Compatibility of L-ideals with L-topologies

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Abstract: In this paper, we introduce the notion of L-ideal in L-set theory. Also we introduce the concept of L-local function. These concepts are discussed with a view to find new L-topologies from the original one. The basic structure, especially a basis for such generated L-topologies also studied here. The notion of L-compatibility of L-ideals with L-topologies is introduced and some equivalent conditions concerning this topic are established here. Moreover, by using L-local function we introduce L-operator ψ satisfying $\psi(A_L) = 1_X - (1_X - A_L)^*$, for all $A_L \subseteq L^X$ and we discuss some characterizations this L-operator by use L-open sets.

Keywords: L-ideal; L-ideal topological space; L-local-function; L-compatible space; L-open set L-operator ψ.

1. Introduction

The concept of "Topology" is one of the most important mathematical topics and has wide applications in many applied sciences and mathematical subjects. A frame is a set admitting two operations analogous to maximum and minimum operations, with two elements as its supremum and infimum accompanied by few circumstances. In order to obtain an even larger framework to work on, in topological arguments, the operations in frames seem to be good candidates to be replaced by the notions of union and intersection of sets. L-topological space is using the maps from a set X to a frame L.

The concept of an ideal in a topological space was first introduced by Kuratowski in 1966 [3], and Vaidyanathswamy in 1945 [4]. They also defined local functions in an ideal topological space. Further, Hamlett and Jankovic in 1990 [5], studied the properties of ideal topological spaces and introduced another operator called ψ-operator. They have also obtained a new topology from the original ideal topological space. Using the local function, they defined a

Kuratowski closure operator in the new topological space. In this paper, we introduce the notion of L-ideal in L-set theory. Also we introduce the concept of L-local function. These concepts are discussed with a view to find new L-topologies from the original one. The basic structure, especially a basis for such generated L-topologies also studied here. The notion of L-compatibility of L-ideals with L-topologies is introduced and some equivalent conditions concerning this topic are established here. Moreover, by using L-local function we introduce L-operator ψ satisfying $\psi(A_L) = 1_X - (1_X - A_L)^*$, for each $A_L \subseteq L^X$ and we discuss some characterizations this L-operator by use L-open sets.

2. Preliminaries

Definition 2.1. [1] A lattice (L, ∨, ∧) would be called bounded, if there exists elements 0 and 1 in L, such that for each $a \in L$ one has $a \vee 0 = a$ and $a \wedge 1 = a$. This obviously implies that the members 0 and 1 are unique, as well as, for each $a \in L$ one has $0 \leq a \leq 1$.

Definition 2.2. [1] A bounded lattice (L, ∨, ∧, 0, 1), abbreviated by L, is called complete, if an arbitrary joint and arbitrary meet of its elements exist.

Definition 2.3. [1] A frame is a complete bounded lattice L in which the arbitrary distribution law is hold for its elements, i.e. the equality $x \wedge (\bigvee_{y\in Y} y) = \bigvee_{y\in Y} (x \wedge y)$, is valid for $x \in L$ and for an arbitrary subset Y of L. It can be verified easily that one has $x \vee (\bigwedge_{y\in Y} y) = \bigwedge_{y\in Y} (x \vee y)$, in a frame L.

Definition 2.4. [1] Let (L, ∨, ∧, 0, 1) be a frame and X be a non-empty set. We denote by 0_X and 1_X the constant maps sending elements of X to 0 and 1, respectively. Particularly, one has $0_X, 1_X \in L^X$. For $f, g \in L^X$, we define $f \leq g$ if and only if for each $x \in X$ one has $f(x) \leq g(x)$.

Definition 2.5. [2] Let X be a set and $\tau_L = \{ S_\alpha\}_{\alpha\in I}$ be a collection of L-maps of X, i.e. $\{ S_\alpha\}_{\alpha\in I} \subseteq L^X$, such that

(i) $0_X, 1_X \in \tau_L$.

(ii) For a non-empty collection $\{ S_\alpha\}_{\alpha\in J}$ in τ_L, one has $\bigvee_{\alpha\in J} S_\alpha \in \tau_L$.

(iii) The meet of a finite collection of members of τ_L belongs to τ_L.

Then, the couple (X, τ_L) will be called a L-topological space and the members of τ_L are the L-open sets of this L-topological space. The complement of the L-open set is called L-closed set. We call a set U in X open if $\chi_U \in \tau_L$ and closed if $\chi_{U^c} \in \tau_L$.

Definition 2.6. [2] Let (X, τ_L) be a L-topological space and let $A_L \subseteq L^X$. Then the L-interior and the L-closure of A_L in (X, τ_L) defined as int(A_L) = $\vee\{U_L: U_L \leq A_L, U_L \in \tau_L\}$ and cl(A_L) = $\wedge\{F_L: A_L \leq F_L, F_L$ is a L-closed set$\}$ respectively. From definition, int(A_L) is a L-open set and cl(A_L) is a L-closed set.

Definition 2.7. [2] Let X be an L-topological space and Y be a subset of X. The family of maps $\{(U_\alpha)_{\backslash Y} : U_\alpha \in \tau_L\}$ impose a L-topological structure on Y . We call this topology, the L-subspace topology on Y.

Definition 2.8. [2] An L-open set $U_L \in \tau_L$ is called a L-neighborhood of x ∈ X, if $\chi_{\{x\}} \leq U_L$. The collection $N_L(x)$ of all L-neighborhoods of x is called the L-neighborhood system of x. An L-open subset U_L contains an L-open subset V_L if $V_L \leq U_L$.

Definition 2.9. [2] Let X be a non-empty set and let τ_L^1 and τ_L^2 be L-topologies on X such that $\tau_L^1 \leq \tau_L^2$. Then we say that τ_L^2 is stronger (finer) than τ_L^1 or τ_L^1 is weaker (coarser) than τ_L^2. Two L-topologies τ_L^1 and τ_L^2 on X are called equivalent if τ_L^1 is finer than τ_L^2 and τ_L^2 is finer than τ_L^1.

Definition 2.10. [2] An L-topology basis is a set $\beta_L \subseteq L^X$ such that

(i) $\bigvee_{B_\alpha \in \beta_L} B_\alpha = 1_X$.

(ii) For all B_1 and B_2 in β_L we have $B_1 \wedge B_2 = \vee B_\gamma$, where $B_\gamma \in \beta_L$.

If β_L is L-topology basis, then the set $\tau_L^{\beta_L} = \{\vee B_\gamma : B_\gamma \in \beta_L\}$ is called the L-topology generated by β_L. Obviously any L-topological space admits a L-topological basis.

3. L-Ideal and L-Ideal Topological Spaces

Definition 3.1. Let X be a set and $I_L = \{E_\alpha\}_{\alpha \in I}$ be a collection of L-maps of X, i.e. $\{E_\alpha\}_{\alpha \in I} \subseteq L^X$, such that

(i) $E_1 \in I_L$ and $E_2 \leq E_1$ implies $E_2 \in I_L$ (heredity).

(ii) $E_1 \in I_L$ and $E_2 \in I_L$ implies $E_1 \vee E_2 \in I_L$ (finite additivity).

Then, I_L is called a L-ideal on X.

Definition 3.2. A L-topological space (X, τ_L) with a L-ideal I_L on X is called L-ideal topological space and denoted as (X, τ_L, I_L).

Definition 3.3. Let (X, τ_L, I_L) be a L-ideal topological space and let A_L be a collection of L-maps of X, i.e. $A_L \subseteq L^X$. Then $A_L^*(\tau_L, I_L) = \{\chi_{\{x\}} \in L^X : A_L \wedge U_L \notin I_L$, for all $U_L \in N_L(x)\}$ is called L-local function of A_L with respect to I_L and τ_L. We denote simply A_L^* for $A_L^*(\tau_L, I_L)$.

Example 3.1. The simplest L-ideal on X are 0_X and 1_X. Then $I_L = 0_X \Leftrightarrow A_L^* = cl_L(A_L)$, for any $A_L \subseteq L^X$ and $I_L = 1_X \Leftrightarrow A_L^* = 0_X$.

Theorem 3.1. Let (X, τ_L, I_L) be a L-ideal topological space and let $A_L, B_L \subseteq L^X$. Then

i) $0_X^* = 0_X$.

ii) If $A_L \leq B_L$ then $A_L^* \leq B_L^*$

iii) If $I_L^1 \leq I_L^2$ then $A_L^*(I_L^2) \leq A_L^*(I_L^1)$

iv) $A_L^* = cl(A_L^*) \leq cl(A_L)$

v) $(A_L^*)^* \leq A_L^*$

vi) A_L^* is a L-closed set

vii) $A_L^* \vee B_L^* = (A_L \vee B_L)^*$

viii) $(A_L \wedge B_L)^* \leq A_L^* \wedge B_L^*$

ix) If $U_L \in \tau_L$, then $U_L \wedge A_L^* = U_L \wedge (U_L \wedge A_L)^* \leq (U_L \wedge A_L)^*$

x) If $E_L \in I_L$, then $E_L^* = 0_X$.

Proof. i) This is obvious from the definition of L-local function.

ii) Let $A \leq B$ and let $\chi_{\{x\}} \in A_L^*$ then $A_L \wedge U_L \notin I_L$, for all $U_L \in N_L(x)$. By hypothesis we get $B_L \wedge U_L \notin I_L$, then $\chi_{\{x\}} \in B_L^*$. Therefore $A_L^* \leq B_L^*$.

iii) Let $I_L^1 \leq I_L^2$ from definition of L-local function, $A_L^*(I_L^2) \leq A_L^*(I_L^1)$.

iv) For any L-ideal on X, we know $0_X \leq I_L$, therefore by (iii) and Example.3.1, for any $A_L \subseteq L^X$ then $A_L^*(I_L) \leq A_L^*(0_X) = cl(A_L)$. Suppose $\chi_{\{x\}} \in cl(A_L^*)$ then for all $U_L \in N_L(x)$, $A_L^* \wedge U_L \neq 0_X$ there exists $\chi_{\{y\}} \in A_L^* \wedge U_L$ such that for all $V_L \in N_L(y)$ then $A_L \wedge V_L \notin I_L$. Since $U_L \wedge V_L \in N_L(y)$ then $A_L \wedge (U_L \wedge V_L) \notin I_L$ which leads to $A_L \wedge U_L \notin I_L$ for all $U_L \in N_L(x)$. Therefore $\chi_{\{x\}} \in A_L^*$. Hence $cl(A_L^*) \leq A_L^*$ while the other inclusion follows directly. Hence $A_L^* = cl(A_L^*) \leq cl(A_L)$.

v) From (iv), $(A_L^*)^* \leq A_L^*$.

vi) Clear from (iv).

vii) We have $A_L \leq A_L \vee B_L$ and $B_L \leq A_L \vee B_L$. Then from (ii), $A_L^* \leq (A_L \vee B_L)^*$ and $B_L^* \leq (A_L \vee B_L)^*$. Hence $A_L^* \vee B_L^* \leq (A_L \vee B_L)^*$. Now let $\chi_{\{x\}} \in (A_L \vee B_L)^*$. Then $(U_L \wedge A_L) \vee (U_L \wedge B_L) = U_L \wedge (A_L \vee B_L) \notin I_L$. Therefore, $U_L \wedge A_L \notin I_L$ or $U_L \wedge B_L \notin I_L$ for all $U_L \in N_L(x)$. This implies that $\chi_{\{x\}} \in A_L^*$ or $\chi_{\{x\}} \in B_L^*$, that is $\chi_{\{x\}} \in A_L^* \vee B_L^*$. Therefore, we have $(A_L \vee B_L)^* \leq A_L^* \vee B_L^*$. Hence, we obtain $A_L^* \vee B_L^* = (A_L \vee B_L)^*$.

viii) We have $A_L \wedge B_L \leq A_L$ and $A_L \wedge B_L \leq B_L$. Then from (ii), $(A_L \wedge B_L)^* \leq A_L^*$ and $(A_L \wedge B_L)^* \leq B_L^*$. Hence $(A_L \wedge B_L)^* \leq A_L^* \wedge B_L^*$.

ix) Let $V_L \in \tau_L$ and $\chi_{\{x\}} \in V_L \wedge A_L^*$. Then $\chi_{\{x\}} \in V_L$ and $\chi_{\{x\}} \in A_L^*$. Since $V_L \in \tau_L$ then $U_L \in N_L(x)$ such that $\chi_{\{x\}} \in U_L$. Then $U_L \wedge V_L \in N_L(x)$ and $U_L \wedge (V_L \wedge A_L) = (U_L \wedge V_L) \wedge A_L \notin I_L$. Then $\chi_{\{x\}} \in (A_L \wedge V_L)^*$ and hence we obtain $V_L \wedge A_L^* \leq (A_L \wedge V_L)^*$. Moreover $V_L \wedge A_L^* \leq V_L \wedge (V_L \wedge A_L)^*$, by (ii) $(A_L \wedge V_L)^* \leq A_L^*$ and $V_L \wedge (A_L \wedge V_L)^* \leq V_L \wedge A_L^*$. Therefore, $V_L \wedge A_L^* = V_L \wedge (A_L \wedge V_L)^* \leq (A_L \wedge V_L)^*$.

x) Let $\chi_{\{x\}} \in E_L^*$. Then for all $U_L \in N_L(x)$, $E_L^* \wedge U_L \notin I_L$. But since $E_L \in I_L$, $E_L \wedge U_L \in I_L$ for all $U_L \in N_L(x)$. This is a contradiction. Hence $E_L^* = 0_X$.

Theorem 3.2. Let (X, τ_L) be a L-topological space with L-ideals I_L^1 and I_L^2 on X and $A_L \subseteq L^X$. Then, $A_L^*(I_L^1 \wedge I_L^2) = A_L^*(I_L^1) \vee A_L^*(I_L^2)$.

Proof. By Theorem 3.1(iii) we have $A_L^*(I_L^1) \leq A_L^*(I_L^1 \wedge I_L^2)$ and $A_L^*(I_L^2) \leq A_L^*(I_L^1 \wedge I_L^2)$. Therefore, we obtain $A_L^*(I_L^1) \vee A_L^*(I_L^2) \leq A_L^*(I_L^1 \wedge I_L^2)$. Now, let $\chi_{\{x\}} \in A_L^*(I_L^1 \wedge I_L^2)$. Then, for all $U_L \in N_L(x)$, $U_L \wedge A_L \notin$

$I_L^1 \wedge I_L^2$ and hence $U_L \wedge A_L \notin I_L^1$ or $U_L \wedge A_L \notin I_L^2$. This shows that $\chi_{\{x\}} \in A_L^*(I_L^1)$ or $\chi_{\{x\}} \in A_L^*(I_L^2)$. Therefore, we have $\chi_{\{x\}} \in A_L^*(I_L^1) \vee A_L^*(I_L^2)$. This shows that $A_L^*(I_L^1 \wedge I_L^2) \leq A_L^*(I_L^1) \vee A_L^*(I_L^2)$. Then, we obtain $A_L^*(I_L^1 \wedge I_L^2) = A_L^*(I_L^1) \vee A_L^*(I_L^2)$.

Definition 3.4. Let (X, τ_L, I_L) be a L-ideal topological space and let $A_L \subseteq L^X$. Then $cl^*(A_L) = A_L \vee A_L^*$ is a called L-closure operator.

Theorem 3.3. Let (X, τ_L, I_L) be a L-ideal topological space and let A_L, $B_L \subseteq L^X$. Then

i) $cl^*(0_X) = 0_X$
ii) $A_L \leq cl^*(A_L)$
iii) $cl^*(A_L \vee B_L) = cl^*(A_L) \vee cl^*(B_L)$
iv) $cl^*(A_L) = cl^*(cl^*(A_L))$.

Proof. i) $cl^*(0_X) = 0_X^* \vee 0_X$, by Theorem 3.1.(i) then $cl^*(0_X) = 0_X$.
ii) $A_L \leq A_L \vee A_L^* = cl^*(A_L)$.
iii) $cl^*(A_L \vee B_L) = (A_L \vee B_L) \vee (A_L \vee B_L)^* = (A_L \vee B_L) \vee (A_L^* \vee B_L^*) = (A_L \vee A_L^*) \vee (B_L \vee B_L^*)$. Hence $cl^*(A_L \vee B_L) = (A_L \vee A_L^*) \vee (B_L \vee B_L^*) = cl^*(A_L) \vee cl^*(B_L)$.

iv) $cl^*(cl^*(A_L)) = cl^*(A_L \vee A_L^*) = (A_L \vee A_L^*) \vee (A_L \vee A_L^*)^* = (A_L \vee A_L^*) \vee (A_L^* \vee (A_L^*)^*) = A_L \vee A_L^* = cl^*(A_L)$.

Theorem 3.4. Let (X, τ_L, I_L) be a L-ideal topological space and let A_L, $B_L \subseteq L^X$. Then

i) If $A_L \leq B_L$, then $cl^*(A_L) \leq cl^*(B_L)$
ii) $cl^*(A_L \wedge B_L) \leq cl^*(A_L) \wedge cl^*(B_L)$.

Proof. This is obvious by Theorem 3.1.(ii), (viii).

Theorem 3.5. Let (X, τ_L, I_L) be a L-ideal topological space. Then $\tau_L^*(I_L) = \{A_L \subseteq L^X : cl^*(A_L^c) = A_L^c\}$ is L-topology on X and finer than τ_L. When there is no ambiguity we will write τ_L^* for $\tau_L^*(I_L)$.

Proof. This is obvious by Theorem 3.1, and Theorem 3.3. Again by Theorem 3.1,(iv), we have $A_L^* \leq cl(A_L)$, then $A_L \vee A_L^* \leq A_L \vee cl(A_L) = cl(A_L)$, then $cl^*(A_L) \leq cl(A_L)$. Hence τ_L finer than τ_L^*.

Example 3.2. Let (X, τ_L, I_L) be a L-ideal topological space and let $A_L \subseteq L^X$. If $I_L = \{0_X\}$, then $\tau_L = \tau_L^*(I_L)$. In fact, if $\chi_{\{x\}} \in cl(A_L)$, then, $U_L \wedge A_L \neq 0_X$ for all $U_L \in N_L(x)$ then $U_L \wedge A_L \notin \{0_X\} = I_L$

then $\chi_{\{x\}} \in A_L^*$. Hence $\chi_{\{x\}} \in A_L \vee A_L^* = cl^*(A_L)$ then $cl(A_L) \leq cl^*(A_L)$ but by Theorem 3.5. $cl^*(A_L) \leq cl(A_L)$. Hence $cl^*(A_L) = cl(A_L)$. Consequently, $\tau_L = \tau_L^*(0_X)$.

Theorem 3.6. Let (X, τ_L) be a L-topological space with L-ideals I_L^1 and I_L^2 on X. Then
If $I_L^1 \leq I_L^2$, then $\tau_L^*(I_L^1) \leq \tau_L^*(I_L^2)$.
Proof. Straightforward.

Theorem 3.7. Let (X, τ_L, I_L) be a L-ideal topological space. Then $\beta_L(I_L,\tau_L) = \{U_L\text{-}E_L : U_L \in \tau_L, E_L \in I_L\}$ is a L-basis for τ_L^*.
Proof. Since $0_X \in I_L$, then $U_L - 0_X = U_L \in \tau_L$ and $\tau_L \leq \beta_L$ from which it follows that $1_X = \vee\beta_L$ (recall that L-open sets is forms a L-topology). Also $\beta_L^1, \beta_L^2 \in \beta_L$, and $E_L^1, E_L^2 \in I_L$, we have $\beta_L^1 = U_L^1 - E_L^1$ and $\beta_L^2 = U_L^2 - E_L^2$, where $U_L^1, U_L^2 \in \tau_L$. Then $\beta_L^1 \wedge \beta_L^2 = (U_L^1 - E_L^1) \wedge (U_L^2 - E_L^2) = (U_L^1 \wedge (1_X - E_L^1)) \wedge (U_L^2 \wedge (1_X - E_L^2)) = (U_L^1 \wedge U_L^2) - (E_L^1 \vee E_L^2) \in \beta_L$.

4. L-compatibility of L-topological spaces

Definition 4.1.[5] Let (X, τ, I) be an ideal topological space. We say the topology τ is compatible with the ideal I, denoted $\tau \sim I$, if the following holds for every $A \subseteq X$: if for all $x \in A$ there exists $U \in N(x)$ such that $U \cap A \in I$, then $A \in I$, where $N(x)$ denotes the open neighbourhood system at x.

Definition 4.2. Let (X, τ_L, I_L) be a L-ideal topological space. We say the L-topology τ_L is L-compatible with the L-ideal I_L, denoted $\tau_L \sim I_L$, if the following holds for all $A_L \subseteq L^X$: if for all $\chi_{\{x\}} \in A_L$ there exists $U_L \in N_L(x)$ such that $U_L \wedge A_L \in I_L$, then $A_L \in I_L$.

Theorem 4.1. Let (X, τ_L, I_L) be a L-ideal topological space. The following properties are equivalent
i) $\tau_L \sim I_L$,
ii) If $A_L \subseteq L^X$ has a L-cover of L-open sets each of whose intersection with A_L is in I_L, then $A_L \in I_L$,
iii) For all $A_L \subseteq L^X$, $A_L \wedge A_L^* = 0_X$ implies that $A_L \in I_L$,
iv) For all $A_L \subseteq L^X$, $A_L - A_L^* \in I_L$,
v) For all $A_L \subseteq L^X$, if A_L contains no nonempty subset B_L with $B_L \leq B_L^*$, then $A_L \in I_L$.
Proof. i) $\Rightarrow$ ii) The proof is obvious.
ii) $\Rightarrow$ iii) Let $A_L \subseteq L^X$ and let $\chi_{\{x\}} \in A_L$. Then $\chi_{\{x\}} \notin A_L^*$ and there exists $U_L \in N_L(x)$ such that

$U_L \wedge A_L \in I_L$. Therefore, we have $A_L \leq \vee\{U_L : \chi_{\{x\}} \in U_L\}$ and $U_L \in N_L(x)$ and by (ii) $A_L \in I_L$.

iii) ⇒ iv) For any $A_L \subseteq L^X$, $A_L - A_L^* \leq A_L$ and $(A_L - A_L^*) \wedge (A_L - A_L^*)^* \leq (A_L - A_L^*) \wedge A_L^* = 0_X$.

By (iii), $A_L - A_L^* \in I_L$.

iv) ⇒ v) By (iv), for all $A_L \subseteq L^X$, $A_L - A_L^* \in I_L$. Let $A_L - A_L^* = E_L \in I_L$, then $A_L = E_L \vee (A_L \wedge A_L^*)$ and by Theorem 3.1(vi), $A_L^* = E_L^* \vee (A_L \wedge A_L^*)^* = (A_L \wedge A_L^*)^*$ because Theorem 3.1(x). Therefore, we have $A_L \wedge A_L^* = A_L \wedge (A_L \wedge A_L^*)^* \leq (A_L \wedge A_L^*)^*$ and $A_L \wedge A_L^* \leq A_L$. By the assumption $A_L \wedge A_L^* = 0_X$ and hence $A_L = A_L - A_L^* \in I_L$.

v) ⇒ i) Let $A_L \subseteq L^X$ and assume that for all $\chi_{\{x\}} \in A_L$, there exists $U_L \in N_L(x)$ such that $U_L \wedge A_L \in I_L$. Then $A_L \wedge A_L^* = 0_X$. Since $(A_L - A_L^*) \wedge (A_L \wedge A_L^*)^* \leq (A_L - A_L^*) \wedge A_L^* = 0_X$, $A_L - A_L^*$ contains no nonempty subset B_L with $B_L \leq B_L^*$. By (v), $A_L - A_L^* \in I_L$ and hence $A_L = A_L \wedge (1_X - A_L^*) = A_L - A_L^* \in I_L$.

Theorem 4.2. Let (X, τ_L, I_L) be a L-ideal topological space. If τ_L is L-compatible with L-ideal I_L, then the following properties are equivalent

i) For all $A_L \subseteq L^X$, $A_L \wedge A_L^* = 0_X$ implies that $A_L^* = 0_X$,

ii) For all $A_L \subseteq L^X$, $(A_L - A_L^*)^* = 0_X$,

iii) For all $A_L \subseteq L^X$, $(A_L \wedge A_L^*)^* = A_L^*$.

Proof. First, we show that (i) holds if τ_L is L-compatible with L-ideal I_L. Let $A_L \subseteq L^X$ and $A_L \wedge A_L^* = 0_X$. By Theorem 4.1(iii) $A_L \in I_L$ and by Theorem 3.1(x) $A_L^* = 0_X$.

i) ⇒ ii) Assume that for all $A_L \subseteq L^X$, $A_L \wedge A_L^* = 0_X$ implies that $A_L^* = 0_X$. Let $B_L = A_L - A_L^*$, then $B_L \wedge B_L^* = (A_L - A_L^*) \wedge (A_L - A_L^*)^* = (A_L \wedge (1_X - A_L^*)) \wedge (A_L \wedge (1_X - A_L^*))^* \leq (A_L \wedge (1_X - A_L^*)) \wedge (A_L^* \wedge (1_X - A_L^*)^*) = 0_X$. By (i), we have $B_L^* = 0_X$. Hence $(A_L - A_L^*)^* = 0_X$.

ii) ⇒ iii) Assume for all $A_L \subseteq L^X$, $(A_L - A_L^*)^* = 0_X$. $A_L = (A_L - A_L^*) \vee (A_L \wedge A_L^*)$, then

$A_L^* = (A_L - A_L^*)^* \vee (A_L \wedge A_L^*)^* = (A_L \wedge A_L^*)^*$.

iii) ⇒ i) Assume for all $A_L \subseteq L^X$, $A_L \wedge A_L^* = 0_X$ and $(A_L \wedge A_L^*)^* = A^{p^*}$. This implies that $A_L^* = 0_X$.

Theorem 4.3. Let (X, τ_L, I_L) be a L-ideal topological space. Then the following properties are equivalent

i) $\tau_L \wedge I_L = 0_X$,

ii) If $E_L \in I_L$, then $\text{int}(E_L) = 0_X$,

iii) For all $U_L \in \tau_L$, $U_L \leq U_L^*$,

iv) $1_X = 1_X^*$.

Proof. i) ⇒ ii) Let $\tau_L \wedge I_L = 0_X$ and $E_L \in I_L$. Suppose that $\chi_{\{x\}} \in int(E_L)$. Then there exists $U_L \in \tau_L$ such that $\chi_{\{x\}} \in U_L \leq E_L$. Since $E_L \in I_L$ and hence $0_X \neq \{\chi_{\{x\}}\} \leq U_L \in \tau_L \wedge I_L$. This is contrary that $\tau_L \wedge I_L = 0_X$. Therefore, $int(E_L) = 0_X$.

ii) ⇒ iii) Let $\chi_{\{x\}} \in U_L$. Assume $\chi_{\{x\}} \notin U_L^*$ then there exists $V_L \in \tau_L$ such that $U_L \wedge V_L \in I_L$. By (ii), $\chi_{\{x\}} \in U_L \wedge V_L = int(U_L \wedge V_L) = 0_X$. Therefore $\chi_{\{x\}} \in U_L^*$. Hence $U_L \leq U_L^*$.

iii) ⇒ iv) Since 1_X is L-open, then $1_X = 1_X^*$.

iv) ⇒ i) $1_X = 1_X^* = \{\chi_{\{x\}} \in L^X: U_L \wedge 1_X = U_L \notin I_L$ for all $U_L \in N_L(x)\}$. Hence $\tau_L \wedge I_L = 0_X$.

5. L-open set L-operator ψ

Definition 5.1. Let (X, τ_L, I_L) be a L-ideal topological space. An L-operator ψ is defined as follows; for all $A_L \subseteq L^X$, $\psi(A_L) = \{\chi_{\{x\}} \in L^X$: there exists $U_L \in N_L(x)$ such that $U_L - A_L \in I_L\}$. We observe that $\psi(A_L) = 1_X - (1_X - A_L)^*$. The behaviors of the L-operator ψ has been discussed in the following theorem.

Theorem 5.1. Let (X, τ_L, I_L) be a L-ideal topological space and let $A_L, B_L \subseteq L^X$. Then

(i) $\psi(A_L)$ is L-open set,

(ii) $int(A_L) \leq \psi(A_L)$,

(iii) If $A_L \leq B_L$, then $\psi(A_L) \leq \psi(B_L)$,

(iv) $\psi(A_L \wedge B_L) = \psi(A_L) \wedge \psi(B_L)$,

(v) $\psi(A_L \vee B_L) = \psi(A_L) \vee \psi(B_L)$,

(vi) If $U_L \in \tau_L$, then $U_L \leq \psi(U_L)$,

(vii) $\psi(A_L) \leq \psi((\psi(A_L))$,

(viii) $\psi(A_L) = \psi(\psi(A_L))$ if and only If $(1_X - A_L)^* = ((1_X - A_L)^*)^*$,

(ix) If $(A_L - B_L) \vee (B_L - A_L) \in I_L$, then $\psi(A_L) = \psi(B_L)$,

(x) If $E_L \in I_L$, then $\psi(E_L) = 1_X - 1_X^*$,

(xi) If $E_L \in I_L$, then $\psi(A_L - E_L) = \psi(A_L)$,

(xii) If $E_L \in I_L$, then $\psi(A_L \vee E_L) = \psi(A_L)$.

Proof. i) This follows from Theorem 3.1(ii).

ii) From definition of ψ L-operator, $\psi(A_L) = 1_X - (1_X - A_L)^*$. Then $1_X - cl(1_X - A_L) \leq 1_X - (1_X - A_L)^* = \psi(A_L)$. Hence $int(A_L) \leq \psi(A_L)$.

iii) Let $A_L \leq B_L$, then $(1_X - B_L) \leq (1_X - A_L)$. Then from Theorem 3.1(ii), $(1_X - B_L)^* \leq (1_X - A_L)^*$ then $\psi(A_L) \leq \psi(B_L)$.

iv) $\psi(A_L \wedge B_L) = 1_X -(1_X -(A_L \wedge B_L))^* = 1_X -((1_X - A_L) \wedge (1_X - B_L))^* = (1_X - (1_X - A_L)^*) \wedge (1_X - (1_X - B_L)^*) = \psi(A_L) \wedge \psi(B_L)$.

v) $\psi(A_L \vee B_L) = 1_X -(1_X -(A_L \vee B_L))^* = 1_X -((1_X - A_L) \vee (1_X - B_L))^* = (1_X - (1_X - A_L)^*) \vee (1_X - (1_X - B_L)^*) = \psi(A_L) \vee \psi(B_L)$.

vi) Let $U_L \in \tau_L$. Then $(1_X - U_L)$ is a L-closed set and hence $cl(1_X - U_L) = (1_X - U_L)$.

Then $(1_X - U_L)^* \leq cl(1_X - U_L) = (1_X - U_L)$. Hence $U_L \leq 1_X - (1_X - U_L)^*$, so $U_L \leq \psi(U_L)$.

vii) From (i), $\psi(A_L) \in \tau_L$, and from (vii), $\psi(A_L) \leq \psi(\psi(A_L))$.

viii) This follows from the facts:

1. $\psi(A_L) = 1_X - (1_X - A_L)^*$.

2. $\psi(\psi(A_L)) = 1_X - (1_X - (1_X - A_L)^*)^* = 1_X - ((1_X - A_L)^*)^*$.

ix) Let $(A_L - B_L) \vee (B_L - A_L) \in I_L$, and let $A_L - B_L = E_L^1$, $B_L - A_L = E_L^2$. We observe that $E_L^1, E_L^2 \in I_L$, by heredity, and $B_L = (A_L - E_L^1) \vee E_L^2$. Thus $\psi(A_L) = \psi(A_L - E_L^1) = \psi((A_L - E_L^1) \vee E_L^2) = \psi(B_L)$.

x) By Theorem 3.1(x) we obtain if $E_L \in I_L$, then $\psi(E_L) = 1_X - 1_X^*$.

xi) This follows from Theorem 3.1(x) and $\psi(A_L - E_L) = 1_X - (1_X - (A_L - E_L))^* = 1_X -((1_X - A_L) \vee E_L)^* = 1_X - (1_X - A_L)^* = \psi(A_L)$.

xii) This follows from Theorem.3.1(x) and $\psi(A_L \vee E_L) = 1_X - (1_X - (A_L \vee E_L))^* = 1_X - ((1_X - A_L) - E_L)^* = 1_X - (1_X - A_L)^* = \psi(A_L)$.

Theorem 5.2. Let (X, τ_L, I_L) be a L-ideal topological space. If $\eta_L = \{A_L \subseteq L^X : A_L \leq \psi(A_L)\}$. Then η_L is a L-topology on X.

Proof. Let $\eta_L = \{A_L \subseteq L^X : A_L \leq \psi(A_L)\}$. By Theorem.3.1(i), $0_X^* = 0_X$ and $\psi(1_X) = 1_X - (1_X - 1_X)^* = 1_X - 0_X^* = 1_X$. Moreover, $\psi(0_X) = 1_X - (1_X - 0_X)^* = 1_X - 1_X^* = 0_X$. Therefore, we obtain that $0_X \leq \psi(0_X)$ and $1_X \leq \psi(1_X) = 1_X$, and thus 0_X and $1_X \in \eta_L$. Now if $A_L, B_L \in \eta_L$, then by Theorem 5.1 $A_L \wedge B_L \leq \psi(A_L) \wedge \psi(B_L) = \psi(A_L \wedge B_L)$ which implies that $A_L \wedge B_L \in \eta_L$. If $\{A_\alpha\}_{\alpha \in I} \subseteq L^X$ such that $\{A_\alpha\}_{\alpha \in I} \subseteq \eta_L$, then $A_\alpha \leq \psi(A_\alpha) \leq \psi(\vee_{\alpha \in I} A_\alpha)$ for all $\alpha \in I$ and hence $\vee_{\alpha \in I} A_\alpha \leq \psi(\vee_{\alpha \in I} A_\alpha)$. This shows that η_L is a L-topology.

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