A PROOF OF THE UNION-CLOSE SET CONJECTURE

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Abstract. In this paper we introduce the notion of universe, induced communities and cells with their corresponding spots. Using this language we formulate and prove the union close set conjecture by showing that for any finite universe $\mathbb{U}$ and any induced community $\mathcal{M}_{\mathbb{U}}$ there exist some spot $a \in \mathbb{U}$ such that the density

$$\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a) \geq \frac{1}{2}.$$

1. Introduction

The union-closed set conjecture - roughly speaking - is the assertion that in any collection $\Im_{\mathbb{U}}$ of subsets of a set $\mathbb{U}$ closed under union, it is possible to find an element of $\mathbb{U}$ that lives in as many sets in the collection $\Im_{\mathbb{U}}$. The conjecture was first formulated in 1979 by Peter Frankl, in the equivalent form

Conjecture 1.1. *For any intersection-closed family of sets containing more than one set, there exists an element that lives in at most half of the sets in the family.*

It is easy to see that the above conjecture is equivalent to the conjecture:

Conjecture 1.2 (union-closed set conjecture)**.** *For any union-closed family of sets containing more that one set, there exist an element that lives in at least half of the sets in the family.*

The union-closed set conjecture remains open despite considerable efforts by many authors and several papers just devoted to study the problem. Regardless, the substantial progress with inputs and tools brought to bear are noteworthy. In fact the conjecture is proven for a few special cases. The conjecture is known to hold for families of at **most** forty-six sets [4]. It is also known to hold for families whose union has at most eleven elements [1]. It is also known to hold for families whose smallest set has only one or two elements [2]. The conjecture (see [3]) is also known to hold for family with $(\frac{1}{2}-\epsilon)2^n$ subsets of n elements for some $\epsilon > 0$.
The union-closed set conjecture also has several analogous versions and establishing the truth in those variant would imply the truth of the conjecture. The conjecture has a lattice-theoretic twist, which appeared in [5] with only special cases resolved. A well-known graph-theoretic version can also be found in [6]. Any of these variants could yet be a good terrain for the resolution of this simple-sounding conjecture and an appeal to any will certainly depend on that which seems amenable with the

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tools available.
In this paper, we verify the truth of the union-closed set conjecture using an elementary tool. We transform the problem to an entirely new language of density.

2. **The notion of universe, community and cells**

In this section we introduce the notion of cells, community and their corresponding universe. We study some elementary properties of this notion.

Definition 2.1. Let $\mathbb{U}$ be a set and consider the collection

$$\mathcal{M} := \bigcup_{i=1}^{n} \{\mathbb{A}_i |\ \mathbb{A}_i \cup \mathbb{A}_j \subseteq \mathbb{U},\ i \neq j\}.$$

Then we say the collection $\mathcal{M}$ is a **community** induced by set $\mathbb{U}$ if and only if for any $\mathbb{A}_i, \mathbb{A}_j \in \mathcal{M}$ then $\mathbb{A}_i \cup \mathbb{A}_j \in \mathcal{M}$. We call $\mathbb{U}$ the **universe** of the community. We call each $\mathbb{A}_j$ in the community a **cell** and each $a \in \mathbb{A}_j$ a **spot** in the cell. We say a cell $\mathbb{A}_i$ in the community admits an **embedding** in the community if there exists another different cell $\mathbb{A}_j$ in the same community such that $\mathbb{A}_j \subset \mathbb{A}_i$.

Proposition 2.2. *Let $\mathbb{U}$ be a universe with $|\mathbb{U}| = n$ and $\mathcal{M}_{\mathbb{U}}$ be a community induced by the universe. Then we have*

$$|\mathcal{M}_{\mathbb{U}}| \leq 2^n.$$

Proof. Let $\mathbb{U}_{\mathfrak{F}}$ be the power set induced by the universe $\mathbb{U}$. It is easy to see that $\mathbb{U}_{\mathfrak{F}}$ is the largest community induced by the universe with size

$$|\mathbb{U}_{\mathfrak{F}}| = 2^n$$

so that $|\mathcal{M}_{\mathbb{U}}| \leq 2^n$. □

Proposition 2.3. *The communities induced by a finite universe are **totally ordered**.*

Proof. Let $\mathbb{U}$ be a universe with $|\mathbb{U}| = n$ and let $\mathcal{M}_{i_{\mathbb{U}}}$ and $\mathcal{M}_{j_{\mathbb{U}}}$ be any two of distinct communities induced by the universe. Then it follows that the communities must differ by at least one cell so that without loss of generality with $|\mathcal{M}_{j_{\mathbb{U}}}| \leq |\mathcal{M}_{i_{\mathbb{U}}}|$ we can write

$$|\mathcal{M}_{j_{\mathbb{U}}}| \leq |\mathcal{M}_{i_{\mathbb{U}}}| < |\mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}|.$$

We claim that the collection $\mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$ is also a community. Let us pick arbitrarily two cells $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$. We consider three sub-cases: The case $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{M}_{i_{\mathbb{U}}}$ so that $\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{M}_{i_{\mathbb{U}}}$ since $\mathcal{M}_{i_{\mathbb{U}}}$ is a community.
For the case $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{M}_{j_{\mathbb{U}}}$ it must certainly be that $\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{M}_{j_{\mathbb{U}}}$ since $\mathcal{M}_{j_{\mathbb{U}}}$ is also a community. For the last case, where $\mathbb{A}_1 \in \mathcal{M}_{i_{\mathbb{U}}}$ and $\mathbb{A}_2 \in \mathcal{M}_{j_{\mathbb{U}}}$ then

$$\mathbb{A}_1 \cup \mathbb{A}_2 \in \mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}.$$

By choosing a community $\mathcal{M}_{k_{\mathbb{U}}} \neq \mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}}$ with $k \neq i, j$ and $|\mathcal{M}_{k_{\mathbb{U}}}| < |\mathcal{M}_{i_{\mathbb{U}}} \cup \mathcal{M}_{j_{\mathbb{U}}} \cup \mathcal{M}_{k_{\mathbb{U}}}|$ we obtain a five-term inequality by inserting $|\mathcal{M}_{k_{\mathbb{U}}}|$ into the a priori chain. Repeating the argument in this manner establishes the claim. □

3. **Density of spots in a cell**

In this section we introduce the notion of density of spots contained within a **cell**. We launch the following languages.

Definition 3.1. Let $\mathbb{U}$ be a finite universe with $|\mathbb{U}| = n$ and $a_i \in \mathbb{U}$. Let $\mathcal{M}_{\mathbb{U}}$ be the community induced by the universe $\mathbb{U}$. Then we denote the density of the spot a_i in cells in the community $\mathcal{M}_{\mathbb{U}}$ with

$$\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) = \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}|}.$$

Roughly speaking, the union close set conjecture is the assertion that for any collection of union close subset of the set $\mathbb{U}$ there exists some $a \in \mathbb{U}$ that lives in at least half of the subsets in the collection. It turns out that the union close conjecture can pretty much be stated in the language of density of spots as follows:

Conjecture 3.2. *[Union close set conjecture] Let $\mathbb{U}$ be a finite universe with an induced community $\mathcal{M}_{\mathbb{U}}$. Then there exist some $a_i \in \mathbb{U}$ such that*

$$\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) \geq \frac{1}{2}.$$

Remark 3.3. Conjecture 3.2, roughly speaking, can be interpreted as saying that there must always be a spot originating from a universe and contained in as many cells in a typical community. Next we investigate some properties of the notion of density of spots in a cell. The following properties will be useful in the sequel.

Proposition 3.4. *Let $\mathbb{U}$ be a finite universe with $|\mathbb{U}| = n$ and $a_i \in \mathbb{U}$. Let $\mathcal{M}_{\mathbb{U}}$ and $\mathcal{N}_{\mathbb{U}}$ be any two communities induced by the universe $\mathbb{U}$. Then the following properties hold*

(i) $\mathcal{D}_{\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}}(a_i) \leq \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) + \mathcal{D}_{\mathcal{N}_{\mathbb{U}}}(a_i)$.

(ii) $\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) \leq 1 - \mathcal{D}_{\mathcal{M}^c_{\mathbb{U}}}(a_i)$, *where $\mathcal{M}^c_{\mathbb{U}}$ denotes the complement of the collection $\mathcal{M}_{\mathbb{U}}$ in the power set $\mathbb{U}_{\mathfrak{F}}$ induced by the universe $\mathbb{U}$.*

Proof. For (i) we notice that by appealing to Definition 3.1, we can write

$$\begin{aligned}
\mathcal{D}_{\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}}(a_i) &= \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}|} \\
&= \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}|} + \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{N}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}|} \\
&- \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} \cap \mathcal{N}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}|} \\
&\leq \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}|} + \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{N}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}} \cup \mathcal{N}_{\mathbb{U}}|} \\
&\leq \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{M}_{\mathbb{U}}|} + \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{A} \in \mathcal{N}_{\mathbb{U}} |\ a_i \in \mathbb{A}\}}{|\mathcal{N}_{\mathbb{U}}|} \\
&= \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) + \mathcal{D}_{\mathcal{N}_{\mathbb{U}}}(a_i).
\end{aligned}$$

For (ii) it follows similarly that

$$\begin{aligned}\mathcal{D}_{\mathbb{U}_{\Im}}(a_i) &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{A}\in\mathbb{U}_{\Im}|\ a_i\in\mathbb{A}\right\}}{|\mathbb{U}_{\Im}|}\\ &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}\cup\mathcal{M}^c_{\mathbb{U}}|\ a_i\in\mathbb{A}\right\}}{|\mathcal{M}_{\mathbb{U}}\cup\mathcal{M}^c_{\mathbb{U}}|}\\ &= \mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i)+\mathcal{D}_{\mathcal{M}^c_{\mathbb{U}}}(a_i)\end{aligned}$$

by leveraging the property (i) and noting that $\mathcal{M}_{\mathbb{U}}\cap\mathcal{M}^c_{\mathbb{U}}=\emptyset$. By observing that

$$\lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{A}\in\mathbb{U}_{\Im}|\ a_i\in\mathbb{A}\right\}}{|\mathbb{U}_{\Im}|}\leq 1$$

the second part also follows. □

4. **Main results**

In this section we restate and prove the union closet set conjecture in the language of density of spots. Before that we state and prove an important result to be used to verify the union close set conjecture. The proof is quite constructive and inductive in nature, and in most cases can be seen as a cornerstone for establishing the truth of the conjecture albeit purely elementary.

Lemma 4.1. *[Covering Lemma] Let $\mathbb{U}$ be a finite universe with an induced community $\mathcal{M}_{\mathbb{U}}$. Then there exists some spot $a\in\mathbb{U}$ and some $l\in\mathbb{N}$ such that $|\mathcal{M}_{\mathbb{U}}|\leq 2^l-1$ and*

$$\#\left\{\mathbb{A}\in\mathcal{M}_{\mathbb{U}}|\ a\in\mathbb{A}\right\}\geq 2^{l-1}.$$

Proof. First we notice that any community $\mathcal{M}_{\mathbb{U}}$ induced by a finite universe $\mathbb{U}$ containing more that one basic cell must satisfy the inequality $|\mathcal{M}_{\mathbb{U}}|\geq 2$ so that for $|\mathcal{M}_{\mathbb{U}}|=3=2^2-1$, we construct the community using the cells $\mathbb{A}_1,\mathbb{A}_2$ as a building block, with $\mathbb{A}_1\cap\mathbb{A}_2\neq\mathbb{A}_1$ and $\mathbb{A}_1\cap\mathbb{A}_2\neq\mathbb{A}_2$ such that $a\in\mathbb{A}_i$ for some $1\leq i\leq 2$. In particular, by choosing $a\in\mathbb{A}_1$ we construct the community

$$\mathcal{M}_{\mathbb{U}}:=\{\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_1\cup\mathbb{A}_2=\mathbb{A}_3\}$$

with

$$\#\left\{\mathbb{A}_i\in\mathcal{M}_{\mathbb{U}}|\ a\in\mathbb{A}\right\}_{i=1}^{3}\geq 2^{2-1}.$$

Next we construct another community $\mathcal{N}_{\mathbb{U}}$ using the cells of the community $\mathcal{M}_{\mathbb{U}}$ as a building block. It is important to remark that any such community covers the a priori constructed community. Since $\mathbb{A}_i\cup\mathbb{A}_j\in\mathcal{M}_{\mathbb{U}}$ for $1\leq i,j\leq 3$, we choose a cell $\mathbb{A}_k\notin\mathcal{M}_{\mathbb{U}}$ but $\mathbb{A}_k\in\mathcal{N}_{\mathbb{U}}$ such that the cell $\mathbb{A}_k$ do not admit an embedding of the old cell $\mathbb{A}_1\in\mathcal{M}_{\mathbb{U}}$ and vice-versa and produce new cells $\mathbb{A}_1\cup\mathbb{A}_k$, $\mathbb{A}_2\cup\mathbb{A}_k$, $\mathbb{A}_3\cup\mathbb{A}_k$ with a new larger community

$$\mathcal{N}_{\mathbb{U}}:=\{\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_3,\mathbb{A}_k,\mathbb{A}_1\cup\mathbb{A}_k,\mathbb{A}_2\cup\mathbb{A}_k,\mathbb{A}_3\cup\mathbb{A}_k\}$$

so that $|\mathcal{N}_{\mathbb{U}}|\leq 2^3-1$ with

$$\#\left\{\mathbb{A}_i\in\mathcal{N}_{\mathbb{U}}|\ a\in\mathbb{A}_i\right\}\geq 4=2^{3-1}.$$

Let us suppose that for a fixed spot $a \in \mathbb{U}$, it is possible to construct at least a community $\mathcal{N}_{\mathbb{U}}^{r}$ that covers that all the a priori constructed communities in the pool with the size specifications

$$|\mathcal{N}_{\mathbb{U}}^{r}| \leq 2^{l} - 1$$

and

$$\#\{\mathbb{A} \in \mathcal{N}_{\mathbb{U}}^{r} |\ a \in \mathbb{A}\} \geq 2^{l-1}$$

for $l \geq 3$ using this scheme. Next we show we can construct at least another small community using the cells in $\mathcal{N}_{\mathbb{U}}^{r}$ as a building block and yet covering the community $\mathcal{N}_{\mathbb{U}}^{r}$. Let $\mathcal{N}_{\mathbb{U}}^{s}$ be a small community to be constructed so that it covers the community $\mathcal{N}_{\mathbb{U}}^{r}$. By **transitivity** this community also covers all the previously constructed communities in the pool. Let us choose an arbitrary cell $\mathbb{A}_t \in \mathcal{N}_{\mathbb{U}}^{s}$ such that $\mathbb{A}_t \notin \mathcal{N}_{\mathbb{U}}^{r}$ so that $\mathbb{A}_t$ does not admit an embedding of the cell $\mathbb{A}_1 \in \mathcal{N}_{\mathbb{U}}^{r}$ and vice-versa, and construct all the new cells using the old cells $\mathbb{A}_i \in \mathcal{N}_{\mathbb{U}}^{r}$ under the operations of union so that $\mathbb{A}_i \cup \mathbb{A}_t \in \mathcal{N}_{\mathbb{U}}^{s}$. Then we obtain a new closest and larger community $\mathcal{N}_{\mathbb{U}}^{s}$ with size

$$|\mathcal{N}_{\mathbb{U}}^{s}| \leq 2^{l} - 1 + (1 + 2^{l} - 1) = 2 \cdot 2^{l} - 1 = 2^{l+1} - 1$$

with

$$\#\{\mathbb{A} \in \mathcal{N}_{\mathbb{U}}^{s} |\ a \in \mathbb{A}\} \geq 2 \cdot 2^{l-1} = 2^{l}.$$

It follows that for any fixed spot originating from a finite universe we can construct finitely many induced communities of varying sizes with the above size specifications and containing a fixed preassigned spot originating from a finite universe $\mathbb{U}$, thereby ending the proof. □

Remark 4.2. It is crucially important to note that for a finite universe $\mathbb{U}$ with size $|\mathbb{U}| = n$ the constant l appearing in the construction of communities will essentially depend on n. We are now ready to verify the union close set conjecture. It is easy to see that the following result directly implies the truth of the union close set conjecture. It is important to notice that the construction in Lemma 4.1 generates all possible communities with cells containing a fixed spot. Put it differently, we can exploit the above construction to generates all possible communities induced by a finite universe with at least a cell containing a preassigned spot originating from a finite universe. Thus for any designated community induced by a finite universe we only need to choose at least one spot belonging to some cell and appeal to the sizes of the underlying set to verify the union close set conjecture.

Theorem 4.3. *[Density law] Let $\mathbb{U}$ be a finite universe with an induced community $\mathcal{M}_{\mathbb{U}}$. Then there exist some spot $a_i \in \mathbb{U}$ such that*

$$\mathcal{D}_{\mathcal{M}_{\mathbb{U}}}(a_i) \geq \frac{1}{2}.$$

Proof. Let $\mathbb{U}$ be a finite universe with an induced community $\mathcal{M}_{\mathbb{U}}$. Then there exists some spot $a_i \in \mathbb{U}$ contained in some cell $\mathbb{A} \in \mathcal{M}_{\mathbb{U}}$. By appealing to Lemma 4.1 there exists some $l \geq 1$ such that $|\mathcal{M}_{\mathbb{U}}| \leq 2^{l} - 1$ and

$$\#\{\mathbb{A} \in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\} \geq 2^{l-1}$$

so that we have the lower bound

$$\frac{\#\left\{\mathbb{A}\in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\right\}}{|\mathcal{M}_{\mathbb{U}}|} \geq \frac{2^{l-1}}{2^l-1} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2^l}}\right).$$

By taking limits on both sides as $l \longrightarrow \infty$, the result follows immediately. □

It follows that for any finite universe $\mathbb{U}$ with an arbitrary induced community $\mathcal{M}_{\mathbb{U}}$ there exists some spot $a_i \in \mathbb{U}$ for which we can write the lower bound

$$\frac{\#\left\{\mathbb{A}\in \mathcal{M}_{\mathbb{U}} |\ a_i \in \mathbb{A}\right\}}{|\mathcal{M}_{\mathbb{U}}|} \geq \frac{1}{2}\left(\frac{1}{1-o(1)}\right).$$

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