A solution to the Riemann Hypothesis

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Abstract

This paper discloses a proof for the Riemann Hypothesis. We define function $f(s)$ such that sum of $f(s)$ and $f(1-s)$ is zero at the non-trivial zeros of zeta function. Further, $f(s)$ can be written as sum of $-\frac{1}{s}$ and a series function $-\lambda(s)$ which is obtained on the expansion of integration term in the functional equation of zeta function. Either $f(s)$ and $f(1-s)$ are individually zero or $f(s) = -f(1-s)$ for an s, off the critical line if the Riemann hypothesis is false. From geometric analysis of $\lambda(s)$ we find the $f(s) \neq 0$ any where in the critical strip. We also prove by contradiction that $f(s) \neq -f(1-s)$ for any s, off the critical line.

1 Introduction

Form the functional equation of Riemann Zeta function,

$$\zeta(s) := \frac{\pi^{s/2}}{\Pi(s/2)}\left\{\frac{1}{s(s-1)} + \int_1^\infty (x^{s/2-1} + x^{-s/2-1/2}).\left(\frac{\vartheta(x)-1}{2}\right)dx\right\}$$

defined for Re(s) > 0, where

$$\vartheta(x) := \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi x}$$

is the Jacobi-theta function.
Let us define:

$$\Omega(s) = \tfrac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}).\left(\tfrac{\vartheta(x)-1}{2}\right)dx$$

defined in ($0 <$ Re(s) < 1).
On expanding the integral term in $\Omega(s)$, by 'integration by parts' taking $x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}$ as the first function and the Jacobi-theta function term as the second,

$$\Omega(s) = \tfrac{1}{s(s-1)} + \sum_{n=1}^{\infty}\left(0 - \tfrac{2}{s} - \tfrac{2}{1-s}\right)e^{-n^2\pi x} - \int_1^\infty \left(\tfrac{2x^{\frac{s}{2}}}{s} + \tfrac{2x^{-\frac{s}{2}+\frac{1}{2}}}{1-s}\right)\sum_{n=1}^{\infty} e^{-n^2\pi x}(-n^2\pi)dx$$

Repeating integration 'by parts' again and again, taking the Jacobi-theta function part as the second function, one gets a series function:

$$\begin{aligned}\Omega(s) =& \frac{1}{s(s-1)} + \sum_{n=1}^{\infty}\left(-\frac{2}{s} - \frac{2}{1-s}\right)e^{-n^2\pi x}\\ &+ \sum_{n=1}^{\infty}\left(-\frac{1}{\frac{s}{2}(\frac{s}{2}+1)} - \frac{1}{(\frac{1-s}{2})(\frac{1-s}{2}+1)}\right)e^{-n^2\pi x}.n^2\pi\\ &+ \sum_{n=1}^{\infty}\left(-\frac{1}{\frac{s}{2}(\frac{s}{2}+1)} - \frac{1}{(\frac{1-s}{2})(\frac{1-s}{2}+1)}\right)e^{-n^2\pi x}.(n^2\pi)^2 + ...\end{aligned}$$

We define:

$$f(s) = -\frac{1}{s} + \sum_{n=1}^{\infty} e^{-n^2\pi}\left(-\frac{1}{\left(\frac{s}{2}\right)^{\overline{1}}} - \frac{n^2\pi}{\left(\frac{s}{2}\right)^{\overline{2}}} - \frac{(n^2\pi)^2}{\left(\frac{s}{2}\right)^{\overline{3}}} - \frac{(n^2\pi)^3}{\left(\frac{s}{2}\right)^{\overline{4}}} - ...\right)$$

where $(\frac{s}{2})^{\overline{n}} = \frac{s}{2}(\frac{s}{2}+1)(\frac{s}{2}+2)...(\frac{s}{2}+n-1)$ is the rising factorial power function. So, $\Omega(s) = f(s) + f(1-s)$, and $f(s) = -\left(\frac{1}{s} + \lambda(s)\right)$.

Where, $\lambda(s) := \sum\limits_{m=1}^{\infty} \frac{\sum\limits_{n=1}^{\infty} e^{-n^2\pi}\left(n^2\pi\right)^{m-1}}{\left(\frac{s}{2}\right)^{\overline{m}}}$.

2 Geometry of series $\lambda(s)$ on the complex plane:

Plotting $\lambda(s)$ term by term on the complex plane for a given 's', leads us to the value of the series. In the present section we will find that $\lambda(s)$ converges as an anticlockwise inward spiral starting from origin, for values of s in the lower half region of the critical strip.

Let's define $\alpha \in (0, \frac{1}{2})$. Thus, $s = \frac{1}{2} \pm \alpha - i|t|$ represents any point in the lower region of the critical strip to the real axis (excluding the critical line $\Re(s) = \frac{1}{2}$). This is our region of interest because if Riemann hypothesis is false then $\Omega(s)$ is zero in this region.

Now, for $s = \frac{1}{2} \pm \alpha - i|t|$, the argument of the m^{th} term of $\lambda(s)$ is given by:

$$\begin{aligned}\theta_{m,\lambda} &= arg\left(\frac{1}{\frac{s}{2}}\right)^{\overline{m}} = -arg\left(\frac{s}{2}\right)^{\overline{m}}\\ &= \arctan\left(\frac{\frac{|t|}{2}}{\frac{\Re(s)}{2}}\right) + \arctan\left(\frac{\frac{|t|}{2}}{\frac{\Re(s)}{2}+1}\right) + ... + \arctan\left(\frac{\frac{|t|}{2}}{\frac{\Re(s)}{2}+m-1}\right)\end{aligned}$$

So, $\theta_{m,\lambda} < m\theta$, where θ is $\theta_{1,\lambda}$ (argument of the first term OA).

$0 < \theta_{m+1,\lambda} - \theta_{m,\lambda} = \arctan\left(\frac{\frac{|t|}{2}}{\frac{\Re(s)}{2}+m}\right) < \frac{\pi}{2}$ i.e., is positive and acute $\forall m$;

Thus, as m increases, the difference between the arguments of consecutive terms of $\lambda(s)$ decreases.
Since, $\lambda(s)$ converges, in the critical strip,

$$\frac{|T_{m+1}|}{|T_m|} = \frac{\sum\limits_{n=1}^{\infty} e^{-n^2\pi}\left(n^2\pi\right)^m}{|\frac{s}{2}+m|\sum\limits_{n=1}^{\infty} e^{-n^2\pi}\left(n^2\pi\right)^{m-1}} < 1 \;\forall m$$

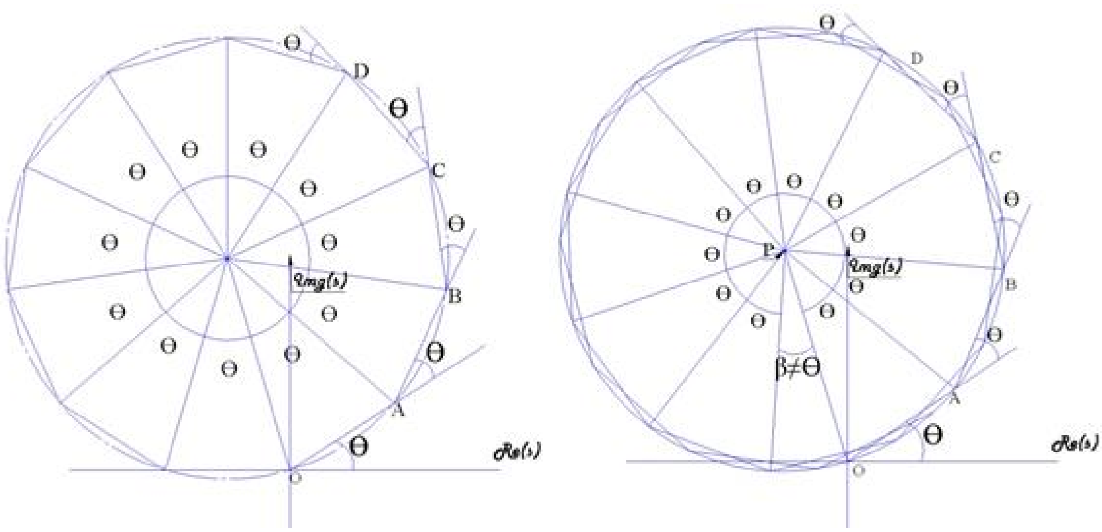
Now, let's try to plot $\lambda(s)$, by plotting some other series.
Take a series,

$$U(s) = \sum_{m=1}^{\infty} T_m(s) = \sum_{m=1}^{\infty} |T_m(s)|e^{i\theta_m}$$

$$OA = T_1(s) = |T_1(s)|e^{i\theta}$$

$$AB = |T_2(s)|e^{2i\theta}$$

$$BC = T_3(s) = |T_3(s)|e^{3i\theta}$$



(a) $U(s) : \frac{|T_{m+1}|}{|T_m|} = 1$, $\theta_m = m\theta$, and $\frac{2\pi}{\theta} \in \mathbb{N}^+$

(b) $U(s) : \frac{|T_{m+1}|}{|T_m|} = 1$, $\theta_m = m\theta$, $\frac{2\pi}{\theta} \notin \mathbb{N}^+$

Figure 1:

Figure 1 shows U(s) series with $\frac{|T_{m+1}|}{|T_m|} = 1$, $\theta_m = m\theta$, i.e. $\frac{T_{m+1}(s)}{T_m(s)} = e^{i\theta}$. Here, either of the following two cases will occur:

- A uniform polygon of sides $\frac{2\pi}{\theta}$: the whole series plots as a single polygon of $\frac{2\pi}{\theta}$ sides (fig. a). In the critical strip the angles θ_m are all acute.

- if θ does not divide 2π, we have a geometry as shown (fig. b) with consecutive cycles going out of phase and there exists $\beta : 0 < \beta < \theta$, and $\beta + (n-1)\theta = 2\pi$, which is the angle formed by the last sector at the center of the polygon on completion of 2π.

But, in none of the cases above, the series converges. Let's define a new series function $\mu(s) = \sum_{m=1}^{\infty} T_{m,\mu}(s)$, such that $\frac{|T_{m+1}|}{|T_m|} < 1$ and, $\theta_m = m\theta$, for $\mu(s)$, where θ is acute.

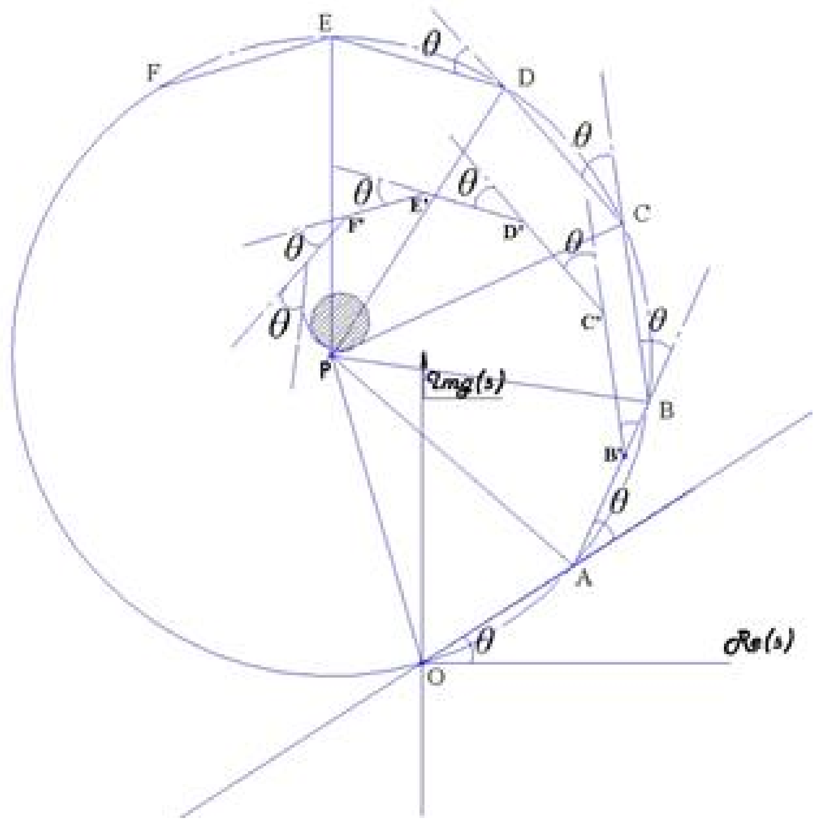


Figure 2: $\mu(s) : \frac{|T_{m+1}|}{|T_m|} < 1$, $\theta_m = m\theta$

As shown in figure 2, we get an inwardly spiraling geometry going anticlockwise from the origin.
The rate of convergence depends on the ratio of the consecutive terms. Thus, on choosing the first term in $\mu(s)$ same as for $U(s)$, it converges at a point inside the circle circumscribing the geometry formed for $\frac{|T_{m+1}|}{|T_m|} = 1$ cases above.

The hashed region shows the region in which the value of the series must lie after first few terms.

Now, $\lambda(s) = \sum_{m=1}^{\infty} T_m(s) = \sum_{m=1}^{\infty} |T_m(s)| e^{i\theta_{m,\lambda}}$.
where, $\theta_{m,\lambda} = arg\left(\frac{1}{\frac{s}{2}}\right)^{\overline{m}} = -arg\left(\frac{s}{2}\right)^{\overline{m}}$

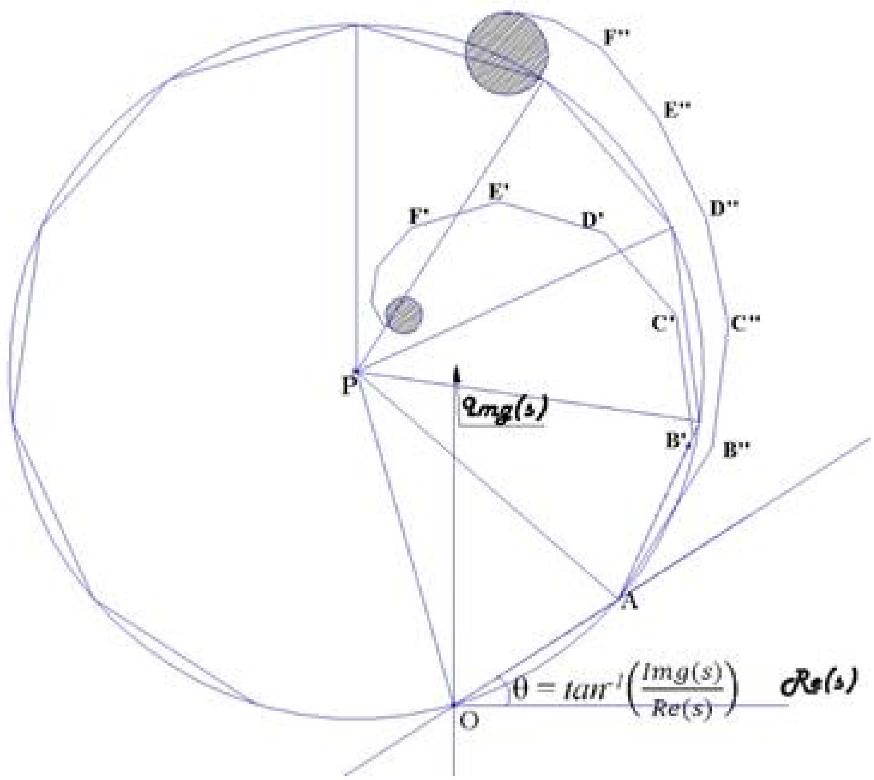


Figure 3: $\lambda(s) : \frac{|T_{m+1}|}{|T_m|} < 1$, $\theta_{m,\lambda} < m\theta$ and $\mu(s)$

We exploit the fact that for both $\mu(s)$ and $\lambda(s)$ we have, $\frac{|T_{m+1}|}{|T_m|} < 1$ and draw $\mu(s)$ with terms having magnitude as that for $\lambda(s)$ but $\theta_m = m\theta$. This corresponds to the spiral OAB'C'D'E'... in figure(3) above. Now, we just have to alter the orientation of the each term relative to the preceding one(starting from the second term). So, from the last case we fix OA and rotate AB' by $(\theta_{2,\lambda} - \theta)$ clockwise about A, to new position AB", so that the acute angle between AB" and line colinear with OA is now $\theta_{2,\lambda}$. Consecutively, rotate C'B" clockwise about B" by $\theta_{3,\lambda} - \theta$ to new position B"C" and D'C" about C" by $\theta_{4,\lambda} - \theta$ and so on. As shown in figure 3, $\lambda(s)$ represented by OAB"C"D"E"... is the unfastened form of the spiral OAB'C'D'E'.... Thus the rate of convergence is reduced. The unfastening is not done uniformly to all line segments representing series terms. Rather, the unfastening monotonously decreases over terms, from $arctan\left(\frac{|t|/2}{\Re(s)/2+1}\right)$ at m= 2 (since OA is fixed) to 0 as $m \to \infty$.

In figure 3, the spiral formed in figure 2 is unfastened as in this case $\theta_{m,\lambda} < m\theta$ and depending on the difference $\theta_{m,\lambda} - m\theta$, we get the rate of unfastening of the spiral. Nonetheless this unfastened spiral also lies in the upper half plane cut by the line col-linear to OA. This is true for all series with $|T_{m+1}| < |T_m|$ and $\theta_{m+1} > \theta m$. The same applies to $\lambda(s)$. Thus, $\lambda(s)$ converges as an anticlockwise inward spiral which always lies in the upper half plane to the line colinear with the first term OA, for $s = \frac{1}{2} \pm \alpha - i|t|$.

Using the above deductions about $f(s)$ and $\lambda(s)$, we look at the conditions for $\Omega(s)$ to be zero for any $\alpha \in \left(0, \frac{1}{2}\right)$.

3 Conditions for the falsification of the Riemann Hypothesis

3.1 Case-1: RH is false if $f(s) = f(1-s) = 0$, for $s = \frac{1}{2} \pm \alpha - i|t|$

Since, $\lambda(s)$ is an anticlockwise and inward - spiral starting from O at origin with first term 'OA', its value always lies in the upper half plane to the line col-linear with OA whereas, $\frac{1}{s}$ lies on this line. Therefore, $\frac{1}{s} + \lambda(s) \neq 0$, $\Rightarrow f(s) \neq 0$. and since, $(f(s))^* = f(s^*)$ thus, $f(s) \neq O$ for any s, off the critical line, $\Re(s) = \frac{1}{2}$. Thus, no such case exists to falsify RH.

3.2 Case-2: RH is false if $f(s) = -f(1-s)$

Let $s = \frac{1}{2} + \alpha + it$,
and $s_\alpha := \alpha + it$
$\Rightarrow s = \frac{1}{2} + s_\alpha$ and $(1-s) = \frac{1}{2} - s_\alpha$ and we define $g : g(s_\alpha) = f(S)$
Then, $g(-s_\alpha) = f(1-s)$;
and $f(s) = -f(1-s) \Leftrightarrow g(s_\alpha) = -g(-s_\alpha)$.
That is, g is an odd function of s_α.
Now:

$$\frac{1}{\left(\frac{s}{2}\right)^{\overline{2}}} = \frac{1}{\frac{s}{2}\left(\frac{s}{2}+1\right)} = \frac{1}{\frac{s}{2}} - \frac{1}{\frac{s}{2}+1}$$

$$\frac{1}{\left(\frac{s}{2}\right)^{\overline{3}}} = \frac{1}{\frac{s}{2}\left(\frac{s}{2}+1\right)\left(\frac{s}{2}+2\right)} = \frac{1}{2!\,\frac{s}{2}} - \frac{1}{1!\left(\frac{s}{2}+1\right)} + \frac{1}{2!\left(\frac{s}{2}+2\right)}$$

$$\frac{1}{\left(\frac{s}{2}\right)^{\overline{4}}} = \frac{1}{\frac{s}{2}\left(\frac{s}{2}+1\right)\left(\frac{s}{2}+2\right)\left(\frac{s}{2}+3\right)} = \frac{1}{3!\,\frac{s}{2}} - \frac{1}{2!\left(\frac{s}{2}+1\right)} + \frac{1}{2!\left(\frac{s}{2}+2\right)} - \frac{1}{3!\left(\frac{s}{2}+3\right)}$$

Therefore, one gets:

$$f(s) = -\frac{1}{s} - \sum_{n=1}^{\infty} e^{-n^2\pi}\left(\frac{1}{\frac{s}{2}}\right) - \sum_{n=1}^{\infty} e^{-n^2\pi}(n^2\pi)\left(\frac{1}{\frac{s}{2}} - \frac{1}{\left(\frac{s}{2}+1\right)}\right)$$
$$-\sum_{n=1}^{\infty} e^{-n^2\pi}(-n^2\pi)^2\left(\frac{1}{2!\,\frac{s}{2}} - \frac{1}{1!\left(\frac{s}{2}+1\right)} + \frac{1}{2!\left(\frac{s}{2}+2\right)}\right)...-$$

That is:

$$g(s_\alpha) = -\frac{1}{\frac{1}{2}+s_\alpha} - \sum_{n=1}^{\infty} e^{-n^2\pi}\left(\frac{1}{\frac{\frac{1}{2}+s_\alpha}{2}}\right) - \sum_{n=1}^{\infty} e^{-n^2\pi}.n^2\pi\left(\frac{1}{\frac{\frac{1}{2}+s_\alpha}{2}} - \frac{1}{\left(\frac{\frac{1}{2}+s_\alpha}{2}+1\right)}\right)$$
$$-\sum_{n=1}^{\infty} e^{-n^2\pi}.(n^2\pi)^2\left(\frac{1}{2!\,\frac{\frac{1}{2}+s_\alpha}{2}} - \frac{1}{1!\left(\frac{\frac{1}{2}+s_\alpha}{2}+1\right)} + \frac{1}{2!\left(\frac{\frac{1}{2}+s_\alpha}{2}+2\right)}\right)...-$$

and

$$g(-s_\alpha) = -\frac{1}{\frac{1}{2}-s_\alpha} - \sum_{n=1}^{\infty} e^{-n^2\pi}\left(\frac{1}{\frac{\frac{1}{2}-s_\alpha}{2}}\right) - \sum_{n=1}^{\infty} e^{-n^2\pi}.n^2\pi\left(\frac{1}{\frac{\frac{1}{2}-s_\alpha}{2}} - \frac{1}{\left(\frac{\frac{1}{2}-s_\alpha}{2}+1\right)}\right)$$
$$-\sum_{n=1}^{\infty} e^{-n^2\pi}.(n^2\pi)^2\left(\frac{1}{2!\,\frac{\frac{1}{2}-s_\alpha}{2}} - \frac{1}{1!\left(\frac{\frac{1}{2}-s_\alpha}{2}+1\right)} + \frac{1}{2!\left(\frac{\frac{1}{2}-s_\alpha}{2}+2\right)}\right)...-$$

This implies:

$$\begin{aligned}g(s_\alpha)+g(-s_\alpha) = &-\frac{1}{\frac{1}{2^2}-(s_\alpha)^2} - \sum_{n=1}^{\infty} e^{-n^2\pi}\left(\frac{2}{\frac{1}{2^2}-s_\alpha^2}\right)\\ &-\sum_{n=1}^{\infty} e^{-n^2\pi}.n^2\pi\left(\frac{2}{\frac{1}{2^2}-s_\alpha^2} - \frac{2\left(1+\frac{1}{2^2}\right)}{\left(\left(\frac{1}{2^2}+1\right)-s_\alpha^2 2\right)}\right)\\ &-\sum_{n=1}^{\infty} e^{-n^2\pi}\left(n^2\pi\right)^2\left(\frac{2}{2!\left(\frac{1}{2^2}-s_\alpha^2\right)} - \frac{2\left(1+\frac{1}{2^2}\right)}{1!\left(\left(\frac{1}{2^2}+1\right)^2-\left(\frac{s_\alpha}{2}\right)^2\right)}\right.\\ &\left.+\frac{2\left(2+\frac{1}{2^2}\right)}{2!\left(\left(\left(2+\frac{1}{2^2}\right)^2-\left(\frac{s_\alpha}{2}\right)^2\right)\right)}-\right)...\end{aligned}$$

which is an even function of s_α since all terms are functions of s_α^2.

But, the condition that $g(s_\alpha)+g(-s_\alpha)=0 \Rightarrow g(s_\alpha)=-g(-s_\alpha)$, i.e., g is an odd function of s_α. If a, b are two odd functions of s_α then, $a(s_\alpha)+b(s_\alpha)$ can not be an even function of s_α Hence, by contradiction$g(s_\alpha) \neq -g(-s_\alpha)$ for any s_α, or $f(s) \neq -f(1-s)$ for any s, off the critical line.

Thus, the Riemann hypothesis is true.