Proving Basice Theorems about Chords and Segments via High-School-Level Geometric Algebra

December 9, 2020

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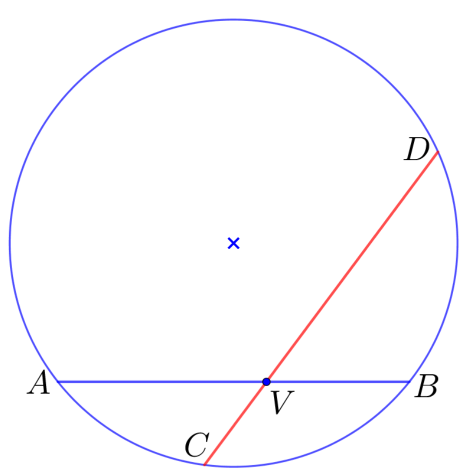
Abstract

We prove the Intersecting-Chords Theorem as a corollary to a relationship, derived via Geometric Algebra, about the product of the lengths of two segments of a single chord. We derive a similar theorem about the product of the lengths of a secant a chord.



"Demonstrate that $(AV)(VB) = (CV)(VD)$*."*

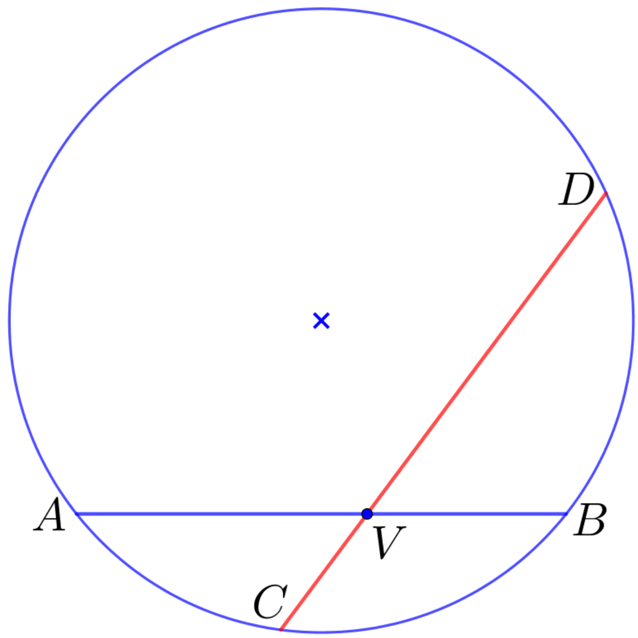


Figure 1: To prove—$(AV)\,(VB) = (CV)\,(VD)$.

1 Introduction

As Proposition 35 in Book III of his *Elements*, Euclid proved the Intersecting-Chords theorem, which in modern terms is stated as in Fig. 1. Here, we will prove that theorem as a corollary to the following:

> The product of the lengths of the two segments into which a point V divides a chord of a circle, is equal to the square of the circle's radius minus the square of the point's distance from the circle's center.

2 Proof

We'll treat the blue chord that we showed in Fig. 1. We set up the proof as in Fig. 2. Point V is an arbitrary point within the circle, and AB is an arbitrary chord with direction $\hat{\mathbf{a}}$. The perpendicular from the circle's center to the chord divides the chord in equal parts (Fig. 3). The "reject" of $\mathbf{v}$ from $\hat{\mathbf{a}}$ is $(\mathbf{v} \wedge \hat{\mathbf{a}})\,\hat{\mathbf{a}}$ ([1], p. 120), the length of which is $\|\mathbf{v} \wedge \hat{\mathbf{a}}\|$. Thus, by the Pythagorean Theorem, each half of the chord measures $\sqrt{r^2 - \|\mathbf{v} \wedge \hat{\mathbf{a}}\|^2}$. Hence, the segments AV and VB measure $\sqrt{r^2 - \|\mathbf{v} \wedge \hat{\mathbf{a}}\|^2} + \boldsymbol{v} \cdot \hat{\mathbf{a}}$ and $\sqrt{r^2 - \|\mathbf{v} \wedge \hat{\mathbf{a}}\|^2} - \boldsymbol{v} \cdot \hat{\mathbf{a}}$, respectively. The product of those lengths is

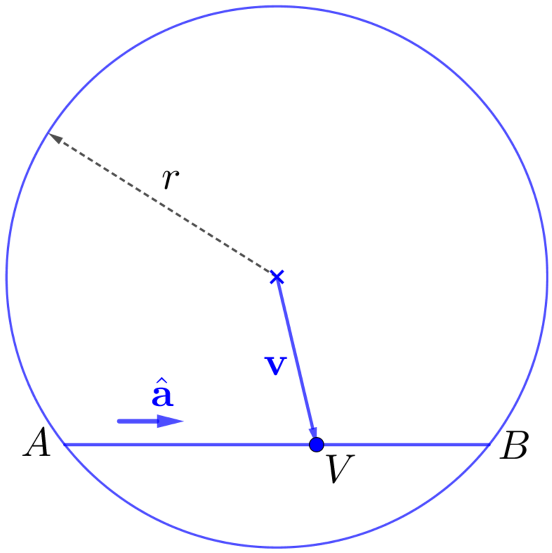


Figure 2: Setting up the proof in GA terms. Point V is an arbitrary point within the circle, and AB is an arbitrary chord, with direction $\hat{\mathbf{a}}$, through V .

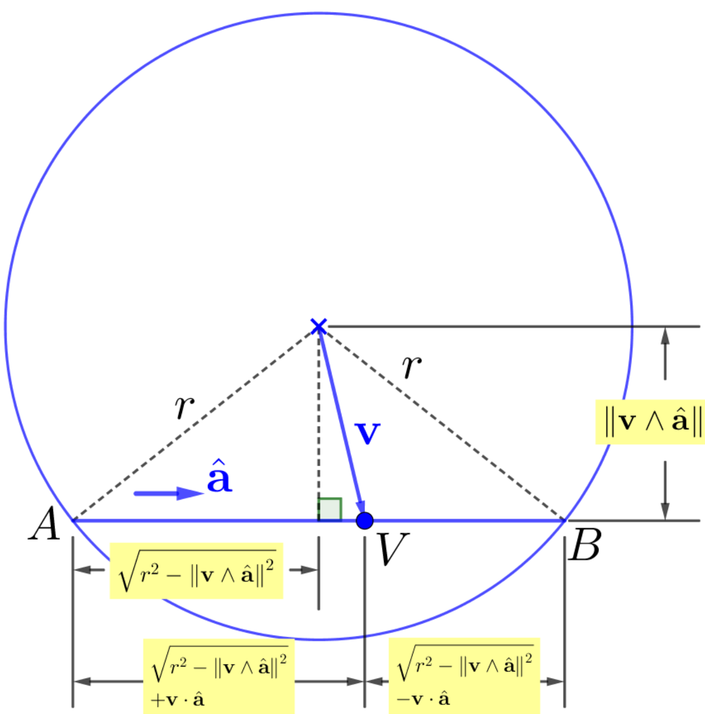


Figure 3: The "reject" of $\mathbf{v}$ from $\hat{\mathbf{a}}$ is $(\mathbf{v} \wedge \hat{\mathbf{a}})\,\hat{\mathbf{a}}$ ([1], p. 120), the length of which is $\|\mathbf{v} \wedge \hat{\mathbf{a}}\|$. Thus, by the Pythagorean theorem, each half of the chord measures $\sqrt{r^2 - \|\mathbf{v} \wedge \hat{\mathbf{a}}\|^2}$. Hence, the segments AV and VB measure $\sqrt{r^2 - \|\mathbf{v} \wedge \hat{\mathbf{a}}\|^2} + \boldsymbol{v} \cdot \hat{\mathbf{a}}$ and $\sqrt{r^2 - \|\mathbf{v} \wedge \hat{\mathbf{a}}\|^2} - \boldsymbol{v} \cdot \hat{\mathbf{a}}$, respectively.

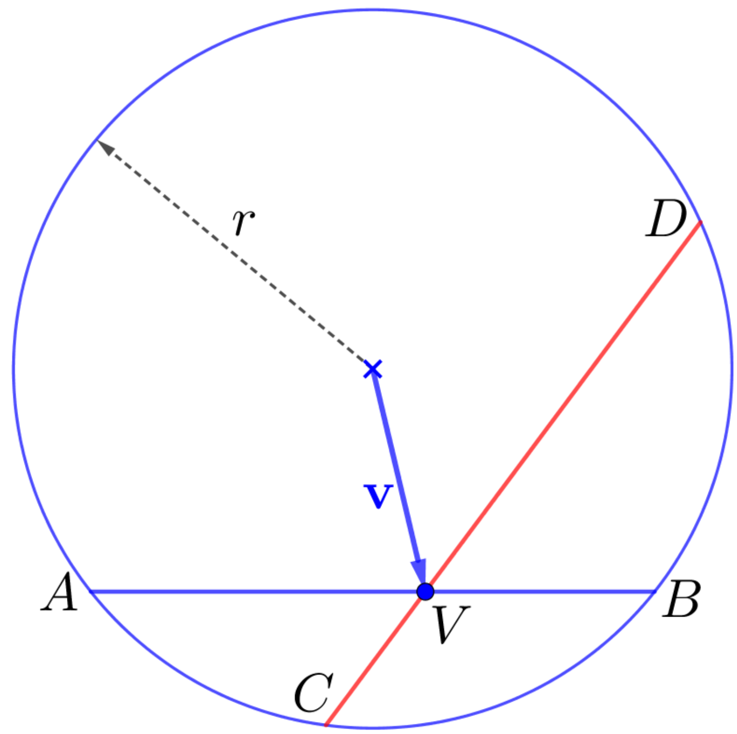


Figure 4: Because V was an arbitrary point, and AB was an arbitrary chord through that point, the result we obtained is valid for every chord through every point within the circle (by the Law of Universal Generalization ([2])). Therefore, $(CV)(VD) = r^2 - v^2 = (AV)(VB)$, and $(CV)(VD) = (AV)(VB)\,\square$.

$$\begin{aligned} r^2 - \|\mathbf{v} \wedge \hat{\mathbf{a}}\|^2 - (\mathbf{v} \cdot \hat{\mathbf{a}})^2 &= r^2 - \left[\|\mathbf{v} \wedge \hat{\mathbf{a}}\|^2 + (\mathbf{v} \cdot \hat{\mathbf{a}})^2\right] \\ &= r^2 - v^2. \end{aligned}$$

Because V was an arbitrary point, and AB was an arbitrary chord through that point, this result is valid for every chord through every point within the circle (by the Law of Universal Generalization ([2])). The Intersecting-Chords Theorem now follows as a corollary (Fig. 4): $(CV)(VD) = r^2 - v^2 = (AV)(VB)$; therefore $(CV)(VD) = (AV)(VB)\,\square$.

Through a similar analysis, we can establish that in Fig 5, $(AB)(AC) = v^2 - r^2$ for any point outside the circle.

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[2] Old Dominion Computer Science, “Universal Generalization” , retrieved 8 December 2020.

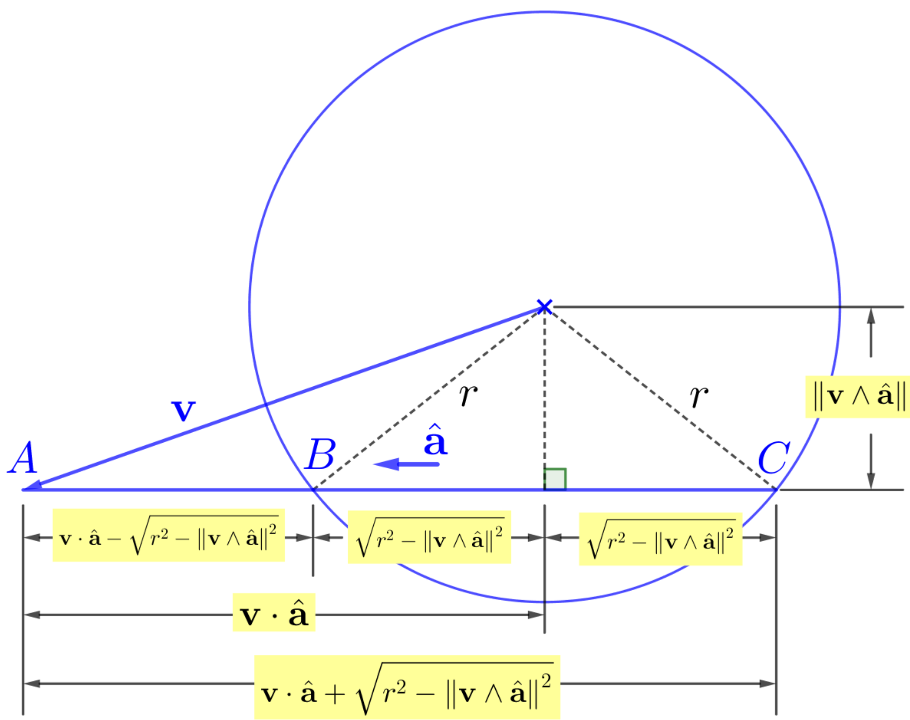


Figure 5: Through an analysis similar to that illustrated in Fig. 3, we can establish that in the present Figure, $(AB)\,(AC) = v^2 - r^2$.