ON THE GENERAL GAUSS CIRCLE PROBLEM

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Abstract. Using the method of compression we show that the number of integral points in a k dimensional sphere of radius $r > 0$ is

$$N_k(r) \gg \sqrt{k} \times r^{k-1+o(1)}.$$

1. Introduction

The Gauss circle problem is a problem that seeks to counts the number of integral points in a circle centered at the origin and of radius r. It is fairly easy to see that the area of a circle of radius $r > 0$ gives a fairly good approximation for the number of such integral points in the circle, since on average each unit square in the circle contains at least an integral point. In particular, by denoting $N(r)$ to be the number of integral points in a circle of radius r, then the following elementary estimate is well-known

$$N(r) = \pi r^2 + |E(r)|$$

where $|E(r)|$ is the error term. The real and the main problem in this area is to obtain a reasonably good estimate for the error term. In fact, it is conjectured that

$$|E(r)| \ll r^{\frac{1}{2}+\epsilon}$$

for $\epsilon > 0$. The first fundamental progress was made by Gauss [3], where it is shown that

$$|E(r)| \leq 2\pi r\sqrt{2}.$$

G.H Hardy and Edmund Landau almost inedependently obtained a lower bound [1] by showing that

$$|E(r)| \neq o(r^{\frac{1}{2}}(\log r)^{\frac{1}{4}}).$$

The current best upper bound (see [2]) is given by

$$|E(r)| \ll r^{\frac{131}{208}}.$$

In this paper we study study a general version of the Gauss circle problem, where we replace the circle with a sphere in any euclidean space of dimension k. In particular, we obtain the following lower bound for the number of integral points in a sphere of radius r

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Theorem 1.1. *Let $N_k(r)$ denotes the number of integral points in a k dimensional sphere of radius $r > 0$. Then $N_k(r)$ satisfies the lower bound*

$$N_k(r) \gg \sqrt{k} \times r^{k-1+o(1)}.$$

1.1. **Notations and conventions.** Through out this paper, we will assume that r is sufficiently large for the radius of a sphere. We write $f(s) \gg g(s)$ if there there exists a constant $c > 0$ such that $f(s) \geq c|g(s)|$ for all s sufficiently large. If the constant depends of some variable, say t, then we denote the inequality by $f(s) \gg_t g(s)$. We write $f(s) = o(g(s))$ if the limits holds $\lim_{s \longrightarrow \infty} \frac{f(s)}{g(s)} = 0$.

2. Preliminaries and background

Definition 2.1. By the compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale $1 \geq m > 0$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. **The mass of compression.** In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.3. By the mass of a compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) - k} \leq \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i, j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is such that $x_i \leq x_j$ for $1 \leq i, j \leq n$.

Lemma 2.4. *The estimate remain valid*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772 \cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m > 0$.

Proposition 2.2. *Let* $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *with* $x_i \neq 0$ *for each* $1 \leq i \leq n$ *and* $x_i \neq x_j$ *for* $i \neq j$, *then the estimates holds*

$$m \log \left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log \left(1 + \frac{n-1}{\operatorname{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\operatorname{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.\end{aligned}$$

□

Definition 2.6. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression of scale $m > 0$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right|$$

Definition 2.7. Let $(x_1,x_2,\ldots,x_n)\in \mathbb{R}^n$ with $x_i\neq 0$ for all $1\leq i\leq n$. Then by the ball induced by $(x_1,x_2,\ldots,x_n)\in \mathbb{R}^n$ under compression of scale $m>0$, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ we mean the inequality

$$\left|\left|\vec{y}-\frac{1}{2}\bigg(x_1+\frac{m}{x_1},x_2+\frac{m}{x_2},\ldots,x_n+\frac{m}{x_n}\bigg)\right|\right|<\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)].$$

A point $\vec{z}=(z_1,z_2,\ldots,z_n)\in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n=2$.

Remark 2.8. In the geometry of balls under compression of scale $m>0$, we will assume implicitly that $1\geq m>0$. The circle induced by points under compression is the ball induced on points when we take $n=2$.

Proposition 2.3. *Let $(x_1,x_2,\ldots,x_n)\in \mathbb{R}^n$ for $n\geq 2$, then we have*

$$\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ \mathbb{V}_1\bigg[\bigg(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\bigg)\bigg]+m^2\mathcal{M}\circ \mathbb{V}_1[(x_1^2,\ldots,x_n^2)]-2mn.$$

In particular, we have the estimate

$$\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ \mathbb{V}_1\bigg[\bigg(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\bigg)\bigg]-2mn+O\bigg(m^2\mathcal{M}\circ \mathbb{V}_1[(x_1^2,\ldots,x_n^2)]\bigg)$$

for $\vec{x}\in \mathbb{N}^n$, where $m^2\mathcal{M}\circ \mathbb{V}_1[(x_1^2,\ldots,x_n^2)]$ is the error term in this case.

Lemma 2.9 (Compression estimate)**.** *Let $(x_1,x_2,\ldots,x_n)\in \mathbb{N}^n$ for $n\geq 2$ and $x_i\neq x_j$ for $i\neq j$, then we have*

$$\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\ll n\sup(x_j^2)+m^2\log\bigg(1+\frac{n-1}{\mathrm{Inf}(x_j)^2}\bigg)-2mn$$

and

$$\mathcal{G}\circ \mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\gg n\mathrm{Inf}(x_j^2)+m^2\log\bigg(1-\frac{n-1}{\sup(x_j^2)}\bigg)^{-1}-2mn.$$

Theorem 2.10. *Let $\vec{z}=(z_1,z_2,\ldots,z_n)\in \mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$. Then $\vec{z}\in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if*

$$\mathcal{G}\circ \mathbb{V}_m[\vec{z}]<\mathcal{G}\circ \mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z}\in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z}=(z_1,z_2,\ldots,z_n)\in \mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$, then it follows that $||\vec{y}||>||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G}\circ \mathbb{V}_m[\vec{z}]\geq \mathcal{G}\circ \mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}||\leq ||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G}\circ \mathbb{V}_m[\vec{z}]<\mathcal{G}\circ \mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.3 that $||\vec{z}||<||\vec{y}||$. It follows that

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\bigg(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\bigg)\right|\right|&<\left|\left|\vec{y}-\frac{1}{2}\bigg(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\bigg)\right|\right|\\&=\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. $\square$

2.2. **Admissible points of balls induced under compression.**

We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 2.11. Let $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with $y_i \neq y_j$ for all $1 \leq i < j \leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| = \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

Remark 2.12. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball.

Theorem 2.13. *The point $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \neq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 2.10, we obtain the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{y}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{z}] < \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.3 that $||\vec{x}|| < ||\vec{y}||$ or $||\vec{y}|| < ||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y}$ is an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. The latter equality follows from assertion that two balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lives on the outer of the indistinguishable balls and must satisfy the inequality

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &= \left|\left|\vec{z} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &= \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}] = \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right|$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. $\square$

Remark 2.14. We note that we can replace the set $\mathbb{N}^n$ used in our construction with $\mathbb{R}^n$ at the compromise of imposing the restrictions $\vec{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$ such that $x_i>1$ for all $1\leq i\leq n$ and $x_i\neq x_j$ for $i\neq j$. The following construction in our next result in the sequel employs this flexibility.

3. **The lower bound**

Theorem 3.1. *Let $N_k(r)$ denotes the number of integral points in the k dimensional sphere of radius $r>0$. Then $N_k(r)$ satisfies the lower bound*

$$N_k(r)\gg\sqrt{k}\times r^{k-1+o(1)}.$$

Proof. Pick arbitrarily a point $(x_1,x_2,\ldots,x_k)=\vec{x}\in\mathbb{R}^k$ with $x_i>1$ for $1\leq i\leq k$ and $x_i\neq x_j$ for $i\neq j$ such that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=2r$. This ensures the ball induced under compression is of radius r. Next we apply the compression of fixed scale $m\leq 1$, given by $\mathbb{V}_m[\vec{x}]$ and construct the ball induced by the compression given by

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])}{2}=r$. By appealing to Theorem 2.13 admissible points $\vec{x}_l\in\mathbb{R}^k$ ($\vec{x}_l\neq\vec{x}$) of the ball of compression induced must satisfy the condition $\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]=2r$. Also by appealing to Theorem 2.10 points $\vec{x}_l\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ must satisfy the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]<\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=2r.$$

For points $\vec{x}_l\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contained in the $2r\times 2r\times\cdots\times 2r$ (k *times*) grid that covers this ball we make the assignment

$$\min_{\vec{x}_l\in(2r)^k}\inf(x_{l_i})_{i=1}^k=r^{o(1)}$$

as $r \longrightarrow \infty$. The number of integral points in the largest ball contained in the $2r \times 2r \times \cdots \times 2r$ (k *times*) grid is

$$\begin{aligned}
N_k(r) &= \sum_{\substack{\vec{x}_l \in (\lfloor 2r \rfloor)^k \subset \mathbb{N}^k \\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_l] \leq 2r}} 1 \\
&\geq \sum_{\vec{x}_l \in (\lfloor 2r \rfloor)^k \subset \mathbb{N}^k} \frac{\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]}{2r} \\
&\gg \sum_{\substack{\vec{x}_l \in (\lfloor 2r \rfloor)^k \subset \mathbb{N}^k \\ 1 \leq i \leq k}} \frac{\sqrt{k}\inf(x_{l_i})}{2r} \\
&= \frac{1}{2r} \sum_{\substack{\vec{x}_l \in (\lfloor 2r \rfloor)^k \subset \mathbb{N}^k \\ 1 \leq i \leq k}} \sqrt{k}\inf(x_{l_i}) \\
&\geq \frac{\sqrt{k}}{2r} \sum_{\substack{\vec{x}_l \in (\lfloor 2r \rfloor)^k \subset \mathbb{N}^k \\ 1 \leq i \leq k}} \min_{\vec{x}_l \in (\lfloor 2r \rfloor)^k} \inf(x_{l_i}) \\
&\gg \frac{\min_{\vec{x}_l \in (2r)^k} \inf(x_{l_i})_{i=1}^k \times \sqrt{k}}{2r} \sum_{\substack{\vec{x}_l \in (\lfloor 2r \rfloor)^k \subset \mathbb{N}^k \\ 1 \leq i \leq k}} 1 \\
&\gg \frac{\min_{\vec{x}_l \in (2r)^k} \inf(x_{l_i})_{i=1}^k \times \sqrt{k}}{2r} \times r^k
\end{aligned}$$

and the lower bound follows by our choice

$$\min_{\vec{x}_l \in (2r)^k} \inf(x_{l_i})_{i=1}^k = r^{o(1)}$$

as $r \longrightarrow \infty$. □

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