Genalgebra: A System of Generalized Algebraic Numbers and Operators[1]

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An historical account of numbers and algebra is narrated. Properties of numbers and operators are reviewed. Expansion of extant number and operator systems is introduced. Philosophy of mathematics is discussed.

Prologue[3]

Hello everyone, thank you for your kind and generous readership //:-D This is a research paper, but I will keep it as entertaining as possible. We'll present several old and new ideas about numbers and operations in this paper. Please enjoy-

[1] This paper is dedicated to the author's family members and friends who also played parental figures, eternal inspirers, continuing educators, and spiritual mentors to him in being there for him when no one else was, who corrected him when he was wrong, and who taught him life lessons and everlasting wisdoms. He also especially thanks the wonderful people that he met online in social media like in twitter, Instagram, linkedIn, facebook, blogger, youtube, vimeo, and online databases like bexpress of Berkeley Law, vixra, social science research network, etc. He also thanks anonymous writers and scholars who contribute to stackexchange.com, Wikipedia.com, and other websites. Started being written on 1/22/2022. He also specially thanks his ex-girlfriends and ex-dates and female friends, who kindly granted him their precious feminine time and helped him to survive in this harsh and cold and lonely world otherwise… He also thank his brotherly male friends too for their brotherly love- He's a secular Christian, politically independent, and a private academic. And yes, he's 100% heterosexual, an openly straight man. He is currently running for the US Senate as an independent Alaskan //:-) Anyways, in the previous paper titled, "An Algebra of Infinity", we should have named it as: "Infinix: An Algebra of Infinity." See https://vixra.org/abs/2201.0135 . Did our distinguished readers notice the similarity of 'alpha' and 'infinity' symbols; and 'zero' and 'omega' symbols in that paper? //!-)

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[3] For the sketch summary of this paper, see https://huhnkielee.blogspot.com/2021/10/disproof-of-dirac-delta-function.html .

I. Introduction to Numbers and Algebra

1. A Story of Numbers

In the beginning, there was number one. There was only one person named Adam[4]. If there is only one thing in the universe, Adam, well. Perhaps one Mr. Adam is here, but not there. So perhaps there existed the concept of existence and nonexistence, 1 and 0. The discovery of zero is a big chapter in the history of mathematics.[5] It took Indian Yogi's to develop the philosophical number of zero.[6]

But, until Indian philosophers discovered zero, humans only had natural numbers. We use base 10 numbers.[7] Why not base 3 number system instead? Any guesses?

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Well, we use base 10 number system, because we have ten fingers, hence the name, 'digits.'

So yes, natural numbers is about counting from one to ten. Modern numeral characters are based on Arabic language in the Middle East, hence the name, Arabic numerals.[8] Even the term, "Al Gebra", has Middle Eastern Arabic origin. A joke in mathematics: algebra does not

[4] See https://en.wikipedia.org/wiki/Adam .
[5] See https://www.history.com/news/who-invented-the-zero .
[6] See https://www.orchidsinternationalschool.com/blog/child-learning/invention-of-zero/ .
[7] See https://en.wikipedia.org/wiki/Radix .
[8] See https://en.wikipedia.org/wiki/Arabic_numerals .

mean "the gibberish." Well, "gebra" in Arabic means "repair" or "assembly" or "combining" or "putting things together."[9] If you look at an algebraic equation, it's an assembly of numbers and characters put together:

$$x^2 + 2xy + y^2 = (x + y)^2$$

On the other hand, "geometry" has Greek origin, which is right next to the Middle East, actually. "Geo Metry" means measuring the earth.[10] One Mr. Euclid wrote about 'elements' and drew beautiful diagrams in his classical book.[11]

Geometric figures like triangles, rectangles, and circles are beautiful. But there are not too easy to draw in word processors. It's easier to type ASCII numbers and letters in a computer.[12] A mathematics teacher of the famed French entomologist, Jean Henri Fabre, once said, "algebra is a mathematics for lazy people."[13] In this paper, we kinda feel lazy, so we'll just do the numbers and characters and talk some gibberish too //!-)

2. Natural Numbers and Addition Operator

Let's get back to the Garden of Eden[14], shall we?

In the beginning, there once was this one person, one Mr. Adam. God made Adam after his own image:[15]

$$1 = 1$$

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[9] See https://www.darsaal.com/dictionary/english-to-arabic/algebra.html ; https://en.wikipedia.org/wiki/Algebra .

[10] See https://math.answers.com/Q/What_does_geometry_mean_in_Greek ; https://en.wikipedia.org/wiki/Geometry .

[11] See https://en.wikipedia.org/wiki/Euclid%27s_Elements .

[12] See https://en.wikipedia.org/wiki/ASCII .

[13] See https://libquotes.com/jean-henri-fabre/quote/lbj8i6b ; http://scihi.org/jean-henri-fabre/ ; https://www.poemhunter.com/jean-henri-fabre/biography/ ; https://en.wikipedia.org/wiki/Jean-Henri_Fabre . This author loved insects when he was a young boy and Monsieur Fabre was one of his big heroes and he read Fabre's biography in Korean language as he grew up in Seoul, and he still remembers a chapter that tells how young Fabre learned mathematics.

[14] See https://en.wikipedia.org/wiki/Garden_of_Eden .

[15] See https://biblehub.com/genesis/1-27.htm ; https://www.thegospelcoalition.org/essay/man-as-the-image-of-god/ ; https://www.openbible.info/topics/god_made_us_perfect .

Then God saw Adam being lonely on earth and created Eve[16] for him:

$$1 + 1 = 2$$

Next, Adam and Eve ate the forbidden fruit of apple[17] and they gave birth to a child:

$$2 + 1 = 3$$

To feed the hungry mouth of three people, Adam has to go to work and Eve had the pain and suffering of pregnancy and childbirth and childrearing: Paradise Lost.[18]

Let's say Adam became a farmer of apple tree orchard. Adam counts his blessings and he starts to count his apples, and the rest is history.

3. Subtraction: First Kind in the World of Identity and Inverse

Adam harvested 5 apples today. He ate one, his wife ate one, and their baby ate one. How many apples are left?

$$1 + 1 + 1 + 1 + 1 \quad = 5$$

$$(1 + 1 + 1) + (1 + 1) = 5 \qquad \text{[Associative Law]}$$

$$3 \quad + \quad 2 \quad = 5$$

[16] See https://en.wikipedia.org/wiki/Eve .
[17] See https://en.wikipedia.org/wiki/Forbidden_fruit .
[18] See https://www.imdb.com/title/tt10306320/ ; https://en.wikipedia.org/wiki/Paradise_Lost ; https://www.poetryfoundation.org/poems/45718/paradise-lost-book-1-1674-version . For a tangentially related subject matter, also see a movie made by this author, "A Therapy for Metrophobia", available at: https://www.youtube.com/watch?v=ltg6mfeG11s .

Adam is not Indian but he's a Middle Easterner, an ancestor of Arabs and Jews. So he didn't quite know the concept of zero at that time. Without zero, there is no such a thing a subtraction.

The concept of identity is related to the concept of inverse.[19] We may take it as granted, but let us step back and visit our elementary schools and sit in an arithmetic class.

$7 + x = 7$

$x = 0$ [Identity of Addition]

Next, we have:

$7 + y = 0$

$y = -7$ [Inverse of 7 in Addition]

So, the story goes like this. Once upon a time, for the very first time, first, there was positive numbers and addition operator. That's all Adam had. Then, thanks to the famed Silk Road[20] that connects the East and the West, Adam learned about the discovery of zero from Indian Yogi's[21]. Adam learned how to inverse the addition operator and named it, 'subtraction' operator. Adam also learned inverse numbers in the domain of addition-subtraction, and named them, 'negative' numbers. What we observe here is this:

An inverse operator creates a new number system.

As the story goes, the descendants of Adam the algebraist grouped natural numbers and their negative counterparts and zero, and named the set of those numbers as 'integers.'

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[19] See https://en.wikipedia.org/wiki/Group_theory ; https://en.wikipedia.org/wiki/Group_(mathematics) .
[20] See https://www.nationalgeographic.org/encyclopedia/silk-road/ ; https://en.wikipedia.org/wiki/Silk_Road .
[21] See https://en.wikipedia.org/wiki/Yogi .

4. Multiplication, Division, and Fraction and Rational Numbers

Let's speed things up a lil bit.[22]

$2 * x = 2$

$x = 1$ [Identity of Multiplication]

$2 * y = 1$

$y * 2 = 1$ [Commutative Law]

$y * 2 / 2 = 1 / 2$

$y = ½$ [Inverse of 2 in Multiplication]

As we can see, the inverse operator of division introduced a new set of numbers that we now call, fractions.

A unique inverse of a number exists only when an operator is commutative, a.k.a., 'abelian.'[23] Let's observe the similarity between the two worlds of addition and multiplication:

$3 + y = 0$

$+ y = 0 - 3$

$y = -3$

[22] See https://www.youtube.com/watch?v=GllEDACUbNo . It's great hiphop rap song in the style of the Middle East and African American culture. This author is a huge fan //:-)

[23] See https://mathworld.wolfram.com/AbelianGroup.html ; https://brilliant.org/wiki/abelian-group/ ; https://en.wikipedia.org/wiki/Abelian_group .

$3 * z = 1$

$* z = 1 / 3$

$z = \div 3$

As we can see, '-' and '/' can be now understood as 'signs' or 'unary operators,' which is a very cool concept in computer science and computer programming, as well as in mathematics[24] //:-)

II. Generalization of Addition Operation

1. Addition, Multiplication, Exponentiation[25]

Adam is an efficient farmer and a good bookkeeper. So, instead of adding up numbers, he invented how to multiply, in order to safe ink and paper:

$2 + 2 + 2 + 2 + 2 = 2 * 5$

Eve is an even smarter lady. Eve invented exponentiation:

$2 * 2 * 2 * 2 * 2 = 2\text{^}5$

Later on, descendants of Adam and Eve, as good mathematicians, took one step further and came up with the 'tetration' operator[26]:

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[24] See https://en.wikipedia.org/wiki/Unary_operation . 'Complement' operator in set theory is also a unary operator. See https://en.wikipedia.org/wiki/Complement_(set_theory) .

[25] See https://en.wikipedia.org/wiki/Ackermann_function ; https://en.wikipedia.org/wiki/Hyperoperation ; https://en.wikipedia.org/wiki/Knuth%27s_up-arrow_notation . This author independently discovered the general operation concept years ago, but later on learned that some other mathematicians discovered similar concept before this author did, lol.

[26] See https://en.wikipedia.org/wiki/Tetration .

$2\wedge(2\wedge(2\wedge(2\wedge2))) = 2\#5$

Mathematics is by and large about generalization. The next step would be rightfully named as 'pentation'[27]:

$2\#(2\#(2\#(2\#2))) = 2@5$

After a while, you'd run out of ASCII special characters. So, let's come up with a neat notational convention, shall we?

$+ = +^1$

$* = +^2$

$\wedge = +^3$

$\# = +^4$

$@ = +^5$

2. Generalization of Inversion

We know addition and multiplication are commutative, but exponentiation is not:

$2 + 3 = 3 + 2 = 5$

$2 * 3 = 3 * 2 = 6$

$2\wedge3 = 8 \neq 9 = 3\wedge2$

[27] See https://en.wikipedia.org/wiki/Pentation .

Exponentiation is not commutative, meaning the ordering of two numbers matters and makes difference. The non-commutativeness of exponentiation is the reason why it has two inverse operators: logarithm[28] and radicals[29].

$$2^\wedge x = 3$$

$$x = log_2 3$$

$$x^\wedge 2 = 3$$

$$x = \sqrt[2]{3}$$

We will call the logarithm, left-inverse operator of exponentiation, because we are passing the left number (2 in the above example) to the other side of the equation. We will call the root operator, right-inverter of exponentiator, because we are passing the right number (2 is on the right side of x in the example above), to the other side of the equation. Also, Logarithm and Left; Root and Right, they rhyme well respectively and it serves as good mnemonic device for us.

In our brand-new notational convention, we will notate the two invertors as follows:

$$log = +^{-3}$$

$$root = +^{3i}$$

As we can see, the left invertor has '-' on the left of the number '3'. The right inversion operator has 'i' on the right side of the number '3'.[30] It's a consistent notational convention.

So. Let's do some exercise:

$$log_2 3 = 2 +^{-3} 3$$

$$\sqrt[2]{3} = 2 +^{3i} 3$$

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[28] See https://en.wikipedia.org/wiki/Logarithm .

[29] See https://en.wikipedia.org/wiki/Nth_root .

[30] Of course, we got the idea of 'i' from the imaginary number, which is kinda like -1. See https://en.wikipedia.org/wiki/Imaginary_number . Later in this paper, we will explore new breeds of imaginary numbers. E.g., 2^x = -1, x = log_x-1, etc.

Or, to make it easier to type, we can simply adopt the existing bracket convention:[31]

$log_2 3 = 2\ [-3]\ 3$

$\sqrt[2]{3} = 2\ [3i]\ 3$

In sum, let's use the bracket notation, as it's easier for us to type. And we'll call the general operators, n-th order operators. We'll call this notational convention as 'genalgebra notation.' Let's make examples:

$2+3 = 2[1]3$ $\quad$ $2-3 = 2[-1]3 = 2[i]3$

$2*3 = 2[2]3$ $\quad$ $2/3 = 2[-2]3 = 2[2i]3$

$2\hat{}3 = 2[3]3$ $\quad$ $log_2 3 = 2[-3]3$ $\quad$ $\sqrt[2]{3} = 2\ [3i]\ 3$

III. Algebra of Sets and Numbers

1. A Preliminary Remark

Traditionally, mathematicians have grouped radicals and logarithmic numbers into a set of 'irrational numbers', which is a subset of the set of real numbers.[32] We argue that such categorization is an imperfect one. In this chapter, we present the correct and complete categorization of algebraic numbers. And we'll show the world how to discover new number systems too //:-)

2. A Preparatory Concept: Algebraic Operation between Sets

We can define an addition between two sets as follows:[33]

[31] See https://en.wikipedia.org/wiki/Hyperoperation
[32] See https://byjus.com/maths/real-numbers/ ; https://en.wikipedia.org/wiki/Real_number .
[33] See https://en.wikipedia.org/wiki/Additive_number_theory .

$A+B = \{x|\ x = a + b, \text{ where } a \in A \text{ and } b \in B\}$

A number is a special of a set, where the set has only one member, which is called a singleton set:

$a = \{a\}$

Then, naturally, we can define an algebraic operation between a set and a number:

$A + x = \{\ a + x \mid a \in A\ \}$

Now let’s make examples for elucidation and illustration.

$\{1, 2\} + \{0, 10, 100,\}$

$= (1 + \{0, 10, 100,\}) \cup (2 + \{0, 10, 100,\})$

$= \{1+0, 1+10, 1+100\} \cup \{2+0, 2+10, 2+100\}$

$= \{1, \quad 11, \quad 101, \quad 2, \quad 12, \quad 102\}$

Notice the distributive property between the ‘union’ set operator and an algebraic operator:

$(A \cup B) + (C \cup D)$

$=(A+C) \cup (A+D) \cup (B+C) \cup (B+D)$

Also note that we used 'addition' as an algebraic operator here, but it can be any algebraic operator, without losing generality.[34] Also note that the union-addition distributive law above resembles Cartesian Product.[35]

More familiarly, the distributive pattern above looks like that of addition and multiplication:

$$(a + b) * (c + d)$$
$$= ac + ad + bc + bd$$

In arithmetic, multiplication has a higher priority than addition, when it comes to the ordering of computation. As we have seen in the previous chapter, addition is more basic than multiplication, and multiplication is more 'advanced' or more 'derivative' than addition. In the distributive pattern of 'union' set operator and 'addition' algebraic operator in the above example, we observed that 'addition' takes precedence over 'union' operation. That means, 'union' is even more basic, more foundational, more fundamental, and more primitive an operator than the 'addition' operator. 'Union' operator is like a primordial soup of mathematical entities, like a group of cavemen craving to get together in a union, or animals gathering to form a group, etc.

3. Infinite Number Sets

Let's define some sets:[36]

$+N = 1 * N = N = \{1, 2, 3, \ldots\}$	[Positive Integers]
$-N = -1 * N = \{\ldots, -3, -2, -1\}$	[Negative Integers]
$0 = \{0\}$	[Zero Singleton]
$I = -N \cup 0 \cup N = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$	[Integers]

[34] See https://en.wikipedia.org/wiki/Without_loss_of_generality .
[35] See https://en.wikipedia.org/wiki/Cartesian_product .
[36] For algebraic operations between a set and a number, see this author's previous math paper, "An Algebra of Infinity", available at: https://vixra.org/abs/2201.0135 . For historical development of set-number algebra, see https://en.wikipedia.org/wiki/Minkowski_addition ; https://en.wikipedia.org/wiki/Sumset ; https://en.wikipedia.org/wiki/Additive_number_theory ; https://en.wikipedia.org/wiki/Freeman_Dyson .

$I^{0+} = 0 \cup N$ [Nonnegative Integers]

$I^{-0} = -N \cup 0$ [Nonpositive Integers]

Now, let's think about rational numbers. Rational number set, Q, can be defined as follows:

$$Q = \{ x \mid \quad x = a/b, \quad such\ that \quad a \in I \quad and \quad b \in I \}$$

Next, let's define a set function whose inputs are two numbers, and whose output is a set:

$$Q(a, b) = \{ x \mid \quad x \in Q \quad and \quad a < x \le b \}$$

Then, the set of all the rational numbers between 0 and 1, which excludes 0 and includes 1 can be defined as follows:

$$Q(0, 1) = \{ 1/1, 1/2, 1/3, 1/4, 1/5, \ldots \} = 1 / \{1, 2, 3, 4, 5, \ldots\} = 1 / N$$

Next, the set of all the rational numbers between 1 and 2, which excludes 1 and includes 2 can be defined as follows:

$$Q(1, 2) = \{ 1+1/1, 1+1/2, 1+1/3, \ldots \} = 1+ 1 / \{1, 2, 3, 4, 5, \ldots\} = 1+ 1 / N$$

Likewise, we can see the pattern:

$$Q(0, 1) = 0 + 1/N$$

$$Q(1, 2) = 1 + 1/N$$

$$Q(2, 3) = 2 + 1/N$$

$$Q(3, 4) = 3 + 1/N$$

Next, let us express the set of all positive rational numbers:

$$Q^{+} = \{ x \mid \quad x \in Q \quad and \quad 0 < x \}$$
$$= Q(0, 1) \cup Q(1, 2) \cup Q(2, 3) \cup Q(3, 4) \cup \ldots$$
$$= (0 + 1/N) \cup (1 + 1/N) \cup (2 + 1/N) \cup (3 + 1/N) \cup \ldots$$
$$= \{0, 1, 2, 3, \ldots\} + 1/N$$
$$= I^{0+} + 1/N$$

Then, we got the entire rational number set, Q, as follows:

$$Q = Q^{-} \cup 0 \cup Q^{+}$$
$$= -Q^{+} \cup 0 \cup Q^{+}$$

Let's look at the negative rational number set:

$$-Q^{+} = -(I^{0+} + 1/N)$$
$$= (-1 * I^{0+} + -1 * 1/N)$$
$$= I^{-0} - 1/N$$

Putting it altogether, we now we can enjoy the symmetric symphony of rational numbers:

$$Q = -Q^{+} \cup 0 \cup Q^{+}$$
$$= (I^{-0} - 1/N) \cup 0 \cup (I^{0+} + 1/N)$$

Why are we doing all this? For fun //xD It's an intellectual entertainment. Math is a great hobby, a good brain exercise, and a fantastic educational tool. This may not be a groundbreaking moment in the history of mathematics, but we're having fun and we're educating the people as well. Entertainment and education are the two keys of the world peace. If everyone is happy and wise, violence shall cease to exist. //:-)

IV. Generalized Algebraic Number System

1. Generalized System of Inverse Numbers

We have reviewed that an inverse operator introduces a new number system. Subtraction introduced negative integers, and division introduced fractions. The left invertor of exponentiation operator created the logarithmic numbers and the right invertor of power operator opened the door to the radicals.[37]

Why stop here? Let's go ahead and observe the history in the making. This time, it's not just a matter of coming up with a prettier, more efficient notational conventions. We will teach the world how new number systems can be discovered, *ad infinitum*[38].

2. Preliminary Examples: Two Invertors of Exponentiator

Let us go back to the days of junior high school mathematics classes:

$2^3 = 8$

$2^x = 10$

$2^4 = 16$

$\therefore 3 < x < 4$

$x = log_2 10$

$x = 2[-3]10$ [Genalgebra Notation][39]

For root operation,

$2^3 = 8$

$x^3 = 20$

[37] Pun intended, for the sake of a joke //xD

[38] See https://en.wikipedia.org/wiki/Ad_infinitum .

[39] Let's call our generalized algebra as 'genalgebra'.

$3^3 = 27$

$\therefore 2 < x < 3$

$x = \sqrt[3]{20}$

$x = 3[3i]20$ [Genalgebra Notation]

3. Two Invertors of Tetrator, and Beyond

Let's call the tetration operator, tetrator.[40] Let us familiarize with tetration by making examples, in the same fashion as before:

$2[4]3 = 2\text{^}(2\text{^}2) = 2\text{^}4 = 16$

$x[4]3 = 100$

$3[4]3 = 3\text{^}(3\text{^}3) = 3\text{^}27 = 7{,}625{,}597{,}484{,}987$ [more than 7 trillion]

$\therefore 2 < x < 3$

$x = \sqrt[3]{100}_4$ [Super Root Notation][41]

$x = 3[4i]100$ [Genalgebraic Notation]

As we can see, tetration is a very fast function, faster than exponential function. That's why 4th order super root number system is a very dense system of irrational numbers. Continuing from the above example,

$2[4]3 = 16$

$x[4]3 = y$

$3[4]3 = 7{,}625{,}597{,}484{,}987$

$16 < y < 7{,}625{,}597{,}484{,}987$

$x = \sqrt[3]{y}_4 = 3[4i]y$

$2 < x < 3$

[40] See https://en.wikipedia.org/wiki/Tetration .
[41] See https://en.wikipedia.org/wiki/Tetration .

Between 2 and 3, there are so many 4^{th} order super root numbers that can be expressed in the 4^{th} order super root numbers using integers only. Let's give a name to such a set as 'integer-based super root set' or ISR:

$$ISR(2, 4, 3) = \{ x \mid x[4]3 = y,\ 2[4]3 < y < (2+1)[4]3 \}$$

$$| ISR(2, 4, 3) | = (2+1)[4]3 - \quad 2[4]3 - 1$$

$$= 7{,}625{,}597{,}484{,}987 \ - 16 \ \ - 1 = 7{,}625{,}597{,}484{,}970$$

That is, between 2 and 3, there are over 7 trillion irrational numbers that can be expressed as 4^{th} order super root number using two integers. We'll use genalgebra notation, as it's easier to get a better feel for those numbers:

$$ISR(2, 4, 3) = \{3[4i]17,\ 3[4i]18,\ 3[4i]19,\ \ldots,\ 3[4i]7{,}625{,}597{,}484{,}986\}$$

$$=3[4i]\{17,\ 18,\ \ldots,\ 7{,}625{,}597{,}484{,}986\}$$

$$=3[4i]\{\ 2[4]3 + 1,\ 2[4]3 + 2,\ \ldots,\ 3[4]3 - 1\}$$

$$=3[4i]\ I(2[4]3 + 1,\ 3[4]3 - 1)$$

Well, on the last line above, we used an integer range set notation I:

$$I(a, b) = \{x \mid a \le x \le b,\ a \in I,\ b \in I\}$$

We can generalize the interger-based super root set in n-th order system:

$$ISR(a, n, b) = b[ni]\ I(a[n]b + 1,\ (a+1)[n]b - 1)$$

$$| ISR(a, n, b) | = (a+1)[n]b - a[n]b - 1$$

Now, let's look at the other 4th order invertor, super logarithm:[42]

$2[4]3 = 16$

$2[4]x = 1000$

$2[4]4 = 2\wedge(2[4]3) = 2\wedge 16 = 65{,}536$

$\therefore 3 < x < 4$

$x = slog_2 10$ [Slog Notation][43]

$x = 2[-4]10$ [Genalgebra Notation]

As we can see, the 4th order left invertor creates a new number system similar to logarithmic numbers. And the 4th order right invertor also creates a new number system as well, similar to radical numbers. The same holds true for pentation, hexation, and 7th order, 8th order, and so on. From third order and beyond, each level of general operator has two inverse operators. And since one invertor introduces one brand-new number system, operator level of third order or higher creates two new number systems that come with two invertors.

4. Pentator, and Beyond

Let us see what the 5th order genalbegraic operator looks like:

$$
\begin{aligned}
2[5]3 &= 2[4](2[4]2) = 2[4](2[3]2) = 2[4]4 \\
&= 2[3](2[3](2[3]2)) \\
&= 2[3](2[3]4) \\
&= 2[3]16 \\
&= 65{,}536
\end{aligned}
$$

$$2[5]4 = 2[4](2[4](2[4]2)) = 2[4](2[5]3)$$

In general,

[42] See https://en.wikipedia.org/wiki/Super-logarithm .

[43] See https://en.wikipedia.org/wiki/Super-logarithm .

$a[n]b = a[n-1](a[n](b-1))$

For an easier example of the 'genalgebraic recursion formula' above,[44] let's see multiplication and addition:

$2[2]5 = 2[1](2[2]4)$

$2*5 =2+(2*4) = 2 +(2 + (2 + (2 + 2)))$

In general,

a[n]b means there are b number of a's connected by 'b-1' operators of (n-1)-th order in right-associative fashion.[45]

Let's come up with a brand-new notational convention to efficiently and compactly express the 'genalgebraic recursive formula'. We'll use the regular expression[46] convention:

$2[2]5$

$=2+(2+(2+(2+(2))))$

$=\{2+(\}^4\ 2\ \{)\}^4$

$=\{2[1](\}^4\ 2\ \{)\}^4$

Above, $\{2+(\}^4$ means the ordered set of characters, i.e., a word of "2+(", is being repeated four times. And we inserted the unnecessary innermost parenthesis around a number, merely to make the pattern easier to generalize.

Then, in general,

[44] See https://en.wikipedia.org/wiki/Hyperoperation .
[45] See https://en.wikipedia.org/wiki/Hyperoperation .
[46] See https://en.wikipedia.org/wiki/Regular_expression .

$$a[n]b = \{a[n-1](\}^{b-1}\ a\ \{)\}^{b-1}$$

Traditionally, a mathematical formula is not expressed in regular expression. Well, we're breaking a new ground here again, in order to take the mathematics world to the next step in evolution. We encourage future mathematicians to use regular expression when they right a generic mathematical formula.

5. Two New Number Systems in Each n-th Order Echelon

So we have observed that, from 3rd order and beyond, each n-th order operator has two inverse operators, which introduce two brand-new number systems. The two new number systems have the following forms:

$$x = a[-n]b$$

$$y = a[ni]b$$

This fact has not been discovered or discussed ever before, outside of this author's "humanology" series in YouTube, where a lot of brand-new ideas in mathematics and other sciences have been discovered and discussed for the past several years //:-)

V. Generalized Imaginary Number System

1. Preliminary Narrative: Imaginary Number *i*

The discovery of the imaginary number, *i,* involves the following 2nd order polynomial equation:

$$x^2 = -1$$

$$x = \pm\sqrt[2]{-1}$$

$$i \equiv +\sqrt[2]{-1}$$

When a mathematician defined the concept for the first time, other people may have thought it as a fanciful and wasteful venture.[47] Centuries late, however, the imaginary number has proved to be a seed of a very fruitful tree, not just in mathematics but in a field as practical as electrical engineering.[48] It has happened a lot in the history of mathematics that a purely mathematical notion out of curiosity later on finds a very practical application.

2. Next Step: Defining Logarithm of Negative Numbers[49]

Now, let's define new imaginary numbers in the 3rd order echelon in our genalgebraic hierarchy:

$$e^x = -1$$

$$x = log_e(-1) = ln(-1)$$

$$j \equiv ln\text{-}1$$

Once we define our brand-new imaginary number, *j,* we can express other related imaginary numbers in terms of *j:*

$$2^x = -1$$

$$x = log_2(-1)$$

$$=ln(-1) / ln2$$

$$=j / ln2$$

In general,

$$ax = -1$$

[47] See https://en.wikipedia.org/wiki/Imaginary_number .
[48] See https://en.wikipedia.org/wiki/Complex_number .
[49] See https://www.rapidtables.com/math/algebra/ln/Ln_of_Negative_Number.html .

$x = j / ln(a)$

3. <u>Imaginary Numbers in n-th Order Level in Genalgebraic Hierarchy</u>

In general, we can define two imaginary numbers in n-th order echelon in our genalgebraic hierarchy. Let's review how they're defined in the third order echelon:

$x[3]2 = -1$

$x = 2[3i]-1 = \sqrt[2]{-1}$

$e[3]y = -1$

$y = e[-3]-1 = ln(-1)$

In general, we can generate the two imaginary numbers, x and y, in n-th order echelon:

$x[n]a = -1$

$x = a[ni]-1$

$b[n]y = -1$

$y = b[-n]-1$

Genalgebra is a fairly new field in mathematics and the outlook seems promising //:-)

Epilogue[50]

Hello everyone, thank you for your kind and generous readership //:-D We hope you enjoyed the show. Our next article to write and publish will be titled, "Pyradmid Number Theory". There, we'll introduce some interesting concepts about dimensionality in combinatorics.[51]

Thank you for your time and see you later, kind and generous ladies and gentlemen //:-)

[50] This paper was started being written on 1/22/2022. It was finished being written on 1/23/2022 //:-)
[51] See https://en.wikipedia.org/wiki/The_Road_Not_Taken .