About the “Addition” of Scalars and Bivectors in Geometric Algebra…

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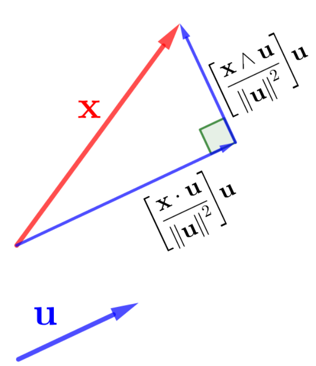
“You can’t add things that are of different types!” This objection to the “addition” of scalars and bivectors—which is voiced by physicists as well as students—has been a barrier to the adoption of Geometric Algebra. We suggest that the source of the objection is not the operation itself, but the expectations raised in critics’ minds by the term “addition”. Indeed, the ways in which this operation interacts with others are unlike those of other “additions”, and might well cause discomfort to the student. This document explores those potential sources of discomfort, and notes that no problems arise from this unusual “addition” because the developers of GA were careful in choosing the objects (e.g. vectors and bivectors) employed in this algebra, and also in defining not only the operations themselves, but their interactions with each other. The document finishes with an example of how this “addition” proves useful.



Left-multiplying the vector $\mathbf{u}^{-1}$ by the “sum” $\mathbf{x}\cdot\mathbf{u}+\mathbf{x}\wedge\mathbf{u}$ transforms $\mathbf{u}^{-1}$ into a vector sum whose resultant is $\mathbf{x}$.

1 Why Do Some People Object to the "Addition" of Scalars and Bivectors in Geometric Algebra? (GA)

For centuries, people from all walks of life have invented symbolic forms for capturing different types of information about the situations that confronted them. Many times,ways were then found to combine those different types of information via mathematical operations. Some of those operations were given the name "addition". Thus we have the addition of numbers; a quite-different graphical operation that its inventors chose to call the "addition" of vectors; and yet another operation that its inventors chose to call the "addition" of matrices. Although these operations differ dramatically in their inputs and procedures, they share two features: the inputs to a given "addition" are all of the same type. The output is of the same type as the inputs. For example, the sum of two real numbers is a real number; the "sum" of two vectors is a vector; and the "sum" of two matrices is a matrix.

Such is not the case for the operation that its inventors called the "addition" of a scalar and a bivector. For our present purposes, we may be well advised to put that name aside for the moment, and to concentrate instead upon the operation's purpose, which is to define a single, convenient mathematical expression that combines two pieces of information about the relative orientations of a pair of vectors. The resulting expression is an example of what GA calls "multivectors", which are a key part of GA because they can be used with great success, thanks to the clarity with which GA defines both the multivectors themselves and the operations that act upon them. (An example of the usefulness of multivectors is given in Section 2.) Nevertheless, the operation's characteristics are unusual for an "addition", and may therefore cause discomfort to an intelligent student.

This brief document explores some of those unusual characteristics, recognizing that the student is the ultimate judge of whether an operation with these characteristics is sufficiently useful as to be worth learning. To make our case that this operation is indeed worth learning, we'll begin with the "sum"

$$\alpha + \mathbf{B},$$

in which α is a scalar and $\mathbf{B}$ is a bivector. Now, let's right-multiply the resulting multivector $(\alpha + \mathbf{B})$ by a vector $\mathbf{v}$ that is parallel to $\mathbf{B}$. (Our conclusions would be the same if we multiplied on the left.)

$$(\alpha + \mathbf{B})\,\mathbf{v} = \alpha\mathbf{v} \; + \; \mathbf{B}\mathbf{v}\;.$$

Because $\mathbf{v}$ is parallel to $\mathbf{B}$, the product $\mathbf{B}\mathbf{v}$ is a vector. So is $\alpha\mathbf{v}$, of course. Therefore, the multiplication by $\mathbf{v}$ did not merely distribute over the "sum" of α and $\mathbf{B}$. Instead, that sum has "morphed" into the familiar "addition of vectors". GA handles such changes consistently, so there's no problem. However, we might want to call students' attention to what just happened. Therefore, let's write the "+" on the right-hand side in red to denote the addition of vectors:

$$(\alpha + \mathbf{B})\,\mathbf{v} = \alpha\mathbf{v} + \mathbf{B}\mathbf{v}\ . \tag{1}$$

Now, to de-fang other potential qualms about the "addition" of scalars and bivectors, let's consider the general case in which $\mathbf{v}$ is not parallel to $\mathbf{B}$. We'll express $\mathbf{B}$ and $\mathbf{v}$ according to the basis used by Macdonald ([1], p. 82). The blue "+" signs indicate another type of "sum": the "addition" of bivectors. (Note that bivectors themselves can be "added" geometrically via a well defined operation ([1], p. 74).

$$\begin{aligned}\mathbf{v} &= \nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2 + \nu_3\mathbf{e}_3\ ,\\ \mathbf{B} &= \beta_1\mathbf{e}_1\mathbf{e}_2 + \beta_2\mathbf{e}_1\mathbf{e}_3 + \beta_3\mathbf{e}_2\mathbf{e}_3\ .\end{aligned}$$

If we substitute these expressions in the right-hand side of Eq. (1), and then expand that side, we'll find ourselves employing yet another type of sum: the familiar addition (green symbols, below) of real numbers.

$$\begin{aligned}(\alpha + \mathbf{B})\,\mathbf{v} = {} & (\alpha\nu_1 + \beta_1\nu_2 + \beta_2\nu_3)\,\mathbf{e}_1\\ & + (\alpha\nu_2 + \beta_3\nu_3 + {}^{-}\beta_1\nu_1)\,\mathbf{e}_2\\ & + (\alpha\nu_3 + {}^{-}\beta_2\nu_1 + {}^{-}\beta_3\nu_2)\,\mathbf{e}_3.\end{aligned}$$

In that equation, the single symbol "+" is used to represent three distinct operations. For a vector $\mathbf{v}$ that is not parallel to $\mathbf{B}$, we have yet another "+" on the right-hand side: the "sum" of a vector (the term in square brackets on the right-hand side) and a trivector:

$$\begin{aligned}(\alpha + \mathbf{B})\,\mathbf{v} = {} & [(\alpha\nu_1 + \beta_1\nu_2 + \beta_2\nu_3)\,\mathbf{e}_1 + (\alpha\nu_2 + \beta_3\nu_3 + {}^{-}\beta_1\nu_1)\,\mathbf{e}_2\\ & + (\alpha\nu_3 + {}^{-}\beta_2\nu_1 + {}^{-}\beta_3\nu_2)\,\mathbf{e}_3]\\ & + \underbrace{(\beta_1\nu_3 + {}^{-}\beta_2\nu_2 + \beta_3\nu_1)\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}_{trivector}.\end{aligned}$$

Like the "sum" with which we started—that of a scalar and a bivector—the "addition" of a vector and a trivector produces a multivector.

Again, and in conclusion, the use of a single symbol "+" to represent four distinct operations causes no problems. Nevertheless, we might help students by pointing out what is going on, and by explaining *why* there is no problem: because the developers of GA were careful in choosing the objects (e.g. vectors and bivectors) employed in this algebra, and also in defining not only the operations themselves, but their interactions with each other.

In the next section, we use the "morphing" of the addition of scalars and bivectors to solve a problem via GA.

2 An Example of How GA Makes Use of the Two Types of Information that are Combined in the "Sum" of a Scalar and a Bivector

We saw above that the multiplication of $\alpha + \mathbf{B}$ by the vector $\mathbf{v}$ produced a sum of two vectors. Now, let's consider another sum of a scalar and a bivector: $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors. We've seen that sum many times, of course: it's often used as the definition of the geometric product $\mathbf{ab}$:

$$\mathbf{ab} \equiv \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}.$$

Here, we'll use that sum in a different way.

We begin by recalling that in GA, every non-zero vector $\mathbf{b}$ has a multiplicative inverse: $\mathbf{b}^{-1} \equiv \mathbf{b}/\|\mathbf{b}\|^2$. We make use of that aspect of GA when we solve for an unknown vector $\mathbf{x}$. First, we find that vector's inner and outer products with some known vector $\mathbf{u}$, then we add those products to form the geometric product $\mathbf{xu}$:

$$\mathbf{xu} = \mathbf{x} \cdot \mathbf{u} + \mathbf{x} \wedge \mathbf{u}.$$

In practice, the quantity that we have written so briefly as "$\mathbf{x} \cdot \mathbf{u} + \mathbf{x} \wedge \mathbf{u}$" can be extensive and complicated. Nevertheless, having obtained it (no matter how complicated it might be), we multiply both sides by $\mathbf{u}^{-1}$:

$$\begin{aligned} \mathbf{xuu}^{-1} &= [\mathbf{x} \cdot \mathbf{u}]\, \mathbf{u}^{-1} + [\mathbf{x} \wedge \mathbf{u}]\, \mathbf{u}^{-1} \\ \mathbf{x} &= \left[\frac{\mathbf{x} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\right] \mathbf{u} + \left[\frac{\mathbf{x} \wedge \mathbf{u}}{\|\mathbf{u}\|^2}\right] \mathbf{u}. \end{aligned}$$

How do we interpret the right-hand side? It's a sum of two vectors (Fig. 1). The first term on that side is a scalar multiple of $\mathbf{u}$ itself. As for the second term, the vector $\mathbf{u}$ is a factor of the bivector $\mathbf{x} \wedge \mathbf{u}$, and therefore is parallel to that bivector. Hence, $\left[\frac{\mathbf{x} \wedge \mathbf{u}}{\|\mathbf{u}\|^2}\right] \mathbf{u}$ is a scalar multiple of a 90° rotation of $\mathbf{u}$.

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).

$$\mathbf{x}\mathbf{u} = \mathbf{x}\cdot\mathbf{u} + \mathbf{x}\wedge\mathbf{u}$$

$$\mathbf{x}\mathbf{u}\mathbf{u}^{-1} = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}^{-1} + (\mathbf{x}\wedge\mathbf{u})\mathbf{u}^{-1}$$

$$\mathbf{x} = \left[\frac{\mathbf{x}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\right]\mathbf{u} + \left[\frac{\mathbf{x}\wedge\mathbf{u}}{\|\mathbf{u}\|^2}\right]\mathbf{u}$$

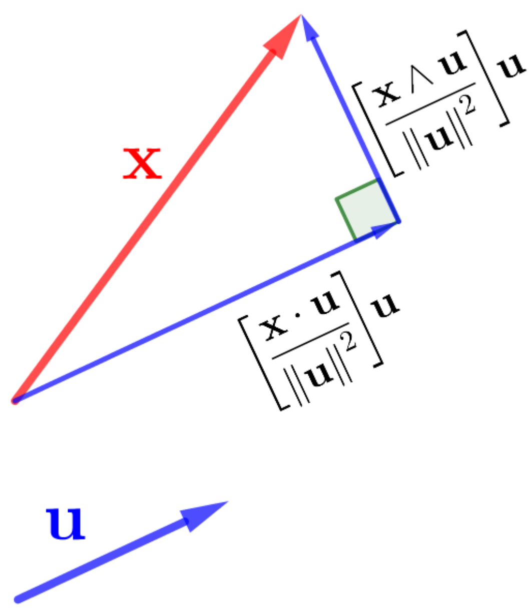


Figure 1: Left-multiplying the vector $\mathbf{u}^{-1}$ by the “sum” $\mathbf{x}\cdot\mathbf{u}+\mathbf{x}\wedge\mathbf{u}$ transforms $\mathbf{u}^{-1}$ into a vector sum whose resultant is $\mathbf{x}$.