The generalized smooth functions Poisson brackets and cohomology

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Abstract

We define the generalized smooth functions and Poisson brackets. We propose a cohomology theory for the generalized functions.

1 The Poisson brackets and cohomology

1.1 Poisson brackets

The Poisson brackets are define for a symplectic manifold (M, ω) [BG] by the formula:

$$\{f, g\} = \omega(df, dg)$$

where $f, g \in \mathcal{C}^\infty(M)$ are smooth functions over the manifold M. Due to the fact that the symplectic form is closed, we have the Jacobi identities:

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$$

1.2 Cohomology

For a manifold M, it is possible to define the cohomology [G] of the differential forms over M with help of the differential d:

$$H^*(M, \mathbf{R}) = H^*(\Lambda^*(TM), d)$$

2 The generalized functions

2.1 Definition

The generalized functions are defined like the commutativ algebra $A(I)$:

$$A(I) = \mathcal{C}^\infty(M)[X_1, X_2, \ldots, X_k]/I$$

where I is an ideal [LB] of the polynomials over the smooth functions

$$\mathcal{C}^\infty(M)[X_1, X_2, \ldots, X_k]$$

such that $A(I)$ is of finite type over $\mathcal{C}^\infty(M)$.

2.2 Example

If I is generated by elements which don't depend on M, then $A(I)$ is a tensor product by the smooth functions and is a trivial fiber bundle in algebras.

$$A(I) = \mathbf{R}[X_1, X_2, \ldots, X_k]/\tilde{I} \otimes_{\mathbf{R}} \mathcal{C}^\infty(M)$$

3 The Poisson brackets

The Poisson brackets for $A(I)$ are defined by the formula:

$$\{a, a'\} = \omega(da, da')$$

where $a, a' \in A(I)$ and $\omega \in \Lambda^2(TA(I))$ is a symplectic form, $TA(I) = Der(A(I))$ are the derivations of $A(I)$.

4 Connections over the generalized functions

A connection ∇ over $A(I)$ is a linear application such that:

$$\nabla_X(f.a) = X(f).a + f.\nabla_X(a)$$

with $a \in A(I)$ and $f \in \mathcal{C}^\infty(M)$, X is a vector field of M.

5 Cohomology of the generalized functions

The cohomology [G] of the algebra of the generalized smooth functions is:

$$H^*(M, I, \mathbf{R}) = H^*(A(I), \mathbf{R})$$

It is a functor from the pairs (M, I), with values in the category of algebras.

References

[BG] R.Bishop, S.Goldberg, "Tensor Analysis on Manifolds", Dover, New-York, 2014.

[G] C.Godbillon, "Eléments de Topologie Algébrique", Hermann, Paris, 1971.

[LB] S.Mac Lane, G.Birkhoff, "Algebra", The Macmillan Company, USA, 1967.