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RESEARCH ARTICLE

Extending Lasenby's embedding of octonions in space-time algebra $Cl(1,3)$, to all three- and four dimensional Clifford geometric algebras $Cl(p,q)$, $n = p + q = 3, 4$[†]

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Summary

We study the embedding of octonions in the Clifford geometric algebra for space-time STA $Cl(1,3)$, as suggested by Anthony Lasenby at AGACSE 2021. As far as possible, we extend the approach to similar octonion embeddings for all three- and four dimensional Clifford geometric algebras $Cl(p,q)$, $n = p + q = 3, 4$. Noticeably, the lack of a quaternionic subalgebra in $Cl(2,1)$, seems to prevent the construction of an octonion embedding in this case, and necessitates a special approach in $Cl(2,2)$. As examples, we present for $Cl(3,0)$ the non-associativity of the octonionic product in terms of multivector grade parts with cyclic symmetry, show how octonion products and involutions can be combined to make the opposite transition from octonions to the Clifford geometric algebra $Cl(3,0)$, and how octonionic multiplication can be represented with (complex) biquaternions or Pauli matrix algebra.

KEYWORDS:
Octonions, Clifford geometric algebra, space-time algebra, biquaternions, Pauli algebra

1 | INTRODUCTION

The algebra of octonions[1] has independently been introduced by Arthur Cayley in 1845[2], and is therefore also called Cayley numbers[4]. Octonions have recently been used for modeling in elementary particle physics[6,20], to generalize the quaternion Fourier transform[10,13,15] to a higher dimensional octonion Fourier transform[2], and for encryption[27,28,29]. One can directly compute with octonions in various computer algebra systems[1,22].

Here we first briefly summarize important octonion algebra properties (see[19], pp. 300–302), assuming $a, b, c, x, y \in \mathbb{O}$.

- Octonions $\mathbb{O}$ form an eight-dimensional bilinear algebra over the reals $\mathbb{R}$ with basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$.
- The multiplication table is given by ($1 \le i, j \le 7$)
$$\mathbf{e}_i \star \mathbf{e}_i = -1, \quad \mathbf{e}_i \star \mathbf{e}_j = -\mathbf{e}_j \star \mathbf{e}_i \text{ for } i \neq j, \qquad \mathbf{e}_i \star \mathbf{e}_{i+1} = \mathbf{e}_{i+3}, \tag{1}$$
where $(i, i+1, i+3)$ can be permuted cyclically and translated modulo 7.

[†]Soli Deo Gloria. This paper is published under the terms of the Creative Peace License[14].
[0]**Abbreviations:** GA, geometric algebra; STA, space-time algebra
[1]Note that they seem to have been first introduced by John T. Graves as octaves and mentioned in a letter to William R. Hamilton[8].
[2]It may be of interest what William Thomson (later Lord Kelvin) wrote in a letter dated 31st July 1864 to Hermann von Helmholtz in the context of the mathematics of electric fields at plate boundaries: *Oh! that the CAYLEYS would devote what skill they have to such things instead of to pieces of algebra which possibly interest four people in the world, certainly not more, and possibly also only the one person who works. It is really too bad that they don't take their part in the advancement of the world and leave the labour of mathematical solutions for people who would spend their time so much more usefully in experimenting.*[26], p. 433.

- Via the Cayley-Dickson doubling process, octonions can be defined from pairs of quaternions $p_1, p_2, q_1, q_2 \in \mathbb{H}$ (note the order of factors, qc(…) is quaternion conjugation):

$$(p_1, q_1) \star (p_2, q_2) = \big(p_1 p_2 - \mathrm{qc}(q_2) q_1,\ \ q_2 p_1 + q_1 \mathrm{qc}(p_2)\big). \tag{2}$$

- $\mathbb{O}$ has no zero divisors, i.e., $ab = 0$ implies $a = 0$ or $b = 0$.
- $\mathbb{O}$ is a division algebra, i.e., $ax = b$ and $ya = b$ have unique solutions x, y for non-zero a.
- $\mathbb{O}$ is non-associative, i.e., in general $a(bc) \neq (ab)c$.
- $\mathbb{O}$ admits unique inverses.
- $\mathbb{O}$ is alternative, i.e., $a(ab) = a^2 b$ and $(ab)b = ab^2$.
- $\mathbb{O}$ is flexible, i.e., $a(ba) = (ab)a$.
- $\mathbb{O}$ is one of only four alternative division algebras over $\mathbb{R}$: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.
- $\mathbb{O}$ has a (positive-definite quadratic form) norm $\| \dots \| : \mathbb{O} \to \mathbb{R}$, the norm is preserved (i.e. admits composition), such that $\|ab\| = \|a\| \|b\|$.
- $\mathbb{O}$ is one of only four unital norm-preserving division algebras over $\mathbb{R}$: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.
- $\mathbb{O}$ is essential for treating *triality*, an automorphism of the universal covering spin group Spin(8) of the rotation group SO(8) or $\mathbb{R}^8$. Triality is not an inner automorphism, nor an orthogonal matrix similarity, nor a linear transformation $Cl(8,0) \to Cl(8,0)$, nor a linear automorphism of SO(8). Triality permutes three elements in the center of $Cl(8,0)$, namely $\{-1, e_{12345678}, -e_{12345678}\}$, with basis vectors e_i, $(1 \leq i \leq 8)$, of $\mathbb{R}^8$. Triality is a restriction of a polynomial mapping $Cl(8,0) \to Cl(8,0)$ of degree two.

Furthermore, like for complex numbers, quaternions and biquaternions, there is a *polar decomposition* for octonions[25]. We finally note previously known embeddings in Clifford algebra for octonions in high dimensions of $Cl(0,7)$, dim = 128, or $Cl(8,0)$, dim = 256, see[19], Chapters 7.4 and 23.

In the present treatment, we take up the recent suggestion of Anthony Lasenby[16,17] for an embedding of octonions in space-time algebra[9], the Clifford geometric algebra of Minkowski space $\mathbb{R}^{1,3}$. We systematically extend this approach to all Clifford geometric algebras $Cl(p,q)$, $n = p + q = 3, 4$. In one instance ($Cl(3,0)$) we also study the detailed expression of the octonionic product in terms of the scalar-, vector-, bivector- and trivector parts of the multivector factors, compute the norm preservation, and investigate how the non-associativity of this product expresses itself in these Clifford geometric algebra grade parts. In all cases the octonionic embedding is specified, complete with full multiplication table and Fano plane diagram visualization, the octonion conjugate is specified in terms of Clifford geometric algebra operations, the octonionic norm is computed explicitly and expressed in Clifford geometric algebra.

The paper is structured as follows. Section 2 reviews the suggestion of Anthony Lasenby[16,17] for the embedding of octonions in 16-dimensional space-time algebra. Section 3 shows how octonion multiplication can be embedded in the eight-dimensional Clifford geometric algebra $Cl(3,0)$ of Euclidean space $\mathbb{R}^3$, complete with the full multivector grade part expression for the product, and includes a special Subsection 3.2 on the well-known non-associativity of octonions explicitly expressed in the example of $Cl(3,0)$, Subsection 3.3 on how octonion products and involutions can be combined to make the opposite transition from octonions to the geometric product of multivectors in Clifford geometric algebra $Cl(3,0)$, Subsection 3.4 on implementing octonion multiplication with (complex) biquaternions (ideal for software implementation), and Subsection 3.5 on how to express the octonion product for $Cl(3,0)$ multivectors with complex two by two matrices. Section 4 shows how the corresponding embedding of octonions works in the algebra of space time with opposite signature $Cl(3,1)$, and Section 5 shows the embedding in anti-Euclidean space $\mathbb{R}^{0,3}$ often preferred in Clifford analysis. Section 6 explains the embedding in $Cl(1,2)$, and Section 7 gives some argument of why it may not be possible to implement the octonionic product in a similar way in $Cl(1,2)$. Section 8 then explains how the octonion product can be embedded in the remaining Clifford geometric algebras with $n = 4$, i.e. for $Cl(0,4)$, $Cl(2,2)$ and $Cl(0,4)$. The conclusion Section 9 contains a summary table for all implementations in Clifford geometric algebras given in this work, listing column wise the algebra itself, the designation of Pauli- and non-Pauli spinors (or what corresponds to them), the octonion conjugation, the octonionic product, the multiplication table number, the Fano plane diagram figure number, the octonionic norm and the section in this work containing the respective detailed definitions and computations. This is followed by acknowledgments and references.

2 | SPACE-TIME ALGEBRA $CL(1,3)$

We now follow the presentation of the subject given by Anthony Lasenby at AGACSE 2021 [16,17], who presented the first known embedding of octonions in space-time algebra.

Space-time algebra was introduced 1966 by David Hestenes in [9], as the Clifford geometric algebra $Cl(1,3)$ of Minkowski space-time vector space $\mathbb{R}^{1,3}$ with four orthonormal basis vectors squaring to

$$e_0^2 = -e_1^2 = -e_2^2 = -e_3^2 = 1. \tag{3}$$

The resulting real space-time algebra has the 16-dimensional multivector basis of one scalar, four vectors, six bivectors, four trivectors and one pseudoscalar

$$\begin{aligned}&\{1, e_0, e_1, e_2, e_3,\\ &\sigma_1 = e_{10}, \sigma_2 = e_{20}, \sigma_3 = e_{30}, I\sigma_1 = -e_{23}, I\sigma_2 = -e_{31}, I\sigma_3 = -e_{12},\\ &Ie_0 = -e_{123}, Ie_1 = -e_{023}, Ie_2 = -e_{031}, Ie_3 = -e_{012}, I = e_{0123}\}.\end{aligned} \tag{4}$$

The even subalgebra of space-time algebra (Hestenes-Dirac) spinors $\psi \in Cl^+(1,3)$ has the real eight-dimensional basis

$$\{1, \sigma_1 = e_{10}, \sigma_2 = e_{20}, \sigma_3 = e_{30}, I\sigma_1 = -e_{23} = \sigma_{23}, I\sigma_2 = -e_{31} = \sigma_{31}, I\sigma_3 = -e_{12} = \sigma_{12}, I = e_{0123} = \sigma_{123}\}, \tag{5}$$

and is isomorphic to $Cl(3,0)$ (and thus to complex biquaternions $\mathbb{C} \otimes \mathbb{H}$), the geometric algebra of Euclidean space $\mathbb{R}^3$ with basis $\{\sigma_1, \sigma_2, \sigma_3\}$.

The spinor basis (also called rotor basis of space-time) can be split into Pauli spinors $\psi_+ = \frac{1}{2}(\psi + e_0\psi e_0)$ that commute with e_0 and have the four-dimensional basis

$$\{1, I\sigma_1 = -e_{23}, I\sigma_2 = -e_{31}, I\sigma_3 = -e_{12}\}, \tag{6}$$

and four-dimensional non-Pauli spinors $\psi_- = \frac{1}{2}(\psi - e_0\psi e_0)$ that anticommute with e_0

$$\{\sigma_1 = e_{10}, \sigma_2 = e_{20}, \sigma_3 = e_{30}, I = e_{0123}\}. \tag{7}$$

Obviously multiplication with the pseudoscalar I (duality in GA) converts a Pauli spinor into a non-Pauli spinor and vice versa.

Lasenby [16,17] introduces the *octonion product* of two STA spinors $\psi, \phi \in Cl^+(1,3)$, a conjugate[3] and a norm as

$$\psi \star \phi = \psi_+\phi_+ + \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+, \quad \psi^* = \widetilde{\psi}_+ - \psi_-, \quad \|\psi\| = \psi \star \psi^* = e_0 \cdot (\psi e_0 \widetilde{\psi}) = \frac{1}{2}(e_0\psi e_0\widetilde{\psi} + \psi e_0 \widetilde{\psi} e_0), \tag{8}$$

where the tilde operation indicates the reverse involution[4], which changes the sign of every bivector in the two bases of Pauli spinors (6) and non-Pauli spinors (7), but leaves the scalar and pseudoscalar invariant. We will see, that every of the four terms in the product $\psi \star \phi$ of (8) corresponds to one 4×4 block in the octonionic multiplication table, as can be seen from

$$\psi_+ \star \phi_+ = \psi_+\phi_+, \qquad \psi_- \star \phi_- = \widetilde{\phi}_-\psi_-, \qquad \psi_+ \star \phi_- = \phi_-\psi_+, \qquad \psi_- \star \phi_+ = \psi_-\widetilde{\phi}_+. \tag{9}$$

The full multiplication table, Table 1 , shows that the first two products in (9) result in Pauli spinors, whereas the last two products result in non-Pauli spinors, respectively.

The definition of the octonionic conjugate ψ^* can be understood to correspond to the usual octonion conjugate by applying it to the bases of Pauli spinors (6) and non-Pauli spinors (7), respectively,

$$\begin{aligned}\{1, I\sigma_1, I\sigma_2, I\sigma_3\}^* &= \{1, I\sigma_1, I\sigma_2, I\sigma_3\}^\sim = \{1, -I\sigma_1, -I\sigma_2, -I\sigma_3\},\\ \{\sigma_1, \sigma_2, \sigma_3, I\}^* &= -\{\sigma_1, \sigma_2, \sigma_3, I\} = \{-\sigma_1, -\sigma_2, -\sigma_3, -I\},\end{aligned} \tag{10}$$

so only the scalars are preserved and all bivectors and the pseudoscalar change sign. Note that the octonion conjugate ψ^* is an anti-involution, i.e.

$$(\psi \star \phi)^* = \phi^* \star \psi^*. \tag{11}$$

[3] Note that the star index here does not mean duality of GA.
[4] Note that by construction $\widetilde{\psi_\pm} = (\widetilde{\psi})_\pm$.

Next, we compute the norm

$$\begin{aligned}\|\psi\| &= \psi \star \psi^* = (\psi_+ + \psi_-) \star (\widetilde{\psi}_+ - \psi_-) = \psi_+\widetilde{\psi}_+ + \widetilde{(-\psi_-)}\psi_- + (-\psi_-)\psi_+ + \psi_-\widetilde{\widetilde{\psi}}_+ = \psi_+\widetilde{\psi}_+ - \widetilde{\psi}_-\psi_- - \psi_-\psi_+ + \psi_-\psi_+ \\ &= \psi_+\widetilde{\psi}_+ - \widetilde{\psi}_-\psi_- = \frac{1}{4}(\psi + e_0\psi e_0)(\widetilde{\psi} + e_0\widetilde{\psi}e_0) - \frac{1}{4}(\widetilde{\psi} - e_0\widetilde{\psi}e_0)(\psi - e_0\psi e_0) \\ &= \frac{1}{4}(\psi\widetilde{\psi} + e_0\psi\widetilde{\psi}e_0 - \widetilde{\psi}\psi - e_0\widetilde{\psi}\psi e_0) + \frac{1}{4}(\psi e_0\widetilde{\psi}e_0 + e_0\psi e_0\widetilde{\psi} + \widetilde{\psi}e_0\psi e_0 + e_0\widetilde{\psi}e_0\psi) \\ &= \frac{1}{4}(2\langle\psi\widetilde{\psi}\rangle - 2\langle\widetilde{\psi}\psi\rangle) + \frac{1}{2}[(\psi e_0\widetilde{\psi})\cdot e_0 + (\widetilde{\psi}e_0\psi)\cdot e_0] = \frac{1}{2}\langle\psi e_0\widetilde{\psi}e_0 + \widetilde{\psi}e_0\psi e_0\rangle = \langle\psi e_0\widetilde{\psi}e_0\rangle \\ &= (\psi e_0\widetilde{\psi})\cdot e_0,\end{aligned} \tag{12}$$

where we used that $\psi\widetilde{\psi}$ is even and invariant under reversion, so it must be a linear combination of scalar and pseudoscalar $\psi\widetilde{\psi} = s + pI$. But $e_0(s + pI)e_0 = s - pI$. So

$$\psi\widetilde{\psi} + e_0\psi\widetilde{\psi}e_0 = s + pI + s - pI = 2s = 2\langle\psi\widetilde{\psi}\rangle, \tag{13}$$

which is used for the equality at the beginning of the fourth equation line of (12). Moreover, we have commutativity of factors in scalar part brackets $\langle\psi\widetilde{\psi}\rangle = \langle\widetilde{\psi}\psi\rangle$, which explains the next equality (second equality on line four of (12)). The commutativity under the scalar part brackets is used again to give the expression at the end of line four of (12), written as inner product at the end of (12).

We first compute the squares of all $Cl^+(1,3)$ basis elements ($k = 1, 2, 3$)

$$\begin{aligned}&1 \star 1 = 1_+1_+ = 1, \quad I\sigma_k \star I\sigma_k = I\sigma_k I\sigma_k = I^2\sigma_k^2 = (-1)(+1) = -1, \\ &\sigma_k \star \sigma_k = \widetilde{\sigma}_k\sigma_k = -\sigma_k^2 = -1, \quad I \star I = \widetilde{I}I = II = -1.\end{aligned} \tag{14}$$

Furthermore, the first equality in (9) shows, that the 4×4 multiplication subtable of Pauli spinors ψ_+ is identical to their geometric algebra product table, i.e.

$$(I\sigma_1) \star (I\sigma_2) = II\sigma_1\sigma_2 = -\sigma_1\sigma_2 = -I\sigma_3, \text{ etc.} \tag{15}$$

By the second equality in (9) we have for the non-Pauli spinors with basis (6) that, apart from main diagonal elements ($j, k = 1, 2, 3,\ j \neq k$)

$$\sigma_j \star \sigma_k = \widetilde{\sigma}_k\sigma_j = -\sigma_k\sigma_j = \sigma_j\sigma_k = -\widetilde{\sigma}_j\sigma_k = -\sigma_k \star \sigma_j, \tag{16}$$

which is again the same as the geometric product and is anti-symmetric. Moreover,

$$I \star \sigma_k = \widetilde{\sigma}_k I = -\sigma_k I = -I\sigma_k, \quad \sigma_k \star I = \widetilde{I}\sigma_k = I\sigma_k = -I \star \sigma_k, \tag{17}$$

which shows the anti-symmetry of the octonionic product for unequal pairs of basis elements of non-Pauli spinors.

Now we look at the products of Pauli spinors on the left $\{1, I\sigma_1, I\sigma_2, I\sigma_3\}$, with non-Pauli spinors on the right $\{\sigma_1, \sigma_2, \sigma_3, I\}$ and obtain from the third equality in (9) that

$$1 \star \psi_- = \psi_- 1 = \psi_-, \quad \psi_+ \star I = I\psi_+, \quad \{1, I\sigma_1, I\sigma_2, I\sigma_3\} \star I = \{I, -\sigma_1, -\sigma_2, -\sigma_3\}, \tag{18}$$

where the third equality set is the result of applying the second equality. Moreover ($j, k = 1, 2, 3,\ j \neq k$)

$$(I\sigma_k) \star \sigma_k = \sigma_k I\sigma_k = I\sigma_k^2 = I, \quad (I\sigma_j) \star \sigma_k = \sigma_k I\sigma_j = I\sigma_k\sigma_j = -I\sigma_j\sigma_k, \tag{19}$$

e.g.,

$$(I\sigma_1) \star \sigma_2 = -I\sigma_1\sigma_2 = -II\sigma_3 = \sigma_3, \quad (I\sigma_2) \star \sigma_1 = -I\sigma_2\sigma_1 = -I(-I\sigma_3) = -\sigma_3, \text{ etc.} \tag{20}$$

At the end, we need to compute the products of non-Pauli spinors on the left $\{\sigma_1, \sigma_2, \sigma_3, I\}$, with Pauli spinors on the right $\{1, I\sigma_1, I\sigma_2, I\sigma_3\}$ and obtain from the fourth equality in (9) that

$$\begin{aligned}&\psi_- \star 1 = \psi_-\widetilde{1} = \psi_- 1 = \psi_-, \\ &I \star \{1, I\sigma_1, I\sigma_2, I\sigma_3\} = I\{1, I\sigma_1, I\sigma_2, I\sigma_3\}^\sim = I\{1, -I\sigma_1, -I\sigma_2, -I\sigma_3\} = \{I, \sigma_1, \sigma_2, \sigma_3\}.\end{aligned} \tag{21}$$

Moreover ($j, k = 1, 2, 3,\ j \neq k$)

$$\sigma_k \star (I\sigma_k) = \sigma_k\widetilde{(I\sigma_k)} = \sigma_k(-I\sigma_k) = -I, \quad \sigma_j \star (I\sigma_k) = \sigma_j\widetilde{(I\sigma_k)} = \sigma_j(-I\sigma_k) = -I\sigma_j\sigma_k, \tag{22}$$

e.g.,

$$\sigma_1 \star (I\sigma_2) = -I\sigma_1\sigma_2 = -II\sigma_3 = \sigma_3, \quad \sigma_2 \star (I\sigma_1) = -I\sigma_2\sigma_1 = -I(-I\sigma_3) = -\sigma_3, \text{ etc.} \tag{23}$$

TABLE 1 Multiplication table for Lasenby octonion embedding in STA $Cl(1,3)$. The upper left 4×4-block corresponds to $\psi_+\phi_+$, the upper right 4×4-block to $\phi_-\psi_+$, the lower left 4×4-block to $\psi_-\widetilde{\phi}_+$, and the lower right 4×4-block to $\widetilde{\phi}_-\psi_-$ of (8) and (9).

Left factors	Right factors: 1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
1	1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
$I\sigma_1$	$I\sigma_1$	-1	$-I\sigma_3$	$I\sigma_2$	I	σ_3	$-\sigma_2$	$-\sigma_1$
$I\sigma_2$	$I\sigma_2$	$I\sigma_3$	-1	$-I\sigma_1$	$-\sigma_3$	I	σ_1	$-\sigma_2$
$I\sigma_3$	$I\sigma_3$	$-I\sigma_2$	$I\sigma_1$	-1	σ_2	$-\sigma_1$	I	$-\sigma_3$
σ_1	σ_1	$-I$	σ_3	$-\sigma_2$	-1	$I\sigma_3$	$-I\sigma_2$	$I\sigma_1$
σ_2	σ_2	$-\sigma_3$	$-I$	σ_1	$-I\sigma_3$	-1	$I\sigma_1$	$I\sigma_2$
σ_3	σ_3	σ_2	$-\sigma_1$	$-I$	$I\sigma_2$	$-I\sigma_1$	-1	$I\sigma_3$
I	I	σ_1	σ_2	σ_3	$-I\sigma_1$	$-I\sigma_2$	$-I\sigma_3$	-1

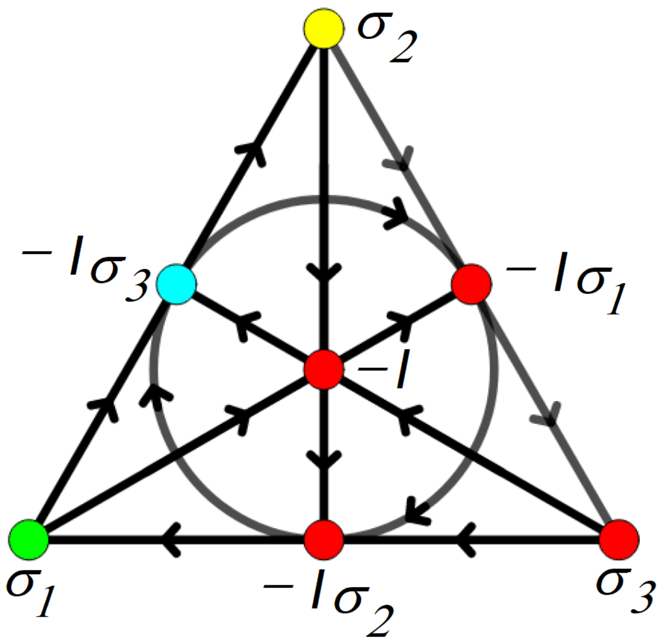

FIGURE 1 Illustration of space-time spinors in $Cl^+(1,3)$ under the octonionic product (8) in Table 1 , as suggested by Lasenby [16,17]. Fano plane depiction adapted from Steve Phelps [21].

We finally summarize all products in the multiplication table for space-time spinors in $Cl^+(1,3)$ under the Lasenby octonion product (8) in Table 1 . The visual depiction of the multiplication relationships of Table 1 in Fig. 1 , clearly shows the isomorphism to octonions.

3 | OCTONIONIC PRODUCT IN $CL(3,0)$

Here we not only show in Subsection 3.1 how to implement the octonionic product in Clifford geometric algebra $Cl(3,0)$, we also study in Subsection 3.2 the non-associativity of the octonion product expressed in geometric algebra in terms of scalar, vector, bivector and trivector components of the factors, in Subsection 3.3 we demonstrate that for octonions one can define an associative product, isomorphic to the geometric product of multivectors in $Cl(3,0)$, in Subsection 3.4 the implementation of octonions with complex biquaternions is shown, and finally in Subsection 3.5 with complex two by two Pauli matrices.

3.1 | Implementation in $Cl(3,0)$

The implementation of the octonion product in $Cl(3,0)$ is of the *widest possible importance*, since it is the perhaps most frequently applied geometric algebra of three-dimensional physical space $\mathbb{R}^3$. And $Cl(3,0)$ is isomorphic to complex quaternions (also called complex biquaternions), that is quaternions with complex coefficients $Cl(3,0) \cong \mathbb{C}\otimes\mathbb{H}$, and isomorphic to the Pauli matrix algebra (of complex two by two matrices) of quantum mechanics. Therefore, without any extra work we obtain embeddings of the octonion product in complex biquaternions and Paul matrix algebra. Since $Cl(3,0)$ is also a subalgebra of conformal geometric algebra (CGA) $Cl(4,1)$, the embedding of the octonion product in $Cl(3,0)$ implies also an embedding in CGA.

Because the even STA subalgebra $Cl^+(1,3)$ of real space-time spinors in $Cl(1,3)$ is isomorphic to the Clifford geometric algebra $Cl(3,0)$ of Euclidean space $\mathbb{R}^3$ with basis elements

$$\{1, \sigma_1, \sigma_2, \sigma_3, I\sigma_1 = \sigma_{23}, I\sigma_2 = \sigma_{31}, I\sigma_3 = \sigma_{12}, I = \sigma_{123}\}, \tag{24}$$

we can also construct in $Cl(3,0)$ an octonionic product, because we can split it in its even subalgebra with basis

$$\{1, \sigma_{23}, \sigma_{31}, \sigma_{12}\}, \tag{25}$$

and the set of odd grade (w.r.t. grades in $Cl(3,0)$) elements

$$\{\sigma_1, \sigma_2, \sigma_3, I = \sigma_{123}\}. \tag{26}$$

But for the construction of the octonion product in $Cl(3,0)$ we need an involution, that has the same effect on vectors σ_k, and bivectors σ_{jk} $(k \neq j)$, as the reversion had in STA $Cl(1,3)$ where all these elements were bivectors. The desired conjugation exists in deed in the form of Clifford conjugation[5] (indicated by an overbar), i.e. the composition of (main) grade involution and reversion, which preserves grades zero and three, but changes the signs of grades one and two in $Cl(3,0)$. We can therefore immediately conclude, that a realization of the octonionic product of M, N in $Cl(3,0)$ is given by

$$M = M_+ + M_-, \quad N = N_+ + N_-, \quad M \star N = M_+N_+ + \overline{N_-}M_- + N_-M_+ + M_-\overline{N_+}, \tag{27}$$

with even grade parts $M_+, N_+ \in Cl^+(3,0)$ and odd grade parts $M_-, N_- \in Cl^-(3,0)$. The multiplication table is again Table 1 , with octonionic product illustration in Fig. 1 .

Remark 1. Note that in the octonion product of (27), the first two terms are of even grade, and the last two are of odd grade

$$(M \star N)_+ = M_+N_+ + \overline{N_-}M_-, \qquad (M \star N)_- = N_-M_+ + M_-\overline{N_+}, \tag{28}$$

since the product of a pair of even (or odd) multivectors in Clifford geometric algebra is even, respectively, the product of an even with an odd multivector is odd. This can also be easily verified from the multiplication table, Table 1 , in terms of multivector grading in $Cl(3,0)$.

In the context of the well studied Clifford geometric algebra $Cl(3,0)$ of three-dimensional Euclidean space $\mathbb{R}^3$, the algebra being isomorphic to complex biquaternions (Hamilton quaternions with complex coefficients), it may help to understand the geometric meaning of the even and odd product parts by expanding them in terms of the graded multivector parts $M_s = \langle M\rangle_0 = \langle M\rangle$, $M_v = \langle M\rangle_1$, $M_b = \langle M\rangle_2$ and $M_t = \langle M\rangle_3$, which are the scalar-, vector-, bivector- and trivector part of M, respectively. The even product part results in (with commutators $[A,B] = AB - BA$)

$$\begin{aligned}(M \star N)_+ &= M_+N_+ + \overline{N_-}M_- = (M_s + M_b)(N_s + N_b) + (-N_v + N_t)(M_v + M_t)\\ &= M_sN_s + N_sM_b + M_sN_b + M_bN_b - N_vM_v + N_tM_v - M_tN_v + M_tN_t\\ &= M_sN_s + M_b\cdot N_b - N_v\cdot M_v + M_tN_t + N_sM_b + M_sN_b + \frac{1}{2}[M_b, N_b] - N_v\wedge M_v + N_tM_v - M_tN_v,\end{aligned} \tag{29}$$

with scalar part

$$(M \star N)_s = M_sN_s + M_b\cdot N_b - N_v\cdot M_v + M_tN_t, \tag{30}$$

[5] Note that by construction $\overline{M_\pm} = (\overline{M})_\pm$.

and bivector part

$$(M \star N)_b = N_s M_b + M_s N_b + \frac{1}{2}[M_b, N_b] - N_v \wedge M_v + N_t M_v - M_t N_v. \tag{31}$$

The odd product part results in

$$\begin{aligned}(M \star N)_- &= N_- M_+ + M_- \overline{N_+} = (N_v + N_t)(M_s + M_b) + (M_v + M_t)(N_s - N_b) \\ &= M_s N_v + N_t M_b + N_v M_b + M_s N_t + N_s M_v - M_t N_b - M_v N_b + N_s M_t \\ &= M_s N_v + N_t M_b + N_s M_v - M_t N_b + N_v \cdot M_b - M_v \cdot N_b + N_v \wedge M_b - M_v \wedge N_b + M_s N_t + N_s M_t, \end{aligned} \tag{32}$$

with vector part

$$(M \star N)_v = M_s N_v + N_t M_b + N_s M_v - M_t N_b + N_v \cdot M_b - M_v \cdot N_b, \tag{33}$$

and trivector part

$$(M \star N)_t = N_v \wedge M_b - M_v \wedge N_b + M_s N_t + N_s M_t. \tag{34}$$

In [19], p. 305, we find the following notation for the octonion product (there $\circ$ replaces $\star$)

$$M \star N = M_s N_s + M_s \mathbf{N} + \mathbf{M} N_s - \mathbf{M} \cdot \mathbf{N} + \mathbf{M} \times \mathbf{N}, \qquad M = M_s + \mathbf{M}, \quad N = N_s + \mathbf{N}, \tag{35}$$

that is only the scalar parts M_s and N_s are split off from $M, N \in Cl(3,0)$. Apart from $M_s N_s$, we can therefore identify the following terms

$$\begin{aligned} &M_s \mathbf{N} = M_s N_v + M_s N_b + M_s N_t, \quad \mathbf{M} N_s = N_s M_v + N_s M_b + N_s M_t, \quad \mathbf{M} \cdot \mathbf{N} = -M_b \cdot N_b + N_v \cdot M_v - M_t N_t \\ &\mathbf{M} \times \mathbf{N} = \frac{1}{2}[M_b, N_b] - N_v \wedge M_v + N_t M_v - M_t N_v + N_t M_b - M_t N_b + N_v \cdot M_b - M_v \cdot N_b + N_v \wedge M_b - M_v \wedge N_b. \end{aligned} \tag{36}$$

The octonion conjugate in $Cl(3,0)$ is given by

$$M^* = \widetilde{M}_+ - M_- = \overline{M}_+ - M_-. \tag{37}$$

Note that the octonion conjugate is an anti-involution, i.e.

$$(M \star N)^* = N^* \star M^*, \tag{38}$$

which can be easily verified for random multivectors $M, N \in Cl(3,0)$, by implementing the octonion product (28) with the Clifford Multivector Toolbox for Matlab [23,24].

Computing the octonion norm further demonstrates the consistency of the implementation and exemplifies how to employ available geometric algebra multivector properties:

$$\begin{aligned} \|M\| &= M \star M^* = (M_+ + M_-) \star (\overline{M}_+ - M_-) \overset{(27)}{=} M_+ \overline{M}_+ + (-\overline{M}_-) M_- - M_- M_+ + M_- \overline{\overline{M}}_+ \\ &= M_+ \overline{M}_+ - \overline{M}_- M_- = (M_s + M_b)(M_s - M_b) - (-M_v + M_t)(M_v + M_t) \\ &= M_s^2 + M_s M_b - M_s M_b - M_b^2 + M_v^2 - M_v M_t + M_v M_t - M_t^2 = M_s^2 - M_b^2 + M_v^2 - M_t^2 \\ &= \langle M \widetilde{M} \rangle = M * \widetilde{M} = \sum_{i=1}^{8} M_i^2, \end{aligned} \tag{39}$$

where $M_i \in \mathbb{R}, 1 \leq i \leq 8$, are the coefficients of M in the $Cl(3,0)$ basis (24). The above computation used the fact that M_s and M_t are in the center of $Cl(3,0)$. $M * \widetilde{M}$ is the scalar product of M and its reverse.

We can furthermore demonstrate explicitly that the octonion product (27) is *norm-preserving*. For that we extract from (39) the following useful equality and symmetry

$$M_+ \overline{M}_+ = M_s^2 - M_b^2 = \langle M_+ \overline{M}_+ \rangle = \langle \overline{M}_+ M_+ \rangle = \overline{M}_+ M_+, \tag{40}$$

which could also be explained by the fact that $Cl^+(3,0)$ is isomorphic to quaternions, and in this isomorphism Clifford conjugation $\overline{(\ldots)}$ acts like quaternion conjugation. Similarly, we can extract from (39) that

$$M_- \overline{M}_- = M_v^2 - M_t^2 = \langle M_- \overline{M}_- \rangle = \langle \overline{M}_- M_- \rangle = \overline{M}_- M_-. \tag{41}$$

Then we can show norm-preservation by direct computation

$$\begin{aligned}
\|M \star N\| &= \langle (M \star N)\widetilde{(M \star N)}\rangle = \langle (M \star N)_+\overline{(M \star N)_+}\rangle - \langle \overline{(M \star N)_-}(M \star N)_-\rangle \\
&= \langle (M_+N_+ + \overline{N}_-M_-)(\overline{N}_+\overline{M}_+ + \overline{M}_-N_-)\rangle - \langle (\overline{M}_+\overline{N}_- + N_+\overline{M}_-)(N_-M_+ + M_-\overline{N}_+)\rangle \\
&= \langle M_+N_+\overline{N}_+\overline{M}_+\rangle + \langle \overline{N}_-M_-\overline{M}_-N_-\rangle + \langle M_+N_+\overline{M}_-N_-\rangle + \langle \overline{N}_-M_-\overline{N}_+\overline{M}_+\rangle \\
&\quad - \langle N_+\overline{M}_-N_-M_+\rangle - \langle \overline{M}_+\overline{N}_-M_-\overline{N}_+\rangle - \langle \overline{M}_+\overline{N}_-N_-M_+\rangle - \langle N_+\overline{M}_-M_-\overline{N}_+\rangle \\
&= (M_+\overline{M}_+)(N_+\overline{N}_+) + (\overline{N}_-N_-)(M_-\overline{M}_-) - (\overline{N}_-N_-)(\overline{M}_+M_+) - (N_+\overline{N}_+)(\overline{M}_-M_-) \\
&= (M_+\overline{M}_+ - \overline{M}_-M_-)\,(N_+\overline{N}_+ - \overline{N}_-N_-) = \|M\|\|N\|,
\end{aligned} \tag{42}$$

where the first two equalities follow from (39) by replacing $M \to (M \star N)$. For the third equality we apply the identities (28). For the fifth equality we apply the identities (40) and (41) as well as the cyclic symmetry of the scalar part of the geometric product. For the sixth equality we apply the symmetries contained in (40) and (41), and for the final equality once more the fourth equality of (39).

Remark 2. Norm preservation could be shown by analogous computations for all other embeddings of octonions in Clifford algebras explained in the current paper, but we hope this example provides sufficient illustration. Strictly speaking, from the algebraic viewpoint, the identity of the multiplication table of the product embedding (27) with that of octonions (see Fig. 1) is fully sufficient to guarantee norm preservation as well.

3.2 | Non-associativity of octonion product in $Cl(3,0)$

The octonion product is known for its non-associativity, distinguishing it from matrix products or the fundamental multivector product in Clifford geometric algebras. It may therefore be of interest to look at the non-associativity of the octonionic product (27) in $Cl(3,0)$, and see how it is expressed in terms of the various multivector grade parts, because the latter interpretation does not exist in canonical octonion algebra. Toward this end, we will first compute for $M, N, P \in Cl(3,0)$ the octonionic triple products $(M \star N) \star P$, $M \star (N \star P)$ and their difference, and then express the latter in terms of the scalar-, vector-, bivector- and trivector parts of M, N, and P.

$$\begin{aligned}
(M \star N) \star P &= (M \star N)_+P_+ + \overline{P}_-(M \star N)_- + P_-(M \star N)_+ + (M \star N)_-\overline{P}_+ \\
&= (M_+N_+ + \overline{N}_-M_-)P_+ + \overline{P}_-(N_-M_+ + M_-\overline{N}_+) + P_-(M_+N_+ + \overline{N}_-M_-) + (N_-M_+ + M_-\overline{N}_+)\overline{P}_+ \\
&= M_+N_+P_+ + \overline{N}_-M_-P_+ + \overline{P}_-N_-M_+ + \overline{P}_-M_-\overline{N}_+ \\
&\quad + P_-M_+N_+ + P_-\overline{N}_-M_- + N_-M_+\overline{P}_+ + M_-\overline{N}_+\overline{P}_+.
\end{aligned} \tag{43}$$

$$\begin{aligned}
M \star (N \star P) &= M_+(N \star P)_+ + \overline{(N \star P)_-}M_- + (N \star P)_-M_+ + M_-\overline{(N \star P)_+} \\
&= M_+(N_+P_+ + \overline{P}_-N_-) + \overline{(P_-N_+ + N_-\overline{P}_+)}M_- + (P_-N_+ + N_-\overline{P}_+)M_+ + M_-\overline{(N_+P_+ + \overline{P}_-N_-)} \\
&= M_+N_+P_+ + M_+\overline{P}_-N_- + \overline{N}_+\overline{P}_-M_- + P_+\overline{N}_-M_- \\
&\quad + P_-N_+M_+ + N_-\overline{P}_+M_+ + M_-\overline{P}_+\overline{N}_+ + M_-\overline{N}_-P_-.
\end{aligned} \tag{44}$$

The difference is

$$\begin{aligned}
(M \star N) \star P - M \star (N \star P) &= [\overline{N}_-M_-, P_+] + [\overline{P}_-N_-, M_+] + [\overline{P}_-M_-, \overline{N}_+] \\
&\quad + P_-\overline{N}_-M_- - M_-\overline{N}_-P_- + N_-[M_+, \overline{P}_+] + M_-[\overline{N}_+, \overline{P}_+] + P_-[M_+, N_+],
\end{aligned} \tag{45}$$

where the first line has even multivectors on the right and the second line consists of odd multivectors. The commutator of two even multivectors occurs thrice, and reduces to the commutator of the bivector parts (because the scalar parts drop out of the commutator computation) which is again a bivector, e.g.,

$$\begin{aligned}
[M_+, N_+] &= [M_b, N_b] = [M_{23}\sigma_{23} + M_{31}\sigma_{31} + M_{12}\sigma_{12}, N_{23}\sigma_{23} + N_{31}\sigma_{31} + N_{12}\sigma_{12}] \\
&= (M_{31}N_{23} - M_{23}N_{31})\sigma_{12} + (M_{23}N_{12} - M_{12}N_{23})\sigma_{31} + (M_{12}N_{31} - M_{31}N_{12})\sigma_{23}.
\end{aligned} \tag{46}$$

Because the Clifford conjugate of a bivector is $\overline{M}_b = -M_b$, the three commutators of the odd part of (45) reduce to

$$\begin{aligned} N_-[M_+, \overline{P}_+] + M_-[\overline{N}_+, \overline{P}_+] + P_-[M_+, N_+] &= -N_-[M_b, P_b] + M_-[N_b, P_b] + P_-[M_b, N_b] \\ &= M_-[N_b, P_b] + N_-[P_b, M_b] + P_-[M_b, N_b], \end{aligned} \tag{47}$$

where we note the cyclic M, N, P-symmetry. Each of these three terms can be further expanded using $N_- = N_v + N_t$ (etc.) as, e.g.,

$$N_-[P_b, M_b] = (N_v + N_t)[P_b, M_b] = N_v \cdot [P_b, M_b] + N_t[P_b, M_b] + N_v \wedge [P_b, M_b], \tag{48}$$

where the first two terms have vector grade and the third term is a trivector. The commutators of the even grade part of (45) can be expanded as, e.g.,

$$\begin{aligned} [\overline{N}_- M_-, P_+] &= [(-N_v + N_t)(M_v + M_t), (P_s + P_b)] = [-N_v M_v - N_v M_t + N_t M_v, P_b] \\ &= -[N_v \wedge M_v, P_b] - M_t[N_v, P_b] + N_t[M_v, P_b] = [M_v \wedge N_v, P_b] - 2M_t(N_v \cdot P_b) + 2N_t(M_v \cdot P_b), \end{aligned} \tag{49}$$

the result being a bivector, and it was used that under the commutator the three scalars P_s, $N_v \cdot M_v$ and $N_t M_t$, do not contribute. In the expansion of the first odd grade part $P_-\overline{N}_- M_- - M_-\overline{N}_- P_-$ the products involving two or three trivector parts drop out, leaving

$$P_-\overline{N}_- M_- - M_-\overline{N}_- P_- = M_v N_v P_v - P_v N_v M_v + 2M_t(N_v \wedge P_v) + 2N_t(P_v \wedge M_v) + 2P_t(M_v \wedge N_v). \tag{50}$$

The first two terms on the right give

$$\begin{aligned} M_v N_v P_v - P_v N_v M_v &= M_v \wedge N_v \wedge P_v - P_v \wedge N_v \wedge M_v + (M_v \wedge N_v) \cdot P_v - P_v \cdot (N_v \wedge M_v) \\ &= 2M_v \wedge N_v \wedge P_v + P_v \cdot (N_v \wedge M_v) - P_v \cdot (N_v \wedge M_v) = 2M_v \wedge N_v \wedge P_v, \end{aligned} \tag{51}$$

where we used $(M_v \cdot N_v)P_v - P_v(N_v \cdot M_v) = 0$ in the first equality. Putting all this together we finally obtain

$$\begin{aligned} &(M \star N) \star P - M \star (N \star P) \\ &= [M_v \wedge N_v, P_b] - 2M_t(N_v \cdot P_b) + 2N_t(M_v \cdot P_b) + [N_v \wedge P_v, M_b] - 2N_t(P_v \cdot M_b) + 2P_t(N_v \cdot M_b) \\ &\quad + [P_v \wedge M_v, N_b] - 2P_t(M_v \cdot N_b) + 2M_t(P_v \cdot N_b) \\ &\quad + 2M_v \wedge N_v \wedge P_v + 2M_t(N_v \wedge P_v) + 2N_t(P_v \wedge M_v) + 2P_t(M_v \wedge N_v) \\ &\quad + N_v \cdot [P_b, M_b] + N_t[P_b, M_b] + N_v \wedge [P_b, M_b] + M_v \cdot [N_b, P_b] + M_t[N_b, P_b] + M_v \wedge [N_b, P_b] \\ &\quad + P_v \cdot [M_b, N_b] + P_t[M_b, N_b] + P_v \wedge [M_b, N_b], \end{aligned} \tag{52}$$

and note that the full result is also invariant under cyclic permutations of M, N, P, and that the first two lines of (52) show the even grade part, and the last three lines the odd grade part. An easy consequence of the cyclic symmetry is

$$(M \star N) \star P - M \star (N \star P) = (N \star P) \star M - N \star (P \star M) = (P \star M) \star N - P \star (M \star N). \tag{53}$$

3.3 | Geometric algebra from octonions

Being able to embed octonions in $Cl(3,0)$, we may ask the question for how in the opposite the geometric algebra multivector product of $Cl(3,0)$ may be obtained from octonions. The simpler question of obtaining quaternions from octonions is easily answered, one just identifies quaternions with the (even) Pauli spinor part of octonions, i.e. $Cl^+(3,0) \cong \mathbb{H} \cong \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are the first three generators of octonions, and note that by (27)

$$M_+ \star N_+ = M_+ N_+, \tag{54}$$

where on the right side $M_+ N_+$ corresponds to the quaternion product (and the product in the even subalgebra $Cl^+(3,0)$). We observe term by term that

$$M_+ N_+ \overset{(27)}{=} M_+ \star N_+, \quad M_- N_- \overset{(27)}{=} N_- \star \overline{M}_-, \quad M_- N_+ \overset{(27)}{=} N_+ \star M_-, \quad \overline{M_+ N_-} \overset{Cl(3,0)}{=} \overline{N}_- \overline{M}_+ \overset{(27)}{=} \overline{N}_- \star M_+, \tag{55}$$

where the octonion overline conjugation of $N_- \star \overline{M}_-$ and $\overline{N}_- \star M_+$, applied to the octonion basis yields

$$\overline{\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}}^{\mathbb{O}} = \{1, -\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3, -\mathbf{e}_4, -\mathbf{e}_5, -\mathbf{e}_6, \mathbf{e}_7\}. \tag{56}$$

According to the multiplication table Table 1 , octonion overline conjugation (56) can be expressed for every $B \in \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$, in product form as

$$\overline{B}^{\mathbb{O}} = (-\mathbf{e}_7 \star B) \star \mathbf{e}_7 = -\mathbf{e}_7 \star (B \star \mathbf{e}_7). \tag{57}$$

The full geometric product in $Cl(3,0)$ can thus be defined from octonions as

$$MN = M_+N_+ + M_-N_- + M_-N_+ + M_+N_- = M_+ \star N_+ + N_- \star \overline{M_-}^{\mathbb{O}} + N_+ \star M_- + \overline{\overline{N_-}^{\mathbb{O}} \star M_+}^{Cl(3,0)}, \tag{58}$$

where in the last term the inner conjugation is octonionic (56), and the outer conjugation is Clifford conjugation in $Cl(3,0)$, after the assignment

$$\{1 \to 1, \mathbf{e}_1 \to \sigma_{23}, \mathbf{e}_2 \to \sigma_{31}, \mathbf{e}_3 \to \sigma_{12}, \mathbf{e}_4 \to \sigma_1, \mathbf{e}_5 \to \sigma_2, \mathbf{e}_6 \to \sigma_3, \mathbf{e}_7 \to \sigma_{123}\}, \tag{59}$$

has been made (compare (24)). Then (58) with assignment (59) yields the Clifford geometric algebra multiplication table of $Cl(3,0)$.

An alternative, even more direct implementation of the fourth geometric product part in (58) can be obtained from

$$-(N_- \star I) \star (M_+ \star I) = -(N_-I) \star (M_+I) \overset{(27)}{=} -M_+IN_-I = -I^2M_+N_- = M_+N_-, \tag{60}$$

observing that according to the multiplication table Table 1 , we have for any $M \in Cl(3,0)$

$$M \star I = MI, \qquad M_+ \star I = M_+I \in Cl^-(3,0), \qquad M_- \star I = M_-I \in Cl^+(3,0). \tag{61}$$

3.4 | Representing octonions with biquaternions

The Clifford geometric algebra $Cl(3,0)$ can be represented with complex biquaternions $\mathbb{C} \otimes \mathbb{H}$, that is quaternions with complex coefficients. The isomorphic complex quaternion basis is

$$\{1, \sigma_{23} \to \boldsymbol{i}, \sigma_{31} \to \boldsymbol{j}, \sigma_{12} \to \boldsymbol{k}, \;\; \sigma_1 \to i\,\boldsymbol{i}, \sigma_2 \to i\boldsymbol{j}, \sigma_3 \to i\,\boldsymbol{k}, I = \sigma_{123} \to i\}, \tag{62}$$

where $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \in \mathbb{H}$ is a real quaternion basis and $i \in \mathbb{C}$ is an extra commutative imaginary unit. The first four elements in (62) correspond to the even grade Pauli spinors (all real), and the last four elements to the odd grade non-Pauli spinors (all with factor i), respectively. That is we have for $M \in Cl(3,0)$, with eight real basis coefficients $M_k \in \mathbb{R}$, $1 \leq k \leq 8$, the isomorphic biquaternion element

$$M = M_+ + M_- = M_1 + M_5\boldsymbol{i} + M_6\boldsymbol{j} + M_7\boldsymbol{k} \,+\, i(M_8 + M_2\boldsymbol{i} + M_3\boldsymbol{j} + M_4\boldsymbol{k}), \qquad M_+ = \mathrm{Re}(M), \quad M_- = i\,\mathrm{Im}(M). \tag{63}$$

The Clifford conjugation of $Cl(3,0)$ corresponds to the quaternion conjugation

$$\overline{M}_+ = \mathrm{qc}(M_+) = M_1 - M_5\boldsymbol{i} - M_6\boldsymbol{j} - M_7\boldsymbol{k}, \qquad \overline{M}_- = \mathrm{qc}(M_-) = i(M_8 - M_2\boldsymbol{i} - M_3\boldsymbol{j} - M_4\boldsymbol{k}). \tag{64}$$

Then the octonionic product $M \star N$ can be embedded in complex biquaternions via

$$\begin{aligned} M \star N &= (M \star N)_+ + (M \star N)_- = M_+N_+ + \mathrm{qc}(N_-)M_- + N_-M_+ + M_-\mathrm{qc}(N_+) \\ &= (M_1 + M_5\boldsymbol{i} + M_6\boldsymbol{j} + M_7\boldsymbol{k})(N_1 + N_5\boldsymbol{i} + N_6\boldsymbol{j} + N_7\boldsymbol{k}) + i^2(N_8 - N_2\boldsymbol{i} - N_3\boldsymbol{j} - N_4\boldsymbol{k})(M_8 + M_2\boldsymbol{i} + M_3\boldsymbol{j} + M_4\boldsymbol{k}) \\ &\quad + i(N_8 + N_2\boldsymbol{i} + N_3\boldsymbol{j} + N_4\boldsymbol{k})(M_1 + M_5\boldsymbol{i} + M_6\boldsymbol{j} + M_7\boldsymbol{k}) + i(M_8 + M_2\boldsymbol{i} + M_3\boldsymbol{j} + M_4\boldsymbol{k})(N_1 - N_5\boldsymbol{i} - N_6\boldsymbol{j} - N_7\boldsymbol{k}), \end{aligned} \tag{65}$$

reducing the computation of the octonionic product to complex quaternionic multiplications. Clearly, in (65) $(M \star N)_+ = \mathrm{Re}(M \star N)$ is simply the real quaternion part of $M \star N$, whereas $(M \star N)_- = i\,\mathrm{Im}(M \star N)$ is the imaginary part of $M \star N$, respectively. This also means that it is very easy to implement octonionic multiplication in any numeric or symbolic software package which can deal with biquaternion (complex quaternions) numbers. We note that this embedding of the octonion product in biquaternions is to some degree similar to the definition of octonions as pairs of quaternions via the Cayley-Dickson doubling process, compare [19], p. 302. Yet there a new imaginary unit is used anticommuting with $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$, as opposed to the commutative $i \in \mathbb{C}$, used in the present subsection.

3.5 Representing octonions with Pauli matrices

The Clifford geometric algebra $Cl(3,0)$ can be represented with complex two by two Pauli matrices, see[19], p. 51, with matrix basis elements

$$\begin{aligned}
&\mathbf{1}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_{23}=i\sigma_1=\begin{pmatrix}0&i\\i&0\end{pmatrix},\quad \sigma_{31}=i\sigma_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \sigma_{12}=i\sigma_3=\begin{pmatrix}i&0\\0&-i\end{pmatrix},\\
&\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad I=\sigma_{123}=i\mathbf{1}=\begin{pmatrix}i&0\\0&i\end{pmatrix},
\end{aligned}\tag{66}$$

where the first line shows the four even grade elements of one scalar and three bivectors, and the second line the four odd grade elements of three vectors and one trivector (up to a sign, the first line multiplied by i or equivalently by I). A general element $M\in Cl(3,0)$ can therefore be written in the Pauli matrix basis with eight real coefficients ($M_k\in\mathbb{R}, 1\leq k\leq 8$), or four complex coefficients as

$$\begin{aligned}
M &= M_+ + M_- = (M_1\mathbf{1}+M_5 i\sigma_1+M_6 i\sigma_2+M_7 i\sigma_3)+(M_8 i\mathbf{1}+M_2\sigma_1+M_3\sigma_2+M_4\sigma_3)\\
&=(M_1+M_8 i)\mathbf{1}+(M_2+iM_5)\sigma_1+(M_3+iM_6)\sigma_2+(M_4+iM_7)\sigma_3\\
&=\begin{pmatrix}(M_1+M_8 i)+(M_4+iM_7) & (M_2+iM_5)-i(M_3+iM_6)\\ (M_2+iM_5)+i(M_3+iM_6) & (M_1+M_8 i)-(M_4+iM_7)\end{pmatrix}.
\end{aligned}\tag{67}$$

If an element of $Cl(3,0)$ is given as complex two by two matrix

$$M=\begin{pmatrix}M_{11}&M_{12}\\M_{21}&M_{22}\end{pmatrix},\tag{68}$$

we can therefore extract from the four complex coefficients $M_{11}, M_{12}, M_{21}, M_{22}\in\mathbb{C}$, the eight real coefficients in the basis (66) as

$$\begin{aligned}
&M_1=\frac{1}{2}\mathrm{Re}(M_{11}+M_{22}),\quad M_2=\frac{1}{2}\mathrm{Re}(M_{12}+M_{21}),\quad M_3=\frac{1}{2}\mathrm{Im}(M_{21}-M_{12}),\quad M_4=\frac{1}{2}\mathrm{Re}(M_{11}-M_{22}),\\
&M_5=\frac{1}{2}\mathrm{Im}(M_{12}+M_{21}),\quad M_6=\frac{1}{2}\mathrm{Re}(M_{12}-M_{21}),\quad M_7=\frac{1}{2}\mathrm{Im}(M_{11}-M_{22}),\quad M_8=\frac{1}{2}\mathrm{Im}(M_{11}+M_{22}).
\end{aligned}\tag{69}$$

The complex matrix for the even part of $M\in Cl(3,0)$ is

$$\begin{aligned}
M_+ &= M_1\mathbf{1}+i(M_5\sigma_1+M_6\sigma_2+M_7\sigma_3)=\begin{pmatrix}M_1+iM_7 & M_6+iM_5\\ -M_6+iM_5 & M_1-iM_7\end{pmatrix},\\
\overline{M}_+ &= M_1\mathbf{1}-i(M_5\sigma_1+M_6\sigma_2+M_7\sigma_3)=-\begin{pmatrix}-M_1+iM_7 & M_6+iM_5\\ -M_6+iM_5 & -M_1-iM_7\end{pmatrix},
\end{aligned}\tag{70}$$

with symmetry[6] for the diagonal elements, respectively the off diagonal elements,

$$M_1-iM_7=\mathrm{cc}(M_1+iM_7),\qquad -M_6+iM_5=-\mathrm{cc}(M_6+iM_5).\tag{71}$$

The complex matrix for the odd part of $M\in Cl(3,0)$ is

$$\begin{aligned}
M_- &= M_8 i\mathbf{1}+M_2\sigma_1+M_3\sigma_2+M_4\sigma_3 = i[M_8\mathbf{1}-i(M_2\sigma_1+M_3\sigma_2+M_4\sigma_3)]\\
&=\begin{pmatrix}M_4+iM_8 & M_2-iM_3\\ M_2+iM_3 & -M_4+iM_8\end{pmatrix}=i\begin{pmatrix}M_8-iM_4 & -M_3-iM_2\\ M_3-iM_2 & M_8+iM_4\end{pmatrix},\\
\overline{M}_- &= -i[-M_8\mathbf{1}-i(M_2\sigma_1+M_3\sigma_2+M_4\sigma_3)]=-i\begin{pmatrix}-M_8-iM_4 & -M_3-iM_2\\ M_3-iM_2 & -M_8+iM_4\end{pmatrix},
\end{aligned}\tag{72}$$

with the corresponding symmetry for the diagonal elements, respectively the off diagonal elements,

$$M_8+iM_4=\mathrm{cc}(M_8-iM_4),\qquad M_3-iM_2=-\mathrm{cc}(-M_3-iM_2).\tag{73}$$

[6] Note that we use cc() for complex conjugation: $\mathrm{cc}(a+ib)=a-ib$.

The octonion product embedding can then be expressed for $M, N \in Cl(3,0)$ in complex matrix form for the (even grade) Pauli part by

$$(M \star N)_+ = M_+N_+ + \overline{N}_-M_- = \begin{pmatrix} M_1 + iM_7 & M_6 + iM_5 \\ -M_6 + iM_5 & M_1 - iM_7 \end{pmatrix} \begin{pmatrix} N_1 + iN_7 & N_6 + iN_5 \\ -N_6 + iM_5 & N_1 - iN_7 \end{pmatrix}$$
$$+ (-i)i \begin{pmatrix} -N_8 - iN_4 & -N_3 - iN_2 \\ N_3 - iN_2 & -N_8 + iN_4 \end{pmatrix} \begin{pmatrix} M_8 - iM_4 & -M_3 - iM_2 \\ M_3 - iM_2 & M_8 + iM_4 \end{pmatrix}, \tag{74}$$

and for the (odd grade) non-Pauli part by

$$(M \star N)_- = N_-M_+ + M_-\overline{N}_+ = i \begin{pmatrix} N_8 - iN_4 & -N_3 - iN_2 \\ N_3 - iN_2 & N_8 + iN_4 \end{pmatrix} \begin{pmatrix} M_1 + iM_7 & M_6 + iM_5 \\ -M_6 + iM_5 & M_1 - iM_7 \end{pmatrix}$$
$$+ i(-1) \begin{pmatrix} M_8 - iM_4 & -M_3 - iM_2 \\ M_3 - iM_2 & M_8 + iM_4 \end{pmatrix} \begin{pmatrix} -M_1 + iM_7 & M_6 + iM_5 \\ -M_6 + iM_5 & -M_1 - iM_7 \end{pmatrix}. \tag{75}$$

The full octonionic product $M \star N$ in complex two by two matrix form is simply the sum of (74) and (75).

4 | OCTONIONIC PRODUCT IN $CL(3, 1)$

We now work in the Clifford geometric algebra $Cl(3,1)$ with opposite signature over the vector space $\mathbb{R}^{3,1}$, found previously relevant for the construction of the space-time Fourier transform in [10,12,13]. For this two pure unit quaternions $f, g \in \mathbb{H}$, $f^2 = g^2 = -1$ are chosen and quaternions $x \in \mathbb{H}$ are split into

$$x_\pm = \frac{1}{2}(x \pm fxg). \tag{76}$$

This split can be fully extended to $Cl(3,1)$ (see its tensor product relation to quaternions below), and in $Cl(3,1)$ the subalgebra generated by time vector e_t, space volume i_3, and hypervolume I,

$$\{1, e_t = e_0, i_3 = e_{123}, I = e_t i_3\}, \tag{77}$$

is isomorphic to quaternions. Choosing for the generalization of the split[7] to $Cl(3,1)$, $f = e_t$, $g = i_3 = e_t^* = e_t I^{-1}$ results in the space-time split related to time axis e_t. In the resulting space-time Fourier transform, this space-time split naturally splits $Cl(3,1)$ multivector valued wave packets into left- and right traveling wave packets.

Furthermore, Patrick Girard et al.[7] find the tensor product of two quaternion algebras $\mathbb{H} \otimes \mathbb{H}$ to be isomorphic to $Cl(3,1)$.[8] Note that Hestenes' choice of $Cl(1,3)$ for STA is algebraically not isomorphic to $Cl(3,1)$.[9]

The orthonormal basis vectors of $\mathbb{R}^{3,1}$ square to

$$-e_0^2 = e_1^2 = e_2^2 = e_3^2 = 1. \tag{78}$$

The 16-dimensional multivector basis of $Cl(3,1)$ is

$$\{1, e_0, e_1, e_2, e_3, s_1 = e_{10}, s_2 = e_{20}, s_3 = e_{30}, Is_1 = -e_{23} = -s_{23}, Is_2 = -e_{31} = -s_{31}, Is_3 = -e_{12} = -s_{12},$$
$$Ie_0 = e_{123}, Ie_1 = e_{023}, Ie_2 = e_{031}, Ie_3 = e_{012}, I = e_{0123} = -s_1s_2s_3\}. \tag{79}$$

The even subalgebra $Cl^+(3,1)$ of spinors is again isomorphic to $Cl(3,0)$ with eight-dimensional basis

$$\{1, s_1, s_2, s_3, Is_1, Is_2, Is_3, I\}. \tag{80}$$

We have the important relationships

$$\widetilde{I} = I, \quad II = -1, \quad s_1s_2 = -Is_3 = -s_2s_1, \quad Is_1Is_2 = -s_1s_2 = Is_3. \tag{81}$$

Furthermore, the even subalgebra of $Cl(3,0) \cong Cl^+(3,1)$ commutes with e_0 and has the basis

$$\{1, Is_1, Is_2, Is_3\} \tag{82}$$

[7] Note that here the asterisk corresponds to duality in geometric algebra $e_t^* = e_t I^{-1}$.

[8] Even without mentioning it, Patrick Girard et al. thus go back to the beginning, i.e. the very way William K. Clifford himself originally constructed geometric algebras in [3].

[9] Remark by Gene McClellan at AGACSE 2021.

TABLE 2 Multiplication table for octonion embedding in $Cl(3,1)$.

Left factors	Right factors							
	1	Is_1	Is_2	Is_3	s_1	s_2	s_3	I
1	1	Is_1	Is_2	Is_3	s_1	s_2	s_3	I
Is_1	Is_1	-1	Is_3	$-Is_2$	I	$-s_3$	s_2	$-s_1$
Is_2	Is_2	$-Is_3$	-1	Is_1	s_3	I	$-s_1$	$-s_2$
Is_3	Is_3	Is_2	$-Is_1$	-1	$-s_2$	s_1	I	$-s_3$
s_1	s_1	$-I$	$-s_3$	s_2	-1	$-Is_3$	Is_2	Is_1
s_2	s_2	s_3	$-I$	$-s_1$	Is_3	-1	$-Is_1$	Is_2
s_3	s_3	$-s_2$	s_1	$-I$	$-Is_2$	Is_1	-1	Is_3
I	I	s_1	s_2	s_3	$-Is_1$	$-Is_2$	$-Is_3$	-1

for Pauli spinors (rotors in space)

$$\psi_+ = \frac{1}{2}(\psi + (e_0^2)e_0\psi e_0) = \frac{1}{2}(\psi - e_0\psi e_0). \tag{83}$$

The odd elements of $Cl(3,0) \cong Cl^+(3,1)$ are the non-Pauli spinors[10]

$$\psi_- = \frac{1}{2}(\psi - (e_0^2)e_0\psi e_0) = \frac{1}{2}(\psi + e_0\psi e_0). \tag{84}$$

with basis elements

$$\{s_1, s_2, s_3, I\}, \tag{85}$$

anti-commute with e_0. We can again formulate an embedding of the octonionic product in $Cl(3,1)$ by defining for two spinors $\psi, \phi \in Cl^+(3,1)$ that

$$\psi \star \phi = \psi_+\phi_+ + \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+. \tag{86}$$

Computing all products explicitly according to (86), we obtain the octonionic multiplication table Table 2 . A visualization diagram for this octonionic multiplication is shown in Fig. 2 . Note that Fig. 1 and Fig. 2 are closely related by replacing σ_k with s_k $(k = 1,2,3)$, and I by $-I$, respectively. Note that the multiplication table shows that the first two terms in (86) result in Pauli spinors, whereas the last two terms result in non-Pauli spinors, respectively.

The octonion conjugate in $Cl(3,1)$, an anti-involution, is again given by

$$\psi^* = \widetilde{\psi}_+ - \psi_-, \qquad (\psi \star \phi)^* = \phi^* \star \psi^*. \tag{87}$$

We now compute the octonion norm in $Cl^+(3,1)$

$$\begin{aligned}
\|\psi\| &= \psi \star \psi^* = (\psi_+ + \psi_-) \star (\widetilde{\psi}_+ - \psi_-) = \psi_+\widetilde{\psi}_+ + \widetilde{(-\psi_-)}\psi_- + (-\psi_-)\psi_+ + \psi_-\widetilde{\widetilde{\psi}}_+ = \psi_+\widetilde{\psi}_+ - \widetilde{\psi}_-\psi_- - \psi_-\psi_+ + \psi_-\psi_+ \\
&= \psi_+\widetilde{\psi}_+ - \widetilde{\psi}_-\psi_- = \frac{1}{4}(\psi - e_0\psi e_0)(\widetilde{\psi} - e_0\widetilde{\psi}e_0) - \frac{1}{4}(\widetilde{\psi} + e_0\widetilde{\psi}e_0)(\psi + e_0\psi e_0) \\
&= \frac{1}{4}(\psi\widetilde{\psi} - e_0\psi\widetilde{\psi}e_0 - \widetilde{\psi}\psi + e_0\widetilde{\psi}\psi e_0) + \frac{1}{4}(-\psi e_0\widetilde{\psi}e_0 - e_0\psi e_0\widetilde{\psi} - \widetilde{\psi}e_0\psi e_0 - e_0\widetilde{\psi}e_0\psi) \\
&= \frac{1}{4}(2\langle\psi\widetilde{\psi}\rangle - 2\langle\widetilde{\psi}\psi\rangle) - \frac{1}{2}[(\psi e_0\widetilde{\psi})\cdot e_0 + (\widetilde{\psi}e_0\psi)\cdot e_0] = -\frac{1}{2}\langle\psi e_0\widetilde{\psi}e_0 + \widetilde{\psi}e_0\psi e_0\rangle = -\langle\psi e_0\widetilde{\psi}e_0\rangle \\
&= -(\psi e_0\widetilde{\psi})\cdot e_0,
\end{aligned} \tag{88}$$

Note that the computation is closely analogous to (12) for $Cl(1,3)$, only several sign changes occur due to $e_0^2 = -1$ in $Cl(3,1)$.

[10]Note that the definition of $\psi_\pm$ provided here is consistent with the corresponding definition in Section 2, because inserting the factor $(e_0^2) = +1$ in Section 2 preserves the definition of $\psi_\pm$.

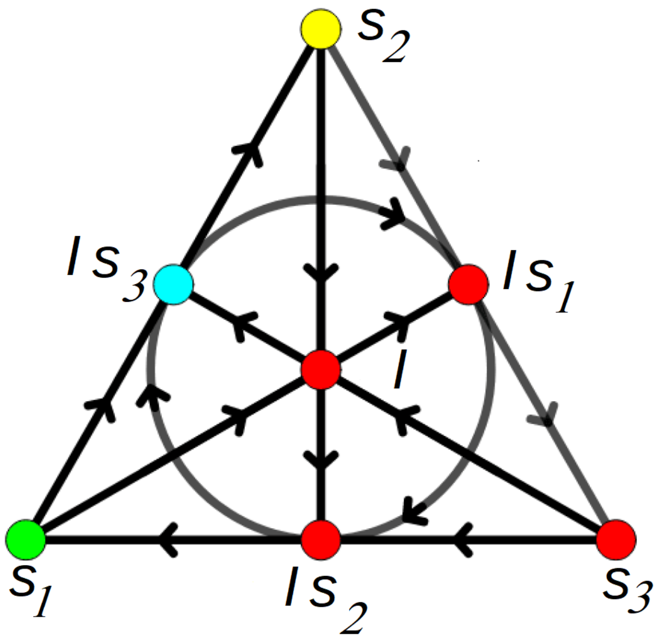

FIGURE 2 Illustration of space-time spinors in $Cl^+(3,1)$ under the octonionic product (86) in Table 2 . Fano plane depiction adapted from Steve Phelps[21].

5 | OCTONIONIC PRODUCT IN $CL(0,3)$

Now we want to pursue the question how far other Clifford algebras $Cl(p,q)$, $n = p+q = 3$, might be suitable for a similar embedding of octonions. First we turn to $Cl(0,3)$ with orthonormal vector basis of $\mathbb{R}^{0,3}$

$$\{e_1, e_2, e_3\}, \quad e_1^2 = e_2^2 = e_3^2 = -1. \tag{89}$$

The eight-dimensional multivector basis of $Cl(0,3)$ has all vectors and bivectors squaring to -1, only the scalar and the central unit trivector pseudoscalar square to $+1$

$$\{1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, I = e_{123}\}. \tag{90}$$

We can split this into the even subalgebra $Cl^+(0,3)$ of spinors (rotors) ψ_+ with basis

$$\{1, e_{23}, e_{31}, e_{12}\}, \tag{91}$$

and odd elements ψ_- of $Cl^-(0,3)$ with basis

$$\{e_1, e_2, e_3, I\}. \tag{92}$$

We define the octonionic product of two multivectors $\psi, \phi \in Cl(0,3)$ as

$$\psi \star \phi = \psi_+\phi_+ + \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+. \tag{93}$$

Computing all products of basis elements of $Cl(0,3)$ under this new product, we obtain the multiplication table Table 3 . A visualization diagram for this octonionic multiplication in $Cl(0,3)$ is shown in Fig. 3 . Note that the first two terms in (93) result in even grade multivectors, whereas the last two terms result in odd multivectors, respectively.

The octonion conjugate in $Cl(0,3)$, an anti-involution, is given by

$$\psi^* = \widetilde{\psi}_+ - \psi_- \qquad (\psi \star \phi)^* = \phi^* \star \psi^*. \tag{94}$$

TABLE 3 Multiplication table for octonion product defined in $Cl(0,3)$.

Left factors	Right factors 1	e_{23}	e_{31}	e_{12}	e_1	e_2	e_3	I
1	1	e_{23}	e_{31}	e_{12}	e_1	e_2	e_3	I
e_{23}	e_{23}	-1	e_{12}	$-e_{31}$	I	$-e_3$	e_2	$-e_1$
e_{31}	e_{31}	$-e_{12}$	-1	e_{23}	e_3	I	$-e_1$	$-e_2$
e_{12}	e_{12}	e_{31}	$-e_{23}$	-1	$-e_2$	e_1	I	$-e_3$
e_1	e_1	$-I$	$-e_3$	e_2	-1	$-e_{12}$	e_{31}	e_{23}
e_2	e_2	e_3	$-I$	$-e_1$	e_{12}	-1	$-e_{23}$	e_{31}
e_3	e_3	$-e_2$	e_1	$-I$	$-e_{31}$	e_{23}	-1	e_{12}
I	I	e_1	e_2	e_3	$-e_{23}$	$-e_{31}$	$-e_{12}$	-1

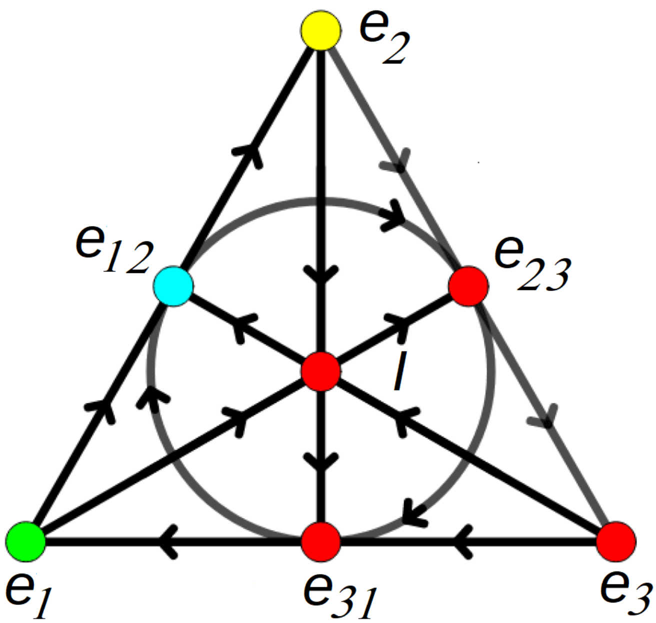

FIGURE 3 Illustration of basis elements of $Cl(0,3)$ under the octonionic product (93) in Table 3 . Fano plane depiction adapted from Steve Phelps[21].

As an application let us compute the octonion norm[11] in $Cl(0,3)$

$$\|\psi\| = \psi \star \psi^* = \psi_+\widetilde{\psi}_+ - \widetilde{\psi}_-\psi_- - \psi_-\psi_+ + \psi_-\widetilde{\widetilde{\psi}}_+ = \psi_+\widetilde{\psi}_+ - \widetilde{\psi}_-\psi_- = (\psi_s+\psi_b)(\psi_s-\psi_b) - (\psi_v-\psi_t)(\psi_v+\psi_t)$$

$$= \psi_s^2 - \psi_s\psi_b + \psi_s\psi_b - \psi_b^2 - \psi_v^2 + \psi_t^2 - \psi_v\psi_t + \psi_v\psi_t = \psi_s^2 - \psi_b^2 - \psi_v^2 + \psi_t^2 = \langle\psi\overline{\psi}\rangle = \sum_{i=1}^{8}\psi_i^2, \tag{95}$$

where $\psi_i \in \mathbb{R}$, $1 \le i \le 8$, are the basis coefficients of ψ in the $Cl(0,3)$ basis (90), and ψ_s, ψ_b, ψ_v and ψ_t, are the scalar-, vector-, bivector- and trivector part of ψ, respectively. Note the use of $\psi_s, \psi_t \in$ center of $Cl(0,3)$.

[11] Note that using the *principal reverse* (see Equation (2.4) on page 2217 of [11]), the composition of reversion with changing the sign of every basis vector factor, allows to write the norm as scalar product $\|\psi\| = \langle\psi \mathrm{pr}(\psi)\rangle = \psi * \mathrm{pr}(\psi) = \psi_+\mathrm{pr}(\psi)_+ + \psi_-\mathrm{pr}(\psi)_-$, using $\mathrm{pr}(\psi_+) = \widetilde{\psi}_+$ and $\mathrm{pr}(\psi_-) = -\widetilde{\psi}_-$.

6 | OCTONIONIC PRODUCT IN $CL(1,2)$

Next we turn to $Cl(1,2)$ with orthonormal vector basis of $\mathbb{R}^{1,2}$

$$\{e_1, e_2, e_3\}, \quad e_1^2 = -e_2^2 = -e_3^2 = 1. \tag{96}$$

The eight-dimensional multivector basis of $Cl(1,2)$ is

$$\{1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, I = e_{123}\}, \tag{97}$$

with squares

$$e_{23}^2 = e_{123}^2 = -1, \quad e_{31}^2 = e_{12}^2 = +1. \tag{98}$$

We can split the basis (97) into a four-dimensional quaternion like subalgebra (for ψ_+) generated by the two vectors of negative square $\{e_2, e_3\}$,

$$\{1, e_2, e_3, e_{23}\}, \tag{99}$$

and the remaining four-dimensional ψ_- set always involving the factor e_1,

$$\{e_1, e_{31}, e_{12}, I = e_{123}\} = e_1\{1, -e_3, e_2, e_{23}\}. \tag{100}$$

We define the octonionic product of two multivectors $\psi, \phi \in Cl(1,2)$ as

$$\psi \star \phi = \psi_+\phi_+ + \overline{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\overline{\phi}_+, \tag{101}$$

where the overbar indicates Clifford conjugation.

Computing all products of basis elements of $Cl(1,2)$ under this new product, we obtain the multiplication table Table 4 . A visualization diagram for this octonionic multiplication in $Cl(1,2)$ is shown in Fig. 4 . The multiplication table shows that the first two product terms in (101) evidently belong to the quaternion like subalgebra (99), whereas the last two belong to the set (100) always involving the factor e_1.

Octonion conjugation in $Cl(1,2)$, an anti-involution, also uses Clifford conjugation (overbar notation)

$$\psi^* = \overline{\psi}_+ - \psi_-, \qquad (\psi \star \phi)^* = \phi^* \star \psi^*. \tag{102}$$

As useful exercise, we compute the octonion norm in $Cl(1,2)$

$$\begin{aligned}
\|\psi\| &= \psi \star \psi^* = \psi_+\overline{\psi}_+ - \overline{\psi}_-\psi_- - \psi_-\psi_+ + \psi_-\overline{\overline{\psi}}_+ = \psi_+\overline{\psi}_+ - \overline{\psi}_-\psi_- \\
&= (\psi_0 + \psi_2 e_2 + \psi_3 e_3 + \psi_{23}e_{23})(\psi_0 - \psi_2 e_2 - \psi_3 e_3 \psi_{23} e_{23}) \\
&\quad - (-\psi_1 e_1 - \psi_{31}e_{31} - \psi_{12}e_{12} + \psi_{123}I)(\psi_1 e_1 + \psi_{31}e_{31} + \psi_{12}e_{12} + \psi_{123}I) \\
&= \psi_0^2 + \psi_2^2 + \psi_3^2 + \psi_{23}^2 + \psi_2\psi_{23}\big(e_2(-e_{23}) + e_{23}(-e_2)\big) + \psi_3\psi_{23}\big(e_3(-e_{23}) + e_{23}(-e_3)\big) \\
&\quad + \psi_1^2 + \psi_{31}^2 + \psi_{12}^2 + \psi_{123}^2 + \psi_1\psi_{123}(e_1 I - I e_1) \\
&= \psi_0^2 + \psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_{23}^2 + \psi_{31}^2 + \psi_{12}^2 + \psi_{123}^2,
\end{aligned} \tag{103}$$

where $\psi_0, ..., \psi_{123} \in \mathbb{R}$, are the eight multivector basis coefficients of ψ in the basis (97) of $Cl(1,2)$. We used that ψ_0 and I are central, and that cross terms always happen to cancel out due to the signs and (anti)commutation properties of basis elements of (97). Just as in Footnote 11 for $Cl(0,3)$, the octonion norm in $Cl(1,2)$ can also be computed using the principal reverse

$$\|\psi\| = \langle \psi\, \mathrm{pr}(\psi)\rangle = \psi * \mathrm{pr}(\psi). \tag{104}$$

7 | THE CASE OF $CL(2,1)$

The Clifford algebra $Cl(2,1)$ is defined over the vector space $\mathbb{R}^{2,1}$ with three orthonormal basis vectors

$$\{e_1, e_2, e_3\}, \quad e_1^2 = e_2^2 = -e_3^2 = 1. \tag{105}$$

The eight-dimensional multivector basis of $Cl(2,1)$ is

$$\{1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, I = e_{123}\}, \tag{106}$$

TABLE 4 Multiplication table for octonion product defined in $Cl(1,2)$.

Left factors	Right factors							
	1	e_2	e_3	e_{23}	e_{31}	e_{12}	e_1	I
1	1	e_2	e_3	e_{23}	e_{31}	e_{12}	e_1	I
e_2	e_2	-1	e_{23}	$-e_3$	I	$-e_1$	e_{12}	$-e_{31}$
e_3	e_3	$-e_{23}$	-1	e_2	e_1	I	$-e_{31}$	$-e_{12}$
e_{23}	e_{23}	e_3	$-e_2$	-1	$-e_{12}$	e_{31}	I	$-e_1$
e_{31}	e_{31}	$-I$	$-e_1$	e_{12}	-1	$-e_{23}$	e_3	e_2
e_{12}	e_{12}	e_1	$-I$	$-e_{31}$	e_{23}	-1	$-e_2$	e_3
e_1	e_1	$-e_{12}$	e_{31}	$-I$	$-e_3$	e_2	-1	e_{23}
I	I	e_{31}	e_{12}	e_1	$-e_2$	$-e_3$	$-e_{23}$	-1

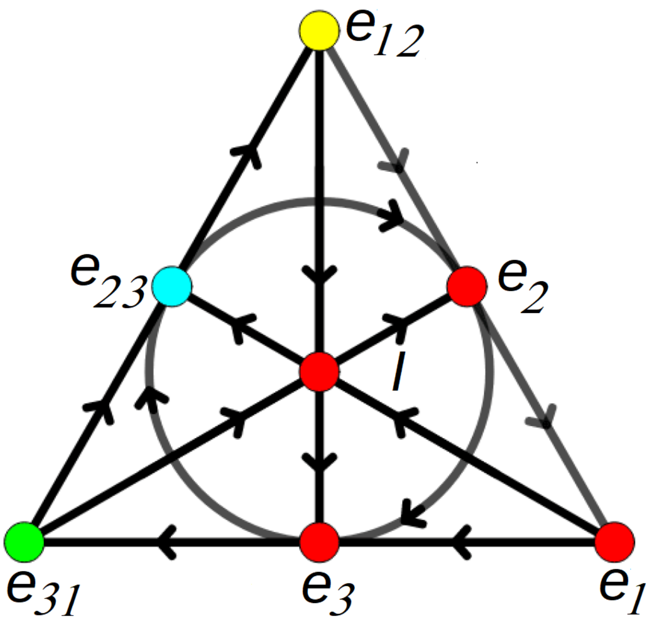

FIGURE 4 Illustration of basis elements of $Cl(1,2)$ under the octonionic product (101) in Table 4 . Fano plane depiction adapted from Steve Phelps[21].

where bivectors and trivectors square to

$$e_{23}^2 = e_{31}^2 = e_{123}^2 = 1, \quad e_{12}^2 = -1. \tag{107}$$

Thus the basis of $Cl(2,1)$ contains only two commuting elements $\{e_3, e_{12}\}$ that square to minus one and their product e_{123} squares to $+1$. Therefore no quaternionic subalgebra can be found in $Cl(2,1)$, which could give rise to the first 4×4 block in the multiplication table of the embedding of an octonionic product. The previous method, introduced by Lasenby, to define an octonion type product, seems therefore not applicable to $Cl(2,1)$.

One could think of taking inspiration from the principal reverse, which is the same as the reverse, except for multiplying e_3 with $\varepsilon_3 = e_3^2 = -1$. Applying this sign change (indicated by a prime) to the basis of the even subalgebra we obtain

$$\{1, e_{23}, e_{31}, e_{12}\}' = \{1, -e_{23}, -e_{31}, e_{12}\}. \tag{108}$$

So would it be possible to introduce an octonionic product in $Cl(2,1)$ in the following way?

$$M \star N = M_+ N_+' + \overline{N_-}' M_- + N_-' M_+ + M_- \overline{N_+}'. \tag{109}$$

This gives the correct diagonal elements when computing the first term $M_+N'_+$:

$$1 \star 1 = 1, \quad e_{23} \star e_{23} = e_{23}e'_{23} = -1, \quad e_{31} \star e_{31} = e_{31}e'_{31} = -1, \quad e_{12} \star e_{12} = e_{12}e_{12} = -1, \tag{110}$$

but the product of e_{23} with e_{12} becomes symmetric (instead of antisymmetric):

$$e_{23} \star e_{12} = e_{23}e'_{12} = e_{23}e_{12} = e_{31}, \qquad e_{12} \star e_{23} = e_{12}e'_{23} = -e_{12}e_{23} = -e_{13} = e_{31}. \tag{111}$$

So the answer is negative again.

8 | OCTONIONIC PRODUCTS IN $CL(P,Q)$, $N = P + Q = 4$

We have already considered the Minkowski space-time algebras $Cl(1,3)$ and $Cl(3,1)$, and all Clifford algebras of three-dimensional spaces $Cl(p,q)$, $n = p + q = 3$. Now we want to consider the even subalgebras of all Clifford algebras $Cl(p,q)$, $n = p + q = 4$ of four-dimensional vector spaces $\mathbb{R}^{p,q}$, $n = p + q = 4$, in order to find the Clifford algebras that permit an octonion product embedding via their even subalgebra, similar to the method proposed by Anthony Lasenby[16,17] for $Cl(1,3)$ and its even subalgebra $Cl^+(1,3)$, isomorphic to $Cl(3,0)$.

For this purpose we can utilize the following even subalgebra isomorphisms, see[19], p. 218.

$$Cl^+(p,q) \cong Cl(p,q-1), \qquad Cl^+(n,0) \cong Cl(0,n-1). \tag{112}$$

We therefore have five isomorphisms for $Cl(p,q)$, $n = p + q = 4$:

$$\begin{aligned} &Cl^+(4,0) \cong Cl(0,3), \quad Cl^+(3,1) \cong Cl(3,0), \quad Cl^+(2,2) \cong Cl(2,1), \quad Cl^+(1,3) \cong Cl(1,2) \cong Cl(3,0), \\ &Cl^+(0,4) \cong Cl(0,3) \cong Cl^+(4,0), \end{aligned} \tag{113}$$

also applying $Cl(p,q) \cong Cl(q+1,p-1)$ of[19], p. 215, and the last isomorphism follows from the first in reverse order.

Remark 3. Because we already found that $Cl(2,1)$ appears not to permit the Lasenby style embedding[16,17] of the octonion product, it would also not work in the even subalgebra of $Cl(2,2)$, according to Section 7. But we note that by excluding one basis vector of positive square, $Cl(2,2)$ is found to have subalgebras isomorphic to $Cl(1,2)$, which would then allow to embed an octonion product as in Section 6. More general, taking the hyperplane subalgebra of $Cl(2,2)$ obtained by excluding any vector dimension of positive square from $Cl(2,2)$, produces an algebra isomorphic to $Cl(1,2)$, which allows following Section 6 to embed an octonion product.

The algebras $Cl(1,3)$ and $Cl(3,1)$ have already been treated in detail in Sections 2 and 4, respectively. So in the following subsections we concentrate on $Cl(4,0)$, $Cl(0,4)$, and $Cl(2,2)$ (following Remark 3), respectively.

8.1 | Octonionic products in $Cl(4,0)$

We denote the orthonormal basis of $\mathbb{R}^4$ by

$$\{e_0, e_1, e_2, e_3\}, \quad e_0^2 = e_1^2 = e_2^2 = e_3^2 = 1. \tag{114}$$

The even subalgebra $Cl^+(4,0)$ has therefore the basis (now I is central)

$$\{1, I\sigma_1 = e_{23}, I\sigma_2 = e_{31}, I\sigma_3 = e_{12}, \sigma_1 = e_{10}, \sigma_2 = e_{20}, \sigma_3 = e_{30}, I = e_{0123}\}, \tag{115}$$

with $(k = 1,2,3)$

$$\widetilde{I} = I, \quad I^2 = 1, \quad \sigma_k^2 = -1, \quad (I\sigma_k)^2 = -1, \quad \sigma_1\sigma_2 = -I\sigma_3, \text{ etc.}, \quad I\sigma_1 I\sigma_2 = -I\sigma_3, \text{ etc.} \tag{116}$$

The $\psi_+ = \frac{1}{2}(\psi + e_0\psi e_0)$ Pauli spinor part (isomorphic to quaternions) commuting with e_0 is then

$$\{1, I\sigma_1, I\sigma_2, I\sigma_3\}, \tag{117}$$

and the $\psi_- = \frac{1}{2}(\psi - e_0\psi e_0)$ (dual) non-Pauli spinor part anti-commuting with e_0 is

$$\{\sigma_1, \sigma_2, \sigma_3, I = \sigma_1\sigma_2\sigma_3\}. \tag{118}$$

TABLE 5 Multiplication table for Lasenby octonion embedding in $Cl(4,0)$.

Left factors	Right factors							
	1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
1	1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
$I\sigma_1$	$I\sigma_1$	-1	$-I\sigma_3$	$I\sigma_2$	$-I$	σ_3	$-\sigma_2$	σ_1
$I\sigma_2$	$I\sigma_2$	$I\sigma_3$	-1	$-I\sigma_1$	$-\sigma_3$	$-I$	σ_1	σ_2
$I\sigma_3$	$I\sigma_3$	$-I\sigma_2$	$I\sigma_1$	-1	σ_2	$-\sigma_1$	$-I$	σ_3
σ_1	σ_1	I	σ_3	$-\sigma_2$	-1	$I\sigma_3$	$-I\sigma_2$	$-I\sigma_1$
σ_2	σ_2	$-\sigma_3$	I	σ_1	$-I\sigma_3$	-1	$I\sigma_1$	$-I\sigma_2$
σ_3	σ_3	σ_2	$-\sigma_1$	I	$I\sigma_2$	$-I\sigma_1$	-1	$-I\sigma_3$
I	I	$-\sigma_1$	$-\sigma_2$	$-\sigma_3$	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	-1

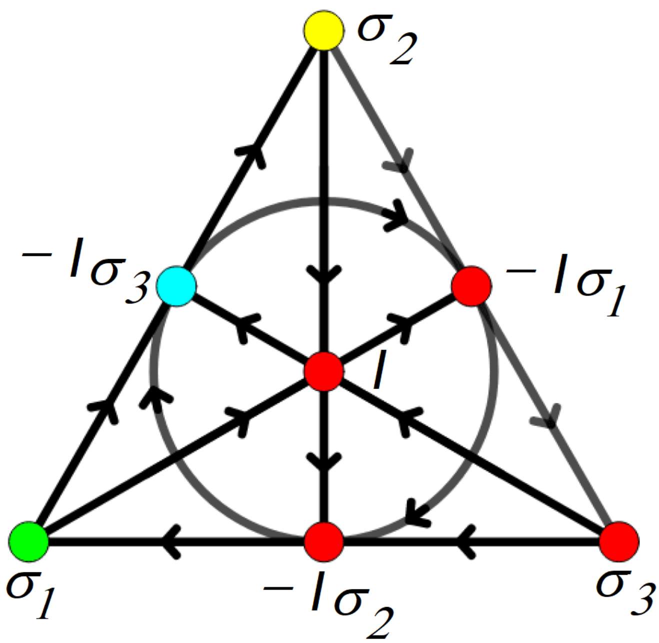

FIGURE 5 Illustration of basis elements of $Cl(4,0)$ under the octonionic product (119) in Table 5 . Fano plane depiction adapted from Steve Phelps[21].

We now use the Ansatz for the octonion product as

$$\psi \star \phi = \psi_+\phi_+ - \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+, \tag{119}$$

where we note the sign difference of the second term with (8). Based on (119), we obtain the following multiplication table Table 5 . A visualization diagram for this octonionic multiplication in $Cl(4,0)$ is shown in Fig. 5 . Note that the multiplication table shows that the first two terms in (119) result in Pauli spinors, whereas the last two terms result in non-Pauli spinors, respectively.

The octonion conjugation in $Cl(4,0)$, an anti-involution, is given by

$$\psi^* = \widetilde{\psi}_+ - \psi_-, \qquad (\psi \star \phi)^* = \phi^* \star \psi^*. \tag{120}$$

The octonionic norm in $Cl^+(4,0)$ is given by

$$\|\psi\| = (\psi e_0 \widetilde{\psi}) \cdot e_0, \tag{121}$$

the computation being the same as in (12) and (13) for $Cl(1,3)$.

TABLE 6 Multiplication table for Lasenby octonion embedding in $Cl(0,4)$.

Left factors	Right factors 1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
1	1	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	σ_1	σ_2	σ_3	I
$I\sigma_1$	$I\sigma_1$	-1	$I\sigma_3$	$-I\sigma_2$	$-I$	$-\sigma_3$	σ_2	σ_1
$I\sigma_2$	$I\sigma_2$	$-I\sigma_3$	-1	$I\sigma_1$	σ_3	$-I$	$-\sigma_1$	σ_2
$I\sigma_3$	$I\sigma_3$	$I\sigma_2$	$-I\sigma_1$	-1	$-\sigma_2$	σ_1	$-I$	σ_3
σ_1	σ_1	I	$-\sigma_3$	σ_2	-1	$-I\sigma_3$	$I\sigma_2$	$-I\sigma_1$
σ_2	σ_2	σ_3	I	$-\sigma_1$	$I\sigma_3$	-1	$-I\sigma_1$	$-I\sigma_2$
σ_3	σ_3	$-\sigma_2$	σ_1	I	$-I\sigma_2$	$I\sigma_1$	-1	$-I\sigma_3$
I	I	$-\sigma_1$	$-\sigma_2$	$-\sigma_3$	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$	-1

8.2 | Octonionic products in $Cl(0,4)$

We denote the orthonormal basis of $\mathbb{R}^{0,4}$ by

$$\{e_0, e_1, e_2, e_3\}, \quad e_0^2 = e_1^2 = e_2^2 = e_3^2 = -1. \tag{122}$$

The even subalgebra $Cl^+(0,4)$ has therefore the basis (now I is central)

$$\{1, I\sigma_1 = e_{23}, I\sigma_2 = e_{31}, I\sigma_3 = e_{12}, \sigma_1 = e_{10}, \sigma_2 = e_{20}, \sigma_3 = e_{30}, I = e_{0123}\}, \tag{123}$$

with ($k = 1,2,3$)

$$\widetilde{I} = I, \quad I^2 = 1, \quad \sigma_k^2 = -1, \quad (I\sigma_k)^2 = -1, \quad \sigma_1\sigma_2 = I\sigma_3, \text{ etc.}, \quad I\sigma_1 I\sigma_2 = I\sigma_3, \text{ etc..} \tag{124}$$

The Pauli spinor part (isomorphic to quaternions)

$$\psi_+ = \frac{1}{2}(\psi + (e_0^2)e_0\psi e_0) = \frac{1}{2}(\psi - e_0\psi e_0), \tag{125}$$

commuting with e_0 has the basis

$$\{1, I\sigma_1, I\sigma_2, I\sigma_3\}, \tag{126}$$

and the (dual) non-Pauli spinor part

$$\psi_- = \frac{1}{2}(\psi - (e_0^2)e_0\psi e_0) = \frac{1}{2}(\psi + e_0\psi e_0), \tag{127}$$

anti-commuting with e_0 has the basis

$$\{\sigma_1, \sigma_2, \sigma_3, I = -\sigma_1\sigma_2\sigma_3\}. \tag{128}$$

We now use the Ansatz for the octonion product as

$$\psi \star \phi = \psi_+\phi_+ - \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+, \tag{129}$$

where we note the sign difference of the second term with (8). Based on (129), we obtain the following multiplication table Table 6 . A visualization diagram for this octonionic multiplication in $Cl(0,4)$ is shown in Fig. 6 . Note that the multiplication table shows that the first two terms in (129) result in Pauli spinors, whereas the last two terms result in non-Pauli spinors, respectively.

The octonion conjugation in $Cl(0,4)$, an anti-involution, is given by

$$\psi^* = \widetilde{\psi}_+ - \psi_-, \qquad (\psi \star \phi)^* = \phi^* \star \psi^*. \tag{130}$$

The octonion norm in $Cl^+(0,4)$ is given by

$$\|\psi\| = -(\psi e_0 \widetilde{\psi}) \cdot e_0, \tag{131}$$

the computation being the same as in Section 4 for $Cl(3,1)$.

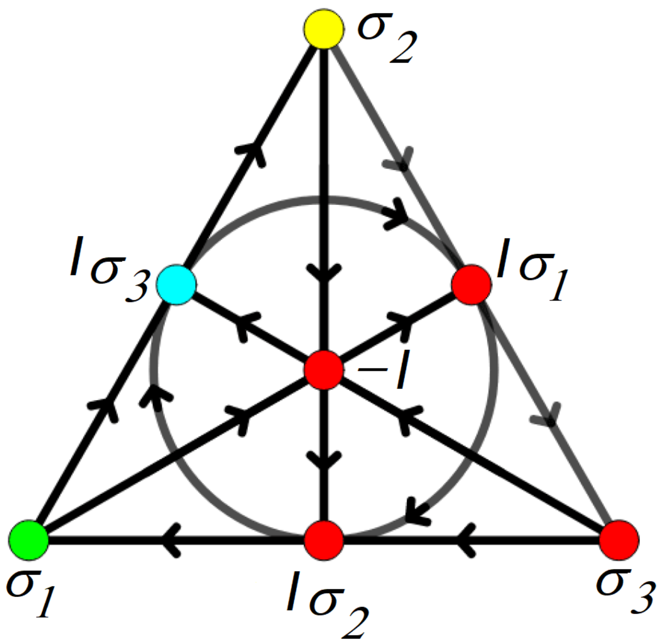

FIGURE 6 Illustration of basis elements of $Cl(0,4)$ under the octonionic product (129) in Table 6 . Fano plane depiction adapted from Steve Phelps [21].

8.3 | Octonionic products in $Cl(2,2)$

As already explained in Remark 3, the even subalgebra $Cl^+(2,2)$ isomorphic to $Cl(2,1)$ may not permit to embed an octonionic product. But we can instead simply take away one vector of positive square from the basis of $\mathbb{R}^{2,2}$, e.g. by removing e_1 and are left with $\{e_0, e_2, e_3\}$ and vector squares

$$e_0^2 = -e_2^3 = -e_3^2 = 1, \tag{132}$$

i.e. a basis for $\mathbb{R}^{1,2}$. By relabeling the basis vectors

$$e'_0 = e_0, \quad e'_1 = e_2, \quad e'_2 = e_3, \tag{133}$$

we can then apply the octonion embedding of Section 6 for obtaining an octonion product in the subalgebra of $Cl(2,2)$, generated by $\{e_0, e_2, e_3\}$.

Because by (133), it would only be a trivial basis element relabeling exercise applied to Section 6, we omit to restate for $Cl(\mathbb{R}^{1,2}) \cong Cl(\{e_0, e_2, e_3\}) \subset Cl(2,2)$ the octonionic product (101), the octonion conjugation (102) and the octonionic norm (103).

9 | CONCLUSIONS

In this paper we have studied A. Lasenby's embedding of octonion multiplication in space-time algebra $Cl(1,3)$ [16,17] and extended it to all Clifford geometric algebras $Cl(p,q)$ of dimensions $n = p + q = 3, 4$ of three and four dimensional quadratic spaces $\mathbb{R}^{p,q}$. A notable exception proved to be $Cl(2,1)$, where the lack of a subalgebra isomorphic to quaternions appears to be the essential barrier. This also means that for the case of $Cl(2,2)$ we are not able to simply use the even subalgebra, but instead need to exclude one basis vector of positive square. In all cases we gave multiplication tables and Fano plane diagrams, and specified the octonion conjugate which enables the computation of the octonion norm as a scalar in geometric algebra (via the scalar-, or the inner product). For $Cl(3,0)$ we additionally studied explicitly the octonionic product non-associativity in terms of the multivector grade parts of the multivector factors involved, showed how to obtain the multivector product of geometric algebra from the octonion product, and how to express the octonionic product using (complex) biquaternions (easiest for numeric and symbolic software implementations) or complex two by two matrices. A summary of the results is compiled in Table 7 . In

TABLE 7 Summary of octonion embeddings in $Cl(p,q)$, $p+q=n=3,4$. Algebra = Clifford geometric algebra selected for embedding, Pauli spinor ψ_+, non-Pauli spinor ψ_-, Conj. = octonion conjugation equivalent, Product = octonionic product, M. Tb. = multiplication table, F. Dg. = Fano plane diagram, Norm = octonion norm computed in Clifford geometric algebra, Sc.= section on the respective Clifford geometric algebra in this paper. $\widetilde{\psi}$ means reversion, $\overline{\psi}$ Clifford conjugation, $\mathrm{pr}(\psi)$ principal reverse and $\langle\psi\rangle$ scalar part, respectively. Section 3 additionally includes embeddings for (complex) biquaternions and Pauli matrix algebra.

Algebra	(non)Pauli spinors	Conj.	Product $\psi \star \phi$	M. Tb.	F. Dg.	Norm	Sc.
$Cl(3,0)$	$\psi_+ \in Cl^+(3,0)$, $\psi_- \in Cl^-(3,0)$	$\overline{\psi}_+ - \psi_-$	$\psi_+\phi_+ + \overline{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\overline{\phi}_+$	Tb. 1	Fig. 1	$\langle\psi\widetilde{\psi}\rangle$	3
$Cl(2,1)$	No implementation found						7
$Cl(1,2)$	$\psi_+ : \{1, e_2, e_3, e_{23}\}$, $\psi_- : \{e_1, e_{31}, e_{12}, I\}$	$\overline{\psi}_+ - \psi_-$	$\psi_+\phi_+ + \overline{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\overline{\phi}_+$	Tb. 4	Fig. 4	$\langle\psi\,\mathrm{pr}(\psi)\rangle$	6
$Cl(0,3)$	$\psi_+ \in Cl^+(0,3)$, $\psi_- \in Cl^-(0,3)$	$\widetilde{\psi}_+ - \psi_-$	$\psi_+\phi_+ + \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+$	Tb. 3	Fig. 3	$\langle\psi\overline{\psi}\rangle$	5
$Cl^+(4,0)$	$\psi_\pm = \frac{1}{2}(\psi \pm e_0\psi e_0)$	$\widetilde{\psi}_+ - \psi_-$	$\psi_+\phi_+ - \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+$	Tb. 5	Fig. 5	$(\psi e_0\widetilde{\psi}) \cdot e_0$	8.1
$Cl^+(3,1)$	$\psi_\pm = \frac{1}{2}(\psi \mp e_0\psi e_0)$	$\widetilde{\psi}_+ - \psi_-$	$\psi_+\phi_+ + \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+$	Tb. 2	Fig. 2	$-(\psi e_0\widetilde{\psi}) \cdot e_0$	4
$Cl(2,2)$	Use subalgebra $Cl(1,2) \cong Cl(\{e_0, e_2, e_3\}) \subset Cl(2,2)$ as in Sec. 6						8.3
$Cl^+(1,3)$	$\psi_\pm = \frac{1}{2}(\psi \pm e_0\psi e_0)$	$\widetilde{\psi}_+ - \psi_-$	$\psi_+\phi_+ + \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+$	Tb. 1	Fig. 1	$(\psi e_0\widetilde{\psi}) \cdot e_0$	2
$Cl^+(0,4)$	$\psi_\pm = \frac{1}{2}(\psi \mp e_0\psi e_0)$	$\widetilde{\psi}_+ - \psi_-$	$\psi_+\phi_+ - \widetilde{\phi}_-\psi_- + \phi_-\psi_+ + \psi_-\widetilde{\phi}_+$	Tb. 6	Fig. 6	$-(\psi e_0\widetilde{\psi}) \cdot e_0$	8.2

space-time algebra there is an immediate interest in the use of the Lasenby octonion embedding for elementary particle physics modeling, an approach which can now be extended to a wide range of Clifford algebras.

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Conflict of interest

The authors declare no potential conflict of interests.

References

1. Abłamowicz R., Fauser B. *CLIFFORD – A Maple Package for Clifford Algebra Computations with Bigebra, SchurFkt, GfG - Groebner for Grassmann, Cliplus, Define, GTP, Octonion, SP, SymGroupAlgebra, and code_support*. http://www.math.tntech.edu/rafal/, December 2008.

2. Błaszczyk L., Snopek KM. *Octonion Fourier Transform of real-valued functions of three variables - selected properties and examples*, Signal Processing, Vol. 136, 2017, pp. 29–37 (2017). DOI: https://doi.org/10.1016/j.sigpro.2016.11.021.

3. Clifford WK. *Applications of Grassmann's Extensive Algebra*. American Journal of Mathematics 1(4), pp. 350–358 (1878).

4. Conway JH., Smitth DA. *On Quaternions and Octonions*, Peters AK., Natick, Massachusetts, 2003, https://the-eye.eu/public/Books/Bibliotik/O/On%20Quaternions%20and%20Octonions%20-%20John%20H.%20Conway.pdf

5. Doran C., Lasenby A. *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge (UK), 2003.

6. Furey N. *Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra*, Physics Letters B, Vol. 785, pp. 84–89 (2018). DOI: https://doi.org/10.1016/j.physletb.2018.08.032.

7. Girard PR., Clarysse P., Pujol R. et al. *Hyperquaternions: A New Tool for Physics*. Adv. Appl. Clifford Algebras 28:68 (2018). DOI: https://doi.org/10.1007/s00006-018-0881-8.

8. Hamilton WR. *Note, by Sir W. R. Hamilton, respecting the researches of John T. Graves, Esq.*, Transactions of the Royal Irish Academy, Vol. 21, pp. 338–341 (1848).

9. Hestenes D. *Space-time Algebra*, 2nd ed., Birkhäuser, Basel, 2015.

10. Hitzer E. *Quaternion Fourier Transform on Quaternion Fields and Generalizations*. Adv. Appl. Clifford Algebras 17, pp. 497–517 (2007). DOI: https://doi.org/10.1007/s00006-007-0037-8.

11. Hitzer E. *General Steerable Two-sided Clifford Fourier Transform, Convolution and Mustard Convolution*. Adv. Appl. Clifford Algebras 27, pp. 2215–2234 (2017). DOI: https://doi.org/10.1007/s00006-016-0687-5.

12. Hitzer E. *Special relativistic Fourier transformation and convolutions*, Mathematical Methods in the Applied Sciences, First published: 04 Mar. 2019, Vol. 42, Issue 7, pp. 2244–2255, 2019, DOI: 10.1002/mma.5502, URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.5502, Preprint: http://vixra.org/pdf/1601.0283v3.pdf.

13. Hitzer E., Sangwine SJ. *The Orthogonal 2D Planes Split of Quaternions and Steerable Quaternion Fourier Transformations*, in Hitzer E., Sangwine SJ. (eds.), *Quaternion and Clifford Fourier transforms and wavelets*, Trends in Mathematics 27, Birkhäuser, Basel, 2013, pp. 15–39. DOI: 10.1007/978-3-0348-0603-9_2, Preprint: http://arxiv.org/abs/1306.2157.

14. Hitzer E. *Creative Peace License*, http://gaupdate.wordpress.com/2011/12/14/the-creative-peace-license-14-dec-2011/, last accessed: 08 June 2021.

15. Hitzer E. *Quaternion and Clifford Fourier transforms*, Foreword by Sangwine SJ., Chapman and Hall/CRC, London, 1st edition, 2021.

16. Lasenby AN. *Some recent GA results in Mathematical Physics and the GA approach to the Fundamental Forces of Nature*, Presentation at AGACSE 2021, YouTube video https://www.youtube.com/watch?v=fFj4E7q4hbY, accessed 07 Oct. 2021.

17. Lasenby AN. *Some recent results for $SU(3)$ and Octonions within the Geometric Algebra approach to the fundamental forces of nature*, submitted to Mathematical Methods in the Applied Sciences, (2022).

18. Lewis CS. *Miracles: a preliminary study*, Collins, London, p. 110, 1947.

19. Lounesto P. *Clifford Algebras and Spinors*, 2nd ed., Lon. Math. Soc. Lect. Note Ser. 286, Cambridge University Press, Cambridge (UK), 2001.

20. Manogue CA., Dray T. *Octonions, E6, and particle physics*, in Quantum Groups, Quantum Foundations, and Quantum Information: a Festschrift for Tony Sudbery 29–30 September 2008, York, UK, J. Phys.: Conf. Ser. 254:012005 (2010). DOI: https://doi.org/10.1088/1742-6596/254/1/012005.

21. Phelps S. *Octonion Multiplication Chart and Fano Plane*, https://www.geogebra.org/m/u35hf2jv, last accessed: 08 Oct. 2021.

22. Sangwine SJ., Le Bihan N. *Quaternion and octonion toolbox for Matlab*, http://qtfm.sourceforge.net/, last accessed 29 Mar. 2016.

23. Sangwine SJ., Hitzer E. Clifford Multivector Toolbox. [Online]. Software library available at: http://clifford-multivector-toolbox.sourceforge.net/ (2015)

24. Sangwine SJ., Hitzer E. Clifford Multivector Toolbox (for MATLAB). Adv. Appl. Clifford Algebras 27, pp. 539–558 (2017). DOI: https://doi.org/10.1007/s00006-016-0666-x.

25. Sangwine SJ., Hitzer E. *Polar Decomposition of Complexified Quaternions and Octonions*. Adv. Appl. Clifford Algebras **30**, 23 (2020). DOI: https://doi.org/10.1007/s00006-020-1048-y.

26. Thompson SP. *The Life of Lord Kelvin*, Vol. I, American Mathematical Society, 2nd edition, Providence, RI, 2004.

27. Wang Y. *Octonion Algebra and Noise-Free Fully Homomorphic Encryption (FHE) Scheme*, Eprint URL: arXiv:1601.06744, 39 pages, Jan. 2016.

28. Wang Y., Malluhi QM. *Remarks on Quaternions/Octonion Based Diffie-Hellman Key Exchange Protocol Submitted to NIST PQC Project*, IACR Cryptol. ePrint Arch. 2017/1258, 8 pages (2017), URL: https://eprint.iacr.org/2017/1258.pdf.

29. Yagisawa M. *Fully Homomorphic Encryption on Octonion Ring*, Cryptology ePrint Archive: Report 2015/733, 42 pages (2015), URL: https://eprint.iacr.org/2015/733.pdf.

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