Each topological space X is of the form $\mathrm{Aut}(Y)\backslash Y$

Pierre-Yves Gaillard

We show that for each topological space X there is a topological space Y such that the quotient space $G\backslash Y$ of Y by the action of the automorphism group G of Y is homeomorphic to X.

For each topological space Y write Y_* for the quotient space of Y by the action of the automorphism group of Y.

Theorem 1. *For each topological space X there is a topological space Y such that Y_* is homeomorphic to X.*

Warning: In this text we interpret the following mathematical notions literally: We regard an ordinal α as the set of ordinals less that α, we regard a cardinal as a particular ordinal, and we regard the elements of the quotient $Q = Z/\sim$ of a set Z by an equivalence relation $\sim$ as being the equivalence classes — in particular each element of Q has a well defined cardinality.

To each couple (S, α) where S is an infinite set and α an ordinal we will attach a topological space $X = \Xi(S, \alpha)$ such that X_* is homeomorphic to the set α equipped with the codiscrete topology.

The set X on which the topology will be defined is the disjoint union

$$X := \bigsqcup_{\beta<\alpha} S^{\beta+1}.$$

The orbits of $\mathrm{Aut}(X)$ in X will be the $S^{\beta+1}$, and each of them will be dense. We can assume $\alpha \geq 1$.

We will define a preorder $\leq$, and then a topology τ on X. We will denote respectively by $\mathrm{Aut}(X)$, $\mathrm{Aut}(X, \leq)$ and $\mathrm{Aut}(X, \tau)$ the group of all bijections $X \to X$, the group of all automorphisms of the preordered set X, and the group of all automorphisms of the topological space X. Consider also the group G of all families $(g_\beta)_{\beta<\alpha}$, where each g_β is a bijection $S \to S$, and the injective morphism $i : G \to \mathrm{Aut}(X)$ be defined by

$$i(g)(x) = (g_\gamma(x_\gamma))_{\gamma<\beta+1} \quad \forall\ x \in S^{\beta+1}.$$

Then $\mathrm{Aut}(X, \leq)$, $\mathrm{Aut}(X, \tau)$ and $i(G)$ are subgroups of $\mathrm{Aut}(X)$. These three subgroups will turn out to coincide.

We define $\leq$ by decreeing that, given

$$x = (x_\delta)_{\delta<\beta+1} \in S^{\beta+1} \text{ and } y = (y_\delta)_{\delta<\gamma+1} \in S^{\gamma+1},$$

we have $x \leq y$ if and only if $\beta = 0$ or

$$\beta \leq \gamma \text{ and } x = (y_\delta)_{\delta<\beta+1}.$$

One checks that this is indeed a preorder, that $i(G)$ is contained in $\operatorname{Aut}(X, \leq)$, and more precisely, that $i(G)$ is the subgroup of all those elements of $\operatorname{Aut}(X, \leq)$ which preserve the $S^{\beta+1}$.

We claim that the inclusion $i(G) \subset \operatorname{Aut}(X, \leq)$ is an equality:

$$i(G) = \operatorname{Aut}(X, \leq). \tag{1}$$

Let g be in $\operatorname{Aut}(X, \leq)$ and x be in $S^{\beta+1}$ with $\beta < \alpha$. It suffices to show $gx \in S^{\beta+1}$. If $\beta = 0$ this is clear because S is the set of those elements x of X which satisfy:

$$(\forall y \in X) \quad (y \leq x \implies x \leq y).$$

If $0 < \beta < \alpha$ set

$$X_x := \{y \in X \setminus S \mid y < x\} \cup \{s\},$$

where s is a fixed element of $S \subset X$. Then the preordered set X_x is in fact a totally ordered set isomorphic to the ordinal β. This implies $gx \in S^{\beta+1}$ as desired. As a result,

$$\text{the } S^{\beta+1} \text{ are the orbits of } \operatorname{Aut}(X, \leq) \text{ in } X. \tag{2}$$

There is a unique topology τ on X such that the closed subsets of X are precisely the intersections of finitely generated upward closed subsets of X. We write $\Xi(S, \alpha)$ for the topological space obtained by equipping X with the topology τ. Let $f : X \to \alpha$ be the map sending $x \in S^{\beta+1}$ to β.

Proposition 2. *The topological space $\Xi(S, \alpha)_*$ is codiscrete. The map f induces a bijection $\Xi(S, \alpha)_* \to \alpha$. The set S is the unique maximal codiscrete subspace of $\Xi(S, \alpha)$ having at least two points.*

Proof. Since

$$\text{the } S^{\beta+1} \text{ are dense in } X, \tag{3}$$

it only remains to show

$$\operatorname{Aut}(X, \leq) = \operatorname{Aut}(X, \tau). \tag{4}$$

We have $\operatorname{Aut}(X, \leq) \subset \operatorname{Aut}(X, \tau)$ because τ is defined in terms of $\leq$. But, since $\leq$ can be recovered from τ because we have $x \leq y$ if and only if y is in the closure of $\{x\}$, Equality (4) holds. Proposition 2 follows from (1), (2), (3) and (4). $\square$

Let now X be an arbitrary topological space. For $x, x' \in X$ write $x \sim x'$ if x and x' have the same closure. This is an equivalence relation. We denote the quotient by Q, i.e. $Q := X/{\sim}$; recall that we regard each element Γ of Q as being literally an equivalence class in X. The proof of Theorem 1 will involve some basic properties of the canonical projection $q : X \to Q$. We state these properties below. The proofs are straightforward and left to the reader.

The map $q : X \to Q$ is surjective, continuous and closed. We have $q^{-1}(\Gamma) = \Gamma$ for all $\Gamma \in Q$ [the first Γ is viewed as an element of Q, the second as a subset of X]. If R is a subset of Q, then $q^{-1}(R) = \bigcup_{\Gamma \in R} \Gamma$. Write $\mathcal{C}_X$ and $\mathcal{C}_Q$ for the set of closed subsets of X and of Q, and denote by

$$\mathcal{C}_X \underset{q^{-1}}{\overset{q_*}{\rightleftarrows}} \mathcal{C}_Q$$

the direct and inverse image maps. Then these two maps are inverse bijections compatible with finite unions and arbitrary intersections. If $C \in \mathcal{C}_X$ then we have $C = \bigcup_{\Gamma \subset C} \Gamma$. Using the general notation $\overline{A} :=$ closure of A, we have

$$q_*\left(\overline{Z}\right) = \overline{q_*(Z)} \quad \text{and} \quad \bigcup_{\Gamma \in \overline{R}} \Gamma = q^{-1}\left(\overline{R}\right) = \overline{q^{-1}(R)}$$

for $Z \subset X$ and $R \subset Q$. For $x \in \Gamma \in Q$ we have

$$\overline{\{x\}} = \overline{\Gamma} = q^{-1}\left(\overline{\{\Gamma\}}\right) = \bigcup_{\Delta \subset \overline{\Gamma}} \Delta.$$

If moreover $x' \in \Gamma' \in Q$ then we have

$$\Gamma \subset \overline{\Gamma'} \iff x \in \overline{\{x'\}} \iff \Gamma \in \overline{\{\Gamma'\}},$$

as well as

$$\Gamma \neq \Gamma' \iff \left(\Gamma \cap \overline{\Gamma'} = \varnothing \text{ or } \overline{\Gamma} \cap \Gamma' = \varnothing\right). \tag{5}$$

Write $|T|$ for the cardinality of any set T. For each $\Gamma \in Q$ choose a bijection

$$\phi_\Gamma : |\Gamma| \to \Gamma$$

[here $|\Gamma|$ denotes the cardinality of Γ viewed as a subset of X]; choose also an infinite set S_Γ in such a way that $\Gamma \neq \Delta$ implies $|S_\Gamma| \neq |S_\Delta|$; and set

$$Y_\Gamma := \Xi(S_\Gamma, |\Gamma|).$$

Let the set [not the topological space] Y be the disjoint union of the Y_Γ, and define $f : Y \to X$ by mapping $y \in (S_\Gamma)^{\beta+1} \subset Y_\Gamma \subset Y$ to $\phi_\Gamma(\beta) \in \Gamma \subset X$.

For any subset A of Y and any equivalence class $\Gamma \in Q$ set $A_\Gamma := A \cap Y_\Gamma$, so that we get $A = \bigsqcup_\Gamma A_\Gamma$. It is easy to see that there is a unique topology on Y such that a subset A of Y is closed if and only if the two conditions below hold

- A_Γ is closed in $Y_\Gamma := \Xi(S_\Gamma, |\Gamma|)$ for all $\Gamma \in Q$,
- the set $\{\Gamma \in Q \mid A_\Gamma = Y_\Gamma\}$ is closed in Q.

We equip Y with this topology.

Note 3. *(a) The topology induced on Y_Γ coincides with that of $\Xi(S_\Gamma, |\Gamma|)$.*

(b) Let y be in $Y_\Gamma \subset Y$. If y is in S_Γ, then $\{y\}$ is dense in Y_Γ, and we have

$$\overline{\{y\}} = \overline{S_\Gamma} = \overline{Y_\Gamma} = \bigsqcup_{\Delta \subset \overline{\Gamma}} Y_\Delta.$$

If y is not in S_Γ, then $\{y\}$ is not dense in Y_Γ, and the closures of $\{y\}$ in Y_Γ and in Y coincide.

(c) In particular two distinct points $y, y' \in Y$ have the same closure in Y if and only if $y, y' \in S_\Gamma \subset Y_\Gamma$ for some $\Gamma \in Q$, and thus

(d) the S_Γ are the only maximal codiscrete subsets of Y having at least two points.

Proof of Theorem 1. Let G be the group of all homeomorphisms $g : Y \to Y$. It suffices to show that the orbits of G coincide with the fibers of $f : Y \to X$. It is even enough to prove that any G-orbit is contained in some fiber of f, that is, given $g \in G$ and $y \in Y$, it suffices to check $f(gy) = f(y)$.

If $f(gy), f(y) \in \Gamma$ for some $\Gamma \in Q$, then the result follows from Proposition 2 and Note 3a. It remains to show that the case

$$f(y) \in \Gamma, \quad f(gy) \in \Gamma', \quad \Gamma \neq \Gamma' \tag{6}$$

is impossible. Assume by contradiction that (6) holds. By (5) we have $\Gamma \cap \overline{\Gamma'} = \varnothing$ or $\overline{\Gamma} \cap \Gamma' = \varnothing$ and we can assume the latter. This implies $f(gy) \notin \overline{\Gamma}$, and thus $gy \notin \overline{Y_\Gamma}$ in view of Note 3b. To derive the desired contradiction it suffices to show

$$g\overline{Y_\Gamma} \subset \overline{Y_\Gamma}. \tag{7}$$

We have

$$gS_\Gamma \subset S_\Gamma. \tag{8}$$

Indeed, since the S_Δ, for $\Delta \in Q$, are the maximal codiscrete subspaces of Y having at least two points by Note 3d, they are permuted by g, and, since they have distinct cardinalities, each of them is preserved by g. Now (7) follows from (8) coupled with the equality $\overline{S_\Gamma} = \overline{Y_\Gamma}$ contained in Note 3b. □