The Chessboard Puzzle

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Abstract

We introduce compact subsets in the plane and in $\mathbb{R}^3$, which we call *Polyorthogon* and *Polycuboid*, respectively. We consider a usual chessboard. We display it by equal bricks or mirrored bricks.

Keywords and phrases: rectangle, cuboid, chessboard *MSC 2020 : 51P99*

1 Introduction

We ask whether a usual chessboard can be displayed by equal bricks formed by the squares of which it is made. Of course, the number of the bricks have to be a devisor of 64. This question can be generalized.

Definition 1. Let a *polyorthogon* be a subset of $\mathbb{R}^2$ such that it is homeomorphic to a circle area $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and it is the union of a finite number of rectangles.
Let a *polycuboid* be a subset of $\mathbb{R}^3$ such that it is homeomorphic to a ball $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ and it is the union of a finite number of cuboids.
We define that the rectangles in a polyorthogon and the cuboids in a polycuboid intersect at most on their boundaries.

A *polyomino* and a *polycube* are well-known. Please see [1] and also Figure 1 and Figure 2.

Remark 1. Note that a polyomino may not be a polyorthogon since the polyomino is not homeomorphic to a circle area, and a polycube may not be a polycuboid.

Remark 2. In a polyorthogon there are exactly two sets of parallel rectangle sides. The second set is perpendicular to the first set.
In a polycuboid there are exactly three sets of parallel cuboid sides. The second set is perpendicular to the first set. The third set is perpendicular both to the second and the first set.

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2 The Chessboard

Definition 2. We write $E \cong F$ if and only if E and F are either polyorthogons or both are polycuboids, and they have the same shape and size.

Please see the picture below. The polyorthogon G is one big square, while H consists of 64 small squares, which form a polyorthogon of the same shape and size. We call H a *chessboard*. It holds $G \cong H$.

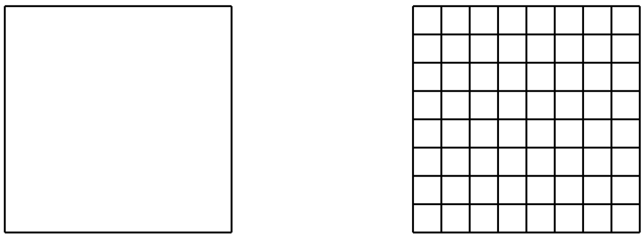

Figure 1:
On the left-hand side, we show two polyorthogons G and H.
Both are also polyominoes.

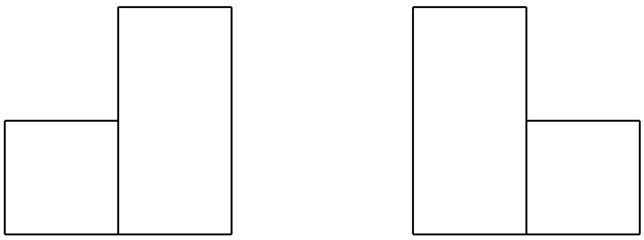

Figure 2:
On the left-hand side, we see a polyorthogon and its mirror image.

Definition 3. We say that 'B *represents* A' or 'B is a *representation* of A' if and only if A is a polyorthogon or a polycuboid or a polyomino or a polycube, $J > 0$, and B is the union

$$B = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_{J-1} \cup B_J \tag{1}$$

and B equals A in shape and size. The sets B_i are homeomorphic to a circle area $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ or a ball $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$. B_i is called a *brick*. Two bricks intersect only on their boundaries or they are disjunct.

We pose an infinite set of questions. The answers can be 'yes' or 'no', where a negative answer generally would be difficult to prove.

We determine that the expression 'X is equal Y' means that we can move and revolve Y such that Y is identical to X.

We use natural numbers L, K.

Question 1. We presume that P either is any polyorthogon or a polyomino. We ask whether there is a set $D_{set} \subset \mathbb{R}^2$ such that D_{set} represents P, and

$$D_{set} \;=\; W_1 \,\cup\, W_2 \,\cup\, W_3 \,\cup \ldots \cup\, W_{L-1} \,\cup\, W_L \tag{2}$$

where W_i is a finite nonempty set of bricks for $i \;=\; 1,2,3,\ldots L-1, L$.
We introduce an abbreviation.
We define that the sum of the cardinalities of the W_is is K, i.e.

$$K \;:=\; \sum_{i=1}^{L} \mathrm{cardinality}(W_i) \tag{3}$$

i.e. P is represented by K bricks. We demand if $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that X is not equal Y, while if $i = j$ X is equal Y.
In the case that D_{set} exists, we say that 'the polyorthogon P (or the polyomino P) is L *placeable* by K bricks', or 'P is L *placeable* by K bricks' in brief. If $L = 1$ and D_{set} exists we say 'P is *placeable* by K bricks'.

Question 2. We presume that P either is a polyorthogon or a polyomino. We ask whether there is a set $E_{set} \subset \mathbb{R}^2$ such that E_{set} represents P, and

$$E_{set} \;=\; W_1 \,\cup\, W_2 \,\cup\, W_3 \,\cup \ldots \cup\, W_{L-1} \,\cup\, W_L \tag{4}$$

where W_i is a finite nonempty set of bricks for $i \;=\; 1,2,3,\ldots L-1, L$. We demand if $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that neither X is equal Y nor X is a mirror image of Y. If $i = j$ either X is equal Y or X is a mirror image of Y. Also rule (3) holds.
In the case that E_{set} exists, we say that 'the polyorthogon P (or the polyomino P) is L *mirror-placeable* by K bricks', or 'P is L *mirror-placeable* by K bricks' in brief. If $L = 1$ and E_{set} exists we say 'P is *mirror-placeable* by K bricks.

Question 3. We presume that Q is a polycuboid or a polycube. We ask whether there is a set $F_{set} \subset \mathbb{R}^3$ such that F_{set} represents Q, and

$$F_{set} \;=\; W_1 \,\cup\, W_2 \,\cup\, W_3 \,\cup \ldots \cup\, W_{L-1} \,\cup\, W_L \tag{5}$$

where W_i is a finite nonempty set of three dimensional bricks for $i \;=\; 1,2,3,\ldots L-1, L$. We demand if $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that X is not equal Y, while if $i = j$ it holds that X is equal Y. Further, we demand also that rule (3) holds.
In the case that F_{set} exists, we say that 'the polycuboid Q (or the polycube Q) is L *sectional* by K bricks', or 'Q is L *sectional* by K bricks' in brief. If $L = 1$ and F_{set} exists we say 'Q is *sectional* by K bricks'.

A similar question can be asked for a polycuboid R or a polycube R, if we define the expression 'R is L *mirror-sectional* by K bricks'.

Proposition 1. *If a polyorthogon or a polyomino or a polycuboid or a polycube, respectively, is L placeable or L sectional by K bricks, it is also L mirror-placeable or L mirror-sectional, respectively, by K bricks.*

Proposition 2. *If a polyorthogon or a polyomino or a polycuboid or a polycube is L mirror-placeable or L mirror-sectional, respectively, by K bricks, it is also* $2 \cdot L$ *placeable or* $2 \cdot L$ *sectional, respectively, by K bricks.*

Proposition 3. *Let P be a polyorthogon or a polycuboid or a polyomino or a polycube, respectively. Then it is placeable by K bricks, or sectional by K bricks, respectively.*

Proof. Take a rectangle or a cuboid, respectively, from P. Take it as a brick. The number K depends on the number of rectangles or cuboids, respectively, in P. □

Proposition 4. *For every natural number K exists a polyorthogon and a polyomino and a polycuboid and a polycube, respectively, that is placeable or sectional, respectively, by K bricks.*

Proof. We consider the most simple polyorthogon or polycuboid, respectively, i.e. a square or a cube, respectively. We take K copies and put them in a row. The constructed polyorthogon and polyomino or polycuboid and polycube, respectively, is placeable or sectional, respectively, by K bricks. □

Proposition 5. *For every natural number K exists a representation of the chessboard by* K^2 *squares.*

Proof. Take the square with sidelength $\frac{8}{K}$. □

We show once more that a chessboard is placeable by some bricks.
There are at least four trivial possibilities, if we use particular squares to represent the chessboard. They have sidelengths 1 or 2 or 4 or 8. We also can use 6 different rectangles to represent the chessboard. They have sidelengths $1 \times 2, 1 \times 4, 1 \times 8, 2 \times 4, 2 \times 8$ and 4×8. Further, there are at least two representations of the chessboard, which we call 'semi-trivial'. The bricks have the shape of the letter 'L'. Two examples are shown below in Figure 3. We add Malus square, which proves that the chessboard is mirror-placeable by 16 bricks which we know already by Proposition 1, and three non-trivial representations.

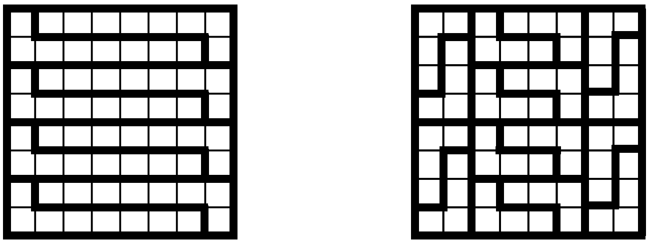

Figure 3:
On the left-hand side, we show two ‘semi-trivial’ representations of a chessboard.

A few years ago we have taught at a primary school. One day a clever child came to us and showed us a representation of a chessboard. Unfortunately we have forgotten the name. Her first name was ‘Malu’. Hence, we dubbed it ‘Malus square’. It is shown now. Malu prompted this paper. Without her it would not have been written.

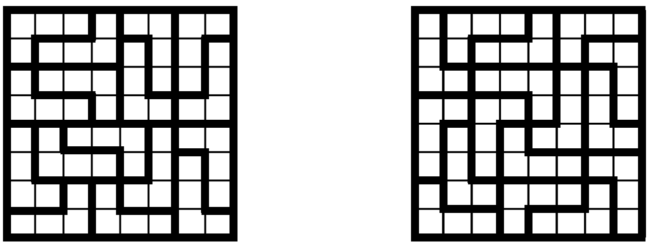

Figure 4:
We see two representations of the chessboard.
The picture on the right hand is Malus square.

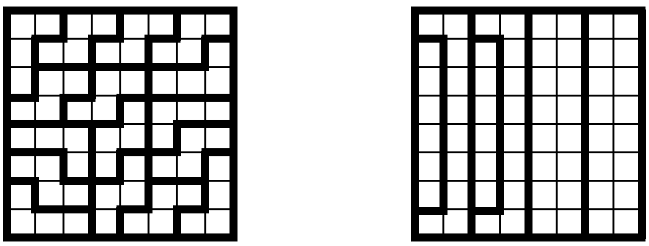

Figure 5:
The pictures on the left-hand side prove that the chessboard is 2 mirror-placeable by 16 bricks and 3 placeable by 6 bricks.

References

[1] Anthony J. Guttmann: *Polygons, Polyominoes and Polycubes*, Springer 2009

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