(y)^2 \cdot m_2 = A(x)^x}} \end{aligned} \tag{5}$$

iii) Finally, in the same way and dividing by the variable z the $\{x, y\}$, where $z \neq 0$ will occur in accordance with the basic relation and in such a use of Theorem 1, the function F which is given by the general form:

$$\mathrm{F(x,y,z)} = \begin{pmatrix} \mathrm{x = (C(z) + \lambda \cdot B(y)) \cdot k} \\ \mathrm{z = (A(x) + B(y)) \cdot k} \\ \mathrm{y = (C(z) - \lambda \cdot A(x)) \cdot k} \end{pmatrix} \text{ with more generally } \mathrm{k} \in Z^+, \lambda \in Q^+ \tag{6}$$

But in accordance with Lemma 3 and relation (3), we have the equivalences

$$\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{A(x) \cdot m_1 = (C(z) + \lambda \cdot B(y)) \cdot k} \\ \mathrm{C(z) \cdot m_3 = (C(z) - \lambda \cdot A(x)) \cdot k} \\ \mathrm{k} = \dfrac{\mathrm{C(z)^{z-1}}}{\mathrm{A(x) + B(z)}} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \mathrm{(A(x)) \cdot (A(x) + B(y)) \cdot A(x) \cdot m_1 = A(x) \cdot (C(z) + \lambda \cdot B(y)) \cdot C(z)^{z-1}} \\ (B(\mathrm{y})) \cdot \mathrm{(A(x) + B(y)) \cdot B(y) \cdot m_2} = (B(y)) \cdot \mathrm{(C(y) - \lambda \cdot A(x)) \cdot C(z)^z - 1} \\ \mathrm{k} = \dfrac{\mathrm{C(z)^{z-1}}}{\mathrm{A(x) + C(z)}} \end{pmatrix}$$

and after operations arrive and clear the system in the final relationship

$$\mathrm{(A(x) + B(y)) \cdot \left\{A(x)^2 \cdot m_1 + \mathit{B}(y)^2 \cdot m_2\right\} = (A(x) + B(y)) \cdot C(z)^z}$$

$$\boxed{\Leftrightarrow \mathrm{A(x)^2 \cdot m_1 + B(y)^2 \cdot m_2 = C(z)^z}} \tag{7}$$

For the case of Conjecture Beal, as we form Diophantine equation $a^x + b^y = c^z$.

They will apply to replacement equivalents

$$A(x) = a, B(y) = b, C(z) = c$$

And therefore arise 3 Diophantine equations of the first degree where calculated variables $\mathrm{m_1}$, $\mathrm{m_2}, \mathrm{m_3}$ and are resolved by known each category

$$\boxed{\begin{aligned} &\mathrm{c^2 \cdot m_3 - a^2 \cdot m_1 = b^y (3)} \\ &\mathrm{c^2 \cdot m_3 - b^2 \cdot m_2 = a^x (5)} \\ &\mathrm{a^2 \cdot m_1 + b^2 \cdot m_2 = c^z (7)} \end{aligned}} \text{ with } m_1, m_2, m_3 \in Z^+$$

But according to Theorem 5 & 6, two of the variables $\{x, y\}$ or $\{x, z\}$ or $\{y, z\}$ will have values through the set $\{3, 5, 7\}$. If therefore the system of primary Diophantos equations $\{\mathrm{i, ii, iii}\}$ import these values we calculate the variables $\mathrm{m_1, m_2, m_3}$ per category and therefore the remaining variables.

$$\mathbf{1^{st}} \boxed{\begin{aligned} &\mathrm{c^2 \cdot m_3 - \alpha^2 \cdot m_1 = b^y \ (i_1)} \\ &\mathrm{a^{x-1} = \alpha \cdot m_1 \ (ii_1)} \\ &\mathrm{c^{z-1} = c \cdot m_3 \ (iii_1)} \end{aligned}} \quad \mathbf{2^{nd}} \boxed{\begin{aligned} &\mathrm{c^2 \cdot m_3 - b^2 \cdot m_2 = \alpha^x \ (i_2)} \\ &\mathrm{b^{y-1} = b \cdot m_2 \ (ii_2)} \\ &\mathrm{c^{z-1} = c \cdot m_3 \ (iii_2)} \end{aligned}} \quad \mathbf{3^{rd}} \boxed{\begin{aligned} &\mathrm{\alpha^2 \cdot m_1 + b^2 \cdot m_2 = c^z \ (i_3)} \\ &\mathrm{b^{y-1} = b \cdot m_2 \ (ii_3)} \\ &\mathrm{\alpha^{x-1} = c \cdot m_1 \ (iii_3)} \end{aligned}}$$

$$\text{with } a, b, c, m_1, m_2, m_3 \in Z^+$$

According to these cases we can use a program to calculate exponents and variables cyclically so that their values are fully calculated. In the following we will see examples of how they are calculated with a program in mathematica.

7. Finding exhibitors theoretically-by Lemma 1 and Theorems (5 & 6).

7.1. Calculates of x, y, z on Diophantine equations

$$\boxed{\begin{aligned} &18^x + 9^y = 9^z, 32^x + 32^y = 4^z, 13^x + 7^y = 2^z, 7^x + 7^y = 98^z \\ &19^x + 38^y = 57^z, 34^x + 51^y = 85^z, 33^x + 66^y = 33^z \end{aligned}}$$

7.2. Diophantine solutions

7.2.1. Example $18^x + 9^y = 9^z$

By analysis under Lemma1 takes the form:

$$18^x + 9^y = 9^z \Leftrightarrow (2 \cdot 9)^x + 9^y = 9^z \Leftrightarrow 2^x + 9^{y-x} = 9^{z-x} \Leftrightarrow 2^x + 3^{2(y-x)} = 3^{2(z-x)}$$

Investigation:

1. Must $x = 3, (y - x) = 0$ as well as $2(z - x) = 2$, from which implies $x = 3, y = 3, z = 4$, which is accepted.
2. If $x = 4, y = 4$ and $z = 5$ is not acceptable.
3. If $x = 2$ then we have 2 cases
   i) $2(z - x) = 4$, and $z = 4$ impossible also because $3^{2(y-x)} = 3^4 - 3^2 = 77$.
   ii) $2(z - x) = 3$, and $z = 7/2$ also impossible not intact.

7.2.2. Example $32^x + 32^y = 4^z$

By analysis under Lemma1 takes the form $1 + 2^{5(y-x)} = 2^{2z-5x}$

Investigation:

1. Must $x - y = 0$ and then $x = y$ and also $2z - 5x = 1 \Leftrightarrow z = (1 + 5x)/2$.If $x = 1$ then $z = 3$ which is accepted. In General It holds for $x = 2k + 1, k$ in Z i.e $z = 5k + 3, x = y = 2k + 1, k$ in Z.

7.2.3. Example $13^x + 7^y = 2^z$

By analysis is the case $1/p + 1/q + 1/w < 1$ with p = 3&q = 2&w > 7. Therefore we have x $= 3$, y $= 2$ and $z = 9$ and takes the form $13^3 + 7^2 = 2^9$.

7.2.4. Example $7^x + 7^y = 98^z$

By analysis under Lemma1 takes the form $1 + 7^{y-x} = (2^z) \cdot 7^{(2z-x)}$

Investigation:

1. If $x = 3, (y - x) = 0$ as well $2z - x = 0$ implies $z = 3/2$ impossible.
2. If $x = 4, (y - x) = 0$ as well $2z - x = 0$, from which entails $z = 4/2 = 2$, which is not accepted
3. If $z = 3, (y - x) = 1$ and $2z - x = 0$ it follows ó $\tau x =$ 6k and $y = 7$ which is accepted
4. If $z = 1, (y - x) = 0$ and $2z - x = 0$ it follows ót $x = 2$ and $y = 2$, which is accepted

7.2.5. Examples $19^x + 38^y = 57^z$

By analysis under Lemma1 takes the form $19^{x-y} + 2^y 19^{y-z} = 3^z$

Investigation:

1. If $x - y = 0, z - x = 0, y = 1$ from where follows $x = y = z = 1$ which is accepted
2. If $x - y = 1, y = 3, y - z = 0, z = 3 => x = 4$ and $y = z = 3$ which is accepted

7.2.6. Example $34^x + 51^y = 85^z$

By analysis under Lemma 1 takes the form $2^x 17^x + 3^y 17^{y-z} = 3^z$

Investigation:

1. If $x - y = 1, y - z = 0, z = 4$ comes from where $x = 5, y = z = 4$ which is accepted
2. If $x - y = 0, y - z = 0, z = 1 => x = 1, y = z = 1$ which is accepted

7.2.7. Example $33^x + 66^y = 33^z$

By analysis under Lemma1 takes the form $3^{x-y} 11^{x-y} + 2^y = 3^{z-y} 11^{z-y}$

Investigation:

1. If $x - y = 0, z - y = 1, y = 5$ from where follows $z = 6, y = x = 5$ which is accepted

8. Calculation of Exhibitors. (Programs for Mathematica)

8.1. The first case

According to the second method of dividing initially with x where $x \neq 0$ the variables $\{y, z\}$, where terms in the equation $x^q + y^p = z^w \Rightarrow$

$$\boxed{\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{y} = (\mathrm{C(z)} \cdot \lambda - \mathrm{A(y)}) \cdot \mathrm{k} \\ \mathrm{x} = (\mathrm{B(x)} - \mathrm{C(z)}) \cdot \mathrm{k} \\ \mathrm{z} = (\mathrm{B(x)} \cdot \lambda - \mathrm{A(y)}) \cdot \mathrm{k} \end{pmatrix}} \text{extra generality } \mathrm{k} \in Z^+, \lambda \in Q^+. \tag{1}$$

And according to the analysis of the second method results in the

$$\boxed{\begin{array}{l} \mathrm{c}^2 \cdot \mathrm{m}_3 - \mathrm{b}^2 \cdot \mathrm{m}_2 = \alpha^{\mathrm{x}} (i) \\ b^{y-1} = \mathrm{b} \cdot \mathrm{m}_2 (ii) \\ \mathrm{c}^{\mathrm{z}-1} = \mathrm{c} \cdot \mathrm{m}_3 (iii) \end{array}} \tag{1}$$

from where with a suitably choice of y according to the first method, we calculate the the $m_2, m_3 \in Z^+$ .That transformed as in program format mathematica accept as the basis $\{a, b, c\}$ of the equation a^m+b^n = c^v with a = 7; b = 7; c = 98;

Program 8.1.1

Clear $[m2, m3, y]$
$a = 7; b = 7; c = 98$
Table[Reduce[$m3^* c$^2 $-$ $m2^* b$^2 $=$ a^(x)&&$m3 > 0$&&$m2 > 0, \{m2, m3\}$, Integers], $\{y, 1, 6, 1\}$]

Results:

{False, $C[1]$\[Element] Integers && $C[1]$ >= 0 && m2 == $195 + 196\ C[1]$ && m3 == $1 + C[1]$

C[1]\[Element] Integers && C[1] >= 0 && m2 == $189 + 196$ C[1] && m3 == $1 +$ C[1],

C[1]\Element] Integers && C[1] >= 0 && m2 == $147 + 196$ C[1] && m3 == $1 + C[1]$

C[1]\[Element] Integers && C[1] >= 0 && m2 == $49 + 196$ C[1] && m3 == $2 + C[1]$

C[1]\[Element] Integers && C[1] >= 0 && m2 == $147 + 196$ C[1] && m3 == $13 +$ C[1]}

Using a program re-try one of the cases the result of the programming 1 and calculate the exhibitors to agree data the initial equation a^m + b^n = c^v.

Continue... Program 8.1.2.

```
Clear[x,y,x1,y1,z1]
a := 7; b := 7; c := 98;
Reduce[x == 13+k&&y = 147+196k&&y1 == 1+Log [b*y] /Log[b]&&z1 = 1+Log [c*x] /Log[c]&&x1 ==
Log [c^2 * x − b^2 * y] /Log[a]&&0 <= k <= 100, {x1, y1, z1}, Integers]
```

Results:

$(\boldsymbol{k} = 85\&\&\boldsymbol{x} = 98\&\&\mathbf{y} = 16807)\&\&\mathbf{x1} = 6\&\&\mathbf{y1} = 7\&\&\mathbf{z1} = 3$

8.2. The Second Case.

According to the second method of dividing a crack where $y_b y \neq 0$ variables $\{x, z\}$, where terms in the equation $x^q + y^p = z^w \Rightarrow$

$$\boxed{f(x,y,z) = \begin{pmatrix} x = (C(z)\cdot\lambda - B(y))\cdot k \\ y = (A(x) - C(z))\cdot k \\ z = (A(x)\cdot\lambda - B(y))\cdot k \end{pmatrix}} \text{ Extra generally } k \in Z^+, \lambda \in Q^+$$

And according to the analysis of the second method results in the

$$\boxed{\begin{array}{l} \mathrm{c}^2\cdot \mathrm{m}_3 - \alpha^2\cdot \mathrm{m}_1 = \mathrm{b}^{\mathrm{y}}(i) \\ a^{x-1} = \alpha\cdot \mathrm{m}_1(ii) \\ \mathrm{c}^{\mathrm{z}-1} = \mathrm{c}\cdot \mathrm{m}_3(iii) \end{array}} \tag{2}$$

from where with a suitably choice of y according to the first method, we calculate the $m_1, m_3 \in Z^+$That transformed as in program format mathematica accept as the basis $\{a, b, c\}$ of the a^m + b^n = c^v with $a = 7; b = 7; c = 98;$

Program 8.2.1

```
Clear [m1, m3]
a := 7; b := 7; c := 98;
Table [Reduce [m3*c^2 − m1*a^2 = b^(y)&& m1 > 0&& m3 > 0, {m1, m3}, Integers], {y, 1, 6, 1}]
```

Results:

{False,

C[1]\ [Element] Integers && C[1] >= 0 && m1 == 195 + 196 C[1] && m3 == 1 + C[1]

C[1]\[Element] lntegers && C[1] >= 0 && m1 == 189 + 196 C[1] && m3 == 1 + C[1]

C[1]\[Element] Integers && C[1] >= 0 && m1 == 147 + 196 C[1] && m3 == 1 + C[1]

C[1]\[Element] Integers && C[1] >= 0 && m1 == 49 + 196 C[1] && m3 == 2 + C[1]

C[1]\[Element] Integers && C[1] >= 0 && m1 == 147 + 196 C[1]&& m3 == 13 + C[1]}

Using a program re-try one of the cases the result of the programming? 1 and calculate the exhibitors to agree dedomana the initial equation a^m + b^n = c^v with $a = 7; b = 7; c = 98$;

Continue... Program 8.2.2

```
Clear[x, y, x1, y1, z1]
a := 7; b := 7; c := 98;
Reduce [z == 13 + k&&x == 147 + 196k&&x1 == 1 + Log [a*x] /Log[a]&&z1 == 1 + Log [c*z] /Log[c]&&y1
== Log [c^2*z − a^2 * x] /Log[b]&&0 <= k <= 100, {x1, y1, z1}, Integers]
```

Results:

$(\boldsymbol{k} = 85\&\&\boldsymbol{z} = 98\&\&\boldsymbol{x} = 16807)\&\&\boldsymbol{x1} = 7\&\&\boldsymbol{y1} = 6\&\&\boldsymbol{z1} = 3$

8.3. The third case.

According to the second method of dividing a initial where $zz \neq 0$ variables $\{x, y\}$, where terms in the equation $x^q + y^p = z^w \Rightarrow$

$$\boxed{\mathrm{f(x,y,z)} = \begin{pmatrix} \mathrm{x = (C(z) + \lambda \cdot B(y)) \cdot k} \\ \mathrm{z = (A(x) + B(y)) \cdot k} \\ \mathrm{y = (C(z) - \lambda \cdot A(x)) \cdot k} \end{pmatrix}} \quad \text{Extra generally with } k \in Z^+, \lambda \in Q^+.$$

And according to the analysis of the second method results in the

$$\boxed{\begin{array}{l} \alpha^2 \cdot \mathrm{m_1} + \mathrm{b^2 \cdot m_2 = c^z} \text{ (i)} \\ \mathrm{b^{y-1} = b \cdot m_2} \text{ (ii)} \\ \alpha^{\mathrm{x}-1} = \alpha \cdot \mathrm{m_1} \text{ (iii)} \end{array}} \tag{3}$$

from where with a suitably choice of y according to the first method, we calculate the $m_1, m_2 \in Z^+$. That transformed as in program format mathematica accept as the basis $\{a, b, c\}$ of the equation a^m+b^n = c^v with $a = 7; b = 7; c = 98$;

Program 8.3.1.

Clear[m1,m3]
a := 7; b := 7; c := 98;
Table[Reduce[$m1^*a$^2 + m2*b^2 == c^(z)&&m1 > 0&&m2 > 0, {m1, m2}, Integers], {z, 1, 6, 1}]

Results:

{m1 == 1 && m2 == 1,

C[1]\ Integers && 1 <= C[1] <= 195 && m1 == C[1] && m2 == 196 − C[1],

C[1]\ Integers && 1 <= C[1] <= 19207 && m1 == C[1] && m2 == 19208 − C[1],

C[1]\ Integers && 1 <= C[1] <= 1882383 && m1 == C[1] && m2 == 1882384 − C[1],

C[1]\ Integers && 1 <= C[1] <= 184473631 && m1 == C[1] && m2 == 184473632 − C[1],

C[1]\ Integers && 1 <= C[1] <= 18078415935 && m1 == C[1] && m2 == 18078415936 − C[1]}

Using a program re-try one of the cases the result of the programming 1 and calculate the exhibitors to $a = 7; b = 7; c = 98;$

Continue Program 8.3.2.

Clear [x, y, x1, y1, z1]
$a := 7; b := 7; c := 98$
Reduce [x == 1&&y == 1&&$y1$ == 1 + Log [b^*y] /Log[b]&&$x1$ == 1 + Log [a^*x] /Log[a]&&$z1$ ==
Log [a^$x1$ + b^$y1$] /Log[c]&&0 <= k <= 100, {$x1, y1, z1$}, Integers]

Results:

$(\boldsymbol{k} \in \mathbb{Z}\&\&\boldsymbol{x} = 1\&\&\boldsymbol{y} = 1)\&\&\boldsymbol{x1} = 2\&\&\boldsymbol{y1} = 2\&\&\boldsymbol{z1} = 1$

Therefore there are 3 solutions $(6, 7, 3), (7, 6, 3)$ and $(2, 2, 1)$.

According to these results it is obvious that we can calculate the exponents in each case if we know the bases.If there is no correspondence then we will not find an integer solution in the second program analysis in each case separately.

8.4.Theorem 7.(F.L.T) For any integer $n > 2$, the equation $\mathbf{x}^n + \mathbf{y}^n = \mathbf{z}^n$ has no positive integer solutions

An equation of the form $x^a + y^b = z^c$ (Beals') to have a solution, according to theorems $\{5, 6\}$, must have at least one exponent equal to 2. And since in Fermat's last theorem we have $a = b = c = n$, it follows directly that the only solution that Fermat's equation $\mathrm{x}^n + \mathrm{y}^n = \mathrm{z}^n$ can have is when n = 2. So for n > 2 there is no solution.

Epilogue

According to the logic for Beals' equation to hold, we obtain a global proof in Theorems 5,6 after considering all cases $\sigma(\mathrm{x}, \mathrm{y}, \mathrm{z}) > 1, \sigma(\mathrm{x}, \mathrm{y}, \mathrm{z}) < 1$ and $\sigma(\mathrm{x}, \mathrm{y}, \mathrm{z}) = 1$ with co-prime bases. So as a final conclusion, at least one exponent of the equation, must be equal to 2 to be solved and not always. This helps in many cases to solve three-variables diophantine equations when considering exponents greater than 3. By this method, Fermat's Last Theorem ($\sigma\,(\mathrm{x}, \mathrm{y}, \mathrm{z}) <= 1$) is very easily and understandably proved as we have seen.

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