Stirling Numbers Via Combinatorial Sums

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Abstract. In this paper, we have derived a formula to find combinatorial sums of the type $\sum_{r=0}^{n} r^k \binom{n}{r}$ where $k \in \mathbb{N}$. The formula is conveniently expressed as a sum of terms multiplied by certain co-efficients. These co-efficients satisfy a recurrence relation, which is also derived in the process of finding the above sum. Upon solving the recurrence, these numbers turn out to be the Stirling Numbers of the first and second kind. Here on, it is trivial to prove the mutual inverse property of both these sequences of numbers due to linear algebra.

Keywords: Combinatorics · Stirling Numbers

1 Introduction

The inquiry into the matter started with the task to find the sum of $\sum_{r=0}^{n} r \binom{n}{r}$.
Using the binomial expansion -

$$\begin{aligned}(1+x)^n &= \sum_{r=0}^{n} x^r \binom{n}{r} \\ &= \binom{n}{0} + x\binom{n}{1} + x^2\binom{n}{2} + \cdots + x^{n-1}\binom{n}{n-1} + x^n\binom{n}{n}\end{aligned}$$

Thus, we have on differentiating -

$$\frac{d}{dx}(1+x)^n = \frac{d}{dx}\left[\binom{n}{0} + x\binom{n}{1} + \cdots + x^n\binom{n}{n}\right] \tag{1}$$

$$\implies n(1+x)^{n-1} = \binom{n}{1} + 2x\binom{n}{2} + 3x^2\binom{n}{3} + \cdots + nx^{n-1}\binom{n}{n} \tag{2}$$

On setting x = 1 in equation 2, we get -

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$$

$$\implies \sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1} \tag{3}$$

which is our required answer. As a logical extension, one can ask what is the sum of $\sum_{r=0}^{n} r^2\binom{n}{r}$

On multiplying equation (2) with x and then differentiating, we obtain -

$$\begin{aligned} nx(1+x)^{n-1} &= x\binom{n}{1} + 2x^2\binom{n}{2} + 3x^3\binom{n}{3} + \cdots \\ &\quad + nx^n\binom{n}{n} \\ \implies \frac{d}{dx}nx(1+x)^{n-1} &= \frac{d}{dx}\left[x\binom{n}{1} + 2x^2\binom{n}{2} + \cdots + nx^n\binom{n}{n}\right] \\ \implies n(1+x)^{n-1} + nx(n-1)(1+x)^{n-2} &= \binom{n}{1} + 2^2x\binom{n}{2} + 3^2x^2\binom{n}{3} + \cdots \\ &\quad + n^2x^{n-1}\binom{n}{n} \end{aligned} \tag{4}$$

Setting x = 1 in the above equation, we get -

$$\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n} = n2^{n-1} + n(n-1)2^{n-2}$$

$$\implies \sum_{r=0}^{n} r^2\binom{n}{r} = n2^{n-1} + n(n-1)2^{n-2} \tag{5}$$

If we use the notation, $S_{nk} = \sum_{r=0}^{n} r^k\binom{n}{r}$, one can notice the pattern in equations (3) and (5) and conjecture that -

$$S_{n3} = n2^{n-1} + n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3}$$

However, it turns out that it is not true because, on multiplying (4) with x and then differentiating -

$$n(1+x)^{n-1} + 3nx(n-1)(1+x)^{n-2} + nx^2(n-1)(n-2)(1+x)^{n-3}$$
$$= \binom{n}{1} + 2^3 x\binom{n}{2} + 3^3 x^2\binom{n}{3} + \cdots + n^3 x^{n-1}\binom{n}{n} \tag{6}$$

On setting $x = 1$ in (6), we get -

$$S_{n3} = n2^{n-1} + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3} \tag{7}$$

As it can be seen, the $n(n-1)2^{n-2}$ term is padded with a co-efficient of 3. Hence, it does not suffice that the conjecture be so straightforward. Since the general method to find S_{nk} would require differentiating k times, we have to allow for co-efficients that multiply with the type of terms given above.

Thus, the task of finding the sum S_{nk} would be reduced to finding such co-efficients to multiply these terms with and there would be no need to differentiate k times explicitly.

2 A General Approach

If we denote $\prod_{i=0}^{k-1}(n-i) = (n)_k$ (A.K.A the falling factorial for $k \geq 1$), and $T_{ni} = (n)_i 2^{n-i}$, then -

$$S_{nk} = \sum_{i=1}^{k} a_{ki}(n)_i 2^{n-i} = \sum_{i=1}^{k} a_{ki} T_{ni} \qquad 1 \leq k \leq n \tag{8}$$

The a_{ki}'s are general co-efficients which are padded to T_{ni} terms. The subscript a_{ki} has been chosen over a_{ni} in hindsight. The reason will be clear in the subsequent sections.

Note that $(n)_n = n!$ and for any $k > n$ $(n)_k = 0 \implies T_{nk} = 0$. We can also assume $(n)_0 = 1$. Hence it can also be stated -

$$S_{nk} = \sum_{i=1}^{k} a_{ki}(n)_i 2^{n-i} = \sum_{i=1}^{n} a_{ki} T_{ni} \qquad k > n \tag{9}$$

The sub-scripts in the formulae (29) and (9) have a subtlety. Once, the value of n for the problem is fixed, the T_{ni}'s are also fixed. The co-efficients a_{ki} however depend on the value of k in the original sum $\sum_{r=0}^{n} r^k \binom{n}{r}$.

Let's say we are evaluating S_{n3}. The final value of the sum is obtained by setting $x = 1$ in polynomial in x of (6) -

$$n(1+x)^{n-1} + 3nx(n-1)(1+x)^{n-2} + nx^2(n-1)(n-2)(1+x)^{n-3}$$

This can be generalised saying that the final series is obtained by setting $x = 1$ in a polynomial $S_{nk}^{(x)}$. In the previous case, this was $S_{n3}^{(x)}$. By analysing the patterns in the previous steps, if $S_{n4}^{(x)}$ had to be derived, we would multiply $S_{n3}^{(x)}$ with x and differentiate with respect to x. Generally -

$$S_{n(k+1)}^{(x)} = \frac{d}{dx}\left[xS_{nk}^{(x)}\right]$$

$$\implies S_{n(k+1)}^{(x)} = S_{nk}^{(x)} + x\frac{d}{dx}S_{nk}^{(x)} \tag{10}$$

where

$$S_{nk}^{(x)} = \sum_{i=1}^{k} a_{ki}(n)_i x^{i-1}(1+x)^{n-i} \tag{11}$$

It is easy to see that every polynomial $S_{nk}^{(x)}$ is a polynomial in x of degree $n - 1 \ \forall\ k \in \mathbb{N}$. This is because of recurrence relation (10) where the next polynomial of the sequence is obtained by first multiplying the previous polynomial by x and then differentiating. The boundary case of $S_{nn}^{(x)}$ can be considered as well. The last term in the sum will be -

$$S_{nn}^{(x)} = \cdots + a_{kn} n! x^{n-1}$$

To find $S_{n(n+1)}^{(x)}$, we need to multiply by x and then differentiate. Referring to equation (11), it can be seen that the last term of sum for $S_{n(n+1)}^{(x)}$ will again be $n!x^{n-1}$. However, it will be multiplied by a different co-efficient, namely $a_{(n+1)n}$. Hence, the clipping of the sum to n in (9) can be understood.

In other words, for $k > n$, the co-efficients a_{km} where $m > n$ may exist, but they are not required to evalute the original sum.

In general, the set of co-efficients $\{a_{(k+1)1}, a_{(k+1)2}\ , \cdots\}$ can be expressed in terms of $\{a_{k1}, a_{k2}, \cdots\}$ because of mixing of terms of equal powers and lowering of the exponent on differentiation in the derivation of the next polynomial of the sequence. The recurrence among polynomials (10) must naturally translate to a recurrence among co-efficients. In other words, we must be able to express the set $\{a_{(k+1)1}, a_{(k+1)2}\ , \cdots\}$ in terms of $\{a_{k1}, a_{k2}, \cdots\}$.

3 Obtaining The Recurrence Relation For a_{ki}

Let us assume $k \leq n$ and evalute all the terms in the sequence $\{a_{k1}, a_{k2}, \cdots, a_{kn}\}$.

The first term of $S_{nk}^{(x)}$ from (11) will always be $n(1+x)^{n-1}$. Due to the fact that $a_{11} = 1$ and (10), this term will always get carried on to the next polynomial sequence, and hence $a_{k1} = 1 \ \forall \ k \in \mathbb{N}$.

The last term is the result of differentiating k times and then multiplying by x. It is a newly generated term in the sequence and not carried over by previous polynomials of the sequence. Thus $a_{kk} = 1$ for $k \leq n$.

Consider the i^{th} term of $S_{nk+1}^{(x)}$ and its co-efficient $a_{(k+1)i}$. Due to (10) and (11), the contributing terms to the i^{th} term from $S_{nk}^{(x)}$ are -

$$\begin{aligned} a_{(k+1)i}(n)_i x^{i-1}(1+x)^{n-i} &= a_{ki}(n)_i x^{i-1}(1+x)^{n-i} \\ &+ a_{ki}(n)_i (1+x)^{n-i} x \frac{d}{dx} x^{i-1} \\ &+ a_{k(i-1)}(n)_{i-1} x^{i-2} x \frac{d}{dx}(1+x)^{n-(i-1)} \end{aligned}$$

$$\begin{aligned} \implies a_{(k+1)i}(n)_i x^{i-1}(1+x)^{n-i} &= i a_{ki}(n)_i x^{i-1}(1+x)^{n-i} \\ &+ a_{k(i-1)}(n)_i x^{i-1}(1+x)^{n-i} \end{aligned} \tag{12}$$

We can extract the recurrence relation by equating the co-efficients in (12) -

$$a_{(k+1)i} = i a_{ki} + a_{k(i-1)} \tag{13}$$

The co-efficients a_{ki} satisfy the same recurrence as that of the famous **Stirling Numbers of the Second Kind** and with the same base cases.

Hence these co-efficients *must* be the Stirling Numbers of the second kind. In their more common notation, they satisfy the recurrence -

$$\left\{ {k+1 \atop i} \right\} = i \left\{ {k \atop i} \right\} + \left\{ {k \atop i-1} \right\} \tag{14}$$

4 Triangle of Stirling Numbers of The Second Kind

It is well known that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$. Here the exponent in r^k is $k = 0$.

Thus keeping in line with (29), we get -

$$S_{n0} = a_{00} T_{n0} \tag{15}$$

Since $(n)_0 = 1$ and $T_{n0} = 2^n$, from (15), it can be seen that -

$$a_{00} = 1 \tag{16}$$

For $k > 0$, the $T_{n0} = 2^n$ term in the expression for S_{nk} is missing. Hence -

$$a_{k0} = 0 \tag{17}$$

Since for $i > k$, the T_{ni} terms do not contribute to the sum as the upper limit of the summation in (29) is k. Thus we can safely define -

$$a_{ki} = 0 \qquad\qquad i > k \tag{18}$$

Thus we can combine (29) and (9) because of (16), (17) and (18) to generalise -

$$S_{nk} = \sum_{i=1}^{n} a_{ki} T_{ni} \qquad\qquad k \in \mathbb{N} \tag{19}$$

We can display these numbers in a triangular fashion (for $0 \le k \le 10$) by building the recurrence -

5 Verifying The Formula for S_{nk}

We can confirm the validity of the method for two examples - one with $k \le n$ and another with $k > n$.

k \ i	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	1	3	1	0	0	0	0
4	0	1	7	6	1	0	0	0
5	0	1	15	25	10	1	0	0
6	0	1	31	90	65	15	1	0
7	0	1	63	301	350	140	21	1

Table 1. Stirling Numbers of the Second Kind

5.1 Case 1 - $n = 5, k = 4$

$$\begin{aligned}\sum_{r=0}^{5} r^4 \binom{5}{r} &= 0^4\binom{5}{0} + 1^4\binom{5}{1} + 2^4\binom{5}{2} + 3^4\binom{5}{3} + 4^4\binom{5}{4} + 5^4\binom{5}{5} \\ &= 0 + (1\times 5) + (16\times 10) + (81\times 10) + (256\times 5) + (625\times 1) \\ &= 2880\end{aligned}$$

From (19), we have -

$$\begin{aligned}S_{54} &= \sum_{i=0}^{5} a_{4i}T_{5i} \\ &= a_{40}T_{50} + a_{41}T_{51} + a_{42}T_{52} + a_{43}T_{53} + a_{44}T_{54} + a_{45}T_{55} \\ &= 0 + (1\times 80) + (7\times 160) + (6\times 240) + (1\times 240) + 0 \\ &= 2880\end{aligned}$$

Hence it matches!

5.2 Case 2 - $n = 3, k = 6$

$$\begin{aligned}\sum_{r=0}^{3} r^6 \binom{3}{r} &= 0^6\binom{3}{0} + 1^6\binom{3}{1} + 2^6\binom{3}{2} + 3^6\binom{3}{3} \\ &= 0 + (1\times 3) + (64\times 3) + (729\times 1) \\ &= 924\end{aligned}$$

From (19), we have -

$$\begin{aligned} S_{36} &= \sum_{i=0}^{3} a_{6i} T_{3i} \\ &= a_{60}T_{30} + a_{61}T_{31} + a_{62}T_{32} + a_{63}T_{33} \\ &= 0 + (1 \times 12) + (31 \times 12) + (90 \times 6) \\ &= 924 \end{aligned}$$

It matches too!

6 Another approach

We shall derive the inverse relation i.e T_{nk} in as a linear sum of S_{nk}'s. From this point on, we shall asume strictly $k \leq n$

$$(1+x)^n = \sum_{r=0}^{n} x^r \binom{n}{r}$$

Differentiating k times -

$$\left[\prod_{i=0}^{k-1}(n-i)\right](1+x)^{n-k} = \sum_{r=k}^{n}\left[\prod_{i=0}^{k-1}(r-i)\right] x^{r-k}\binom{n}{r} \tag{20}$$

The product on the LHS is just the falling factorial. One can expand the product on the RHS as (b_{ki}'s are general co-efficients) -

$$\prod_{i=0}^{k-1}(r-i) = \sum_{i=1}^{k} b_{ki} r^i \tag{21}$$

Plugging in (21) in (20) and multiplying both sides by x^k -

$$\begin{aligned} (n)_k x^k (1+x)^{n-k} &= \sum_{r=k}^{n}\sum_{i=1}^{k} b_{ki} r^i x^r \binom{n}{r} \\ &= \sum_{r=0}^{n}\sum_{i=1}^{k} b_{ki} r^i x^r \binom{n}{r} - \sum_{r=0}^{k-1}\sum_{i=1}^{k} b_{ki} r^i x^r \binom{n}{r} \\ &= \sum_{i=1}^{k} b_{ki} \sum_{r=0}^{n} r^i x^r \binom{n}{r} - \sum_{r=0}^{k-1} x^r \binom{n}{r}\left[\sum_{i=1}^{k} b_{ki} r^i\right] \\ &= \sum_{i=1}^{k} b_{ki} \sum_{r=0}^{n} r^i x^r \binom{n}{r} - \sum_{r=0}^{k-1} x^r \binom{n}{r}\left[\prod_{i=0}^{k-1}(r-i)\right] \end{aligned} \tag{22}$$

Plugging in $x = 1$ in (22) and using our defined notations -

$$\begin{aligned} (n)_k 2^{n-k} &= \sum_{i=1}^{k} b_{ki} \left[\sum_{r=0}^{n} r^i \binom{n}{r} \right] - \sum_{r=0}^{k-1} \binom{n}{r} \left[\prod_{i=0}^{k-1} (r-i) \right] \\ \implies T_{nk} &= \sum_{i=1}^{k} b_{ki} S_{ni} - \sum_{r=0}^{k-1} \binom{n}{r} \left[\prod_{i=0}^{k-1} (r-i) \right] \end{aligned} \tag{23}$$

In the summation indexed by r on the RHS of (23), r can only take values from $\{0, 1, \cdots, k-1\}$. The product vanishes for every value of r as i indexes from 0 to $k-1$. Hence the last summation is identically zero. Ultimately -

$$T_{nk} = \sum_{i=1}^{k} b_{ki} S_{ni} \tag{24}$$

The subscripts of the terms here follow the same pattern as that of (29) but due to different reasons. The b terms are indexed by k first because b_{ki} represents the coefficient of r^i in $\prod_{i=0}^{k-1}(r-i)$ and the maximum power in this product is r^k, and hence the indexing.

One can also observe that (24) is simply the inverse relationship of $S_{nk} = \sum_{i=1}^{k} a_{ki} T_{ni}$.

We can expect to derive, in a similar fashion, a recurrence relation for $\{b_{(k+1)1}, b_{(k+1)2}, \cdots\}$ in terms of $\{b_{k1}, b_{k2}, \cdots\}$.

7 Obtaining The Recurrence Relation For b_{ki}

From the definition of b_{ki}, it can be seen that the co-efficient of the lowest power is $b_{k1} = \prod_{i=1}^{k-1}(-1)^i i = (-1)^{k-1}(k-1)!$. Also, the co-efficient of the highest power is $b_{kk} = 1$.

We have established the base cases and can continue to establish the recurrence relation. From (21), we have -

$$\begin{aligned}
\prod_{i=0}^{k}(r-i) &= \sum_{i=1}^{k+1} b_{(k+1)i} r^i \\
\implies \sum_{i=1}^{k+1} b_{(k+1)i} r^i &= (r-k)\left[\prod_{i=0}^{k-1}(r-i)\right] \\
&= (r-k)\left[\sum_{i=1}^{k} b_{ki} r^i\right] \\
&= \sum_{i=1}^{k} b_{ki} r^{i+1} - \sum_{i=1}^{k} k b_{ki} r^i
\end{aligned}$$

$$\sum_{i=1}^{k+1} b_{(k+1)i} r^i = \sum_{i=2}^{k-1} b_{k(i-1)} r^i - \sum_{i=1}^{k} k b_{ki} r^i \tag{25}$$

The summation limits only differ in the boundary cases which have already been derived. Hence, from (25), we can state a recurrence relation -

$$b_{(k+1)i} = b_{k(i-1)} - k b_{ki} \tag{26}$$

Again, we discover that b_{ki} satisfy the same recurrence as that of **Signed Stirling Numbers of the First Kind** and with the same base cases. As follows from the previous argument, these *must* be the Signed Stirling Numbers of the First Kind. They are called *signed* as some of these numbers are negative.

$$\begin{bmatrix} k+1 \\ i \end{bmatrix} = \begin{bmatrix} k \\ i-1 \end{bmatrix} - k \begin{bmatrix} k \\ i \end{bmatrix} \tag{27}$$

8 Triangle of Signed Stirling Numbers of the First Kind

Setting the index $k = 0$ in (24) and letting i run from 0, we get -

$$T_{n0} = b_{00} S_{n0} \tag{28}$$

Since $T_{n0} = S_{n0} = 2^n$, we can set $b_{00} = 1$

Also, by the recurrence relation (26), we can see that $b_{k1} = (-1)^k (k-1)!$ iff the recurrence satisfied by $i = 1$ is $b_{(k+1)1} = -k b_{k1}$. Thus, $b_{k0} =$ has to be satisfied. In the later sections, this can also shown to be true by linear algebra.

We can again display these numbers in a triangular fashion -

k \ i	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	-1	1	0	0	0	0	0
3	0	2	-3	1	0	0	0	0
4	0	-6	11	-6	1	0	0	0
5	0	24	-50	35	-10	1	0	0
6	0	-120	274	-225	85	-15	1	0
7	0	720	-1764	1624	-735	175	-21	1

Table 2. Signed Stirling Numbers of the First Kind

9 Verifying The Formula For T_{nk}

It has already been stated that $k \leq n$. Hence, we shall verify the formula (24) for two cases :

9.1 Case 1 - $n = 5, k = 3$

$$\begin{aligned}
T_{53} &= 240 \\
S_{51} &= 80 \\
S_{52} &= 240 \\
S_{53} &= 800 \\
b_{31}S_{51} + b_{32}S_{52} + b_{33}S_{53} &= (2 \times 80) - (3 \times 240) + (1 \times 800) \\
&= 240
\end{aligned}$$

Thus it mactches as expected.

9.2 Case 2 - $n = 6, k = 4$

$$\begin{aligned}
T_{64} &= 1440 \\
S_{61} &= 192 \\
S_{62} &= 672 \\
S_{63} &= 2592 \\
S_{64} &= 10752 \\
b_{41}S_{61} + b_{42}S_{62} + b_{43}S_{63} + b_{44}S_{64} &= (-6 \times 192) + (11 \times 672) - (6 \times 2592) + (1 \times 10752) \\
&= 1440
\end{aligned}$$

This too matches as expected!

10 Proving The Inverse Nature of The Two Sequences

We shall concern ourselves only with a square sub-section of the table of Stirling numbers (i.e $1 \leq k \leq n$)

Observe the formula for $T_{nk} = (n)_k 2^{n-k} = n \times (n-1) \times \cdots \times (n-k+1)2^{n-k}$. This can be looked at as a polynomial in n of degree k. The set $\{T_{n1}, T_{n2}, \cdots, T_{nn}\}$ essentially contains polynomials in n of degree $1, 2, \cdots$ upto n. Hence, it can act as a basis for the vector space denoted by $span\left(\{n, n^2, n^3, \cdots, n^n\}\right)$

Similarly, the set $\{S_{n1}, S_{n2}, \cdots, S_{nn}\}$ is also a set of polynomials in n and (29) basically represents a linear transformation in the co-ordinatization of the $\{T_{nk}\}$ basis state.

By using (29) and (24), we get -

$$\begin{aligned}
T_{nk} &= \sum_{i=1}^{k} b_{ki} S_{ni} \\
&= \sum_{i=1}^{k} b_{ki} \left(\sum_{l=1}^{i} a_{il} T_{nl} \right) \\
&= \sum_{i=1}^{k} \sum_{l=1}^{i} b_{ki} a_{il} T_{nl}
\end{aligned}$$

The inner sum runs from $l = 1$ to i. We note that $i \leq k$ due to the outer sum and it does not make a difference to change the upper limit of the inner sum to k. Since $\{T_{nk}\}$ represents a basis state, we must have -

$$\begin{aligned}
\sum_{i=1}^{k} \sum_{l=1}^{k} b_{ki} a_{il} T_{nl} &= \sum_{l=1}^{k} \delta_{kl} T_{nl} \\
&= T_{nk}
\end{aligned}$$

With the Einstein summation convention, we can say $b_{ki} a_{ij} = \delta_{kj}$ and that the square matrices represented by a square-section of the two tables of the Stirling Numbers are inverse with respect to each other.

11 Verifying The Inverse Property

Let us verify our result with a 6×6 sub-matrix of the two tables.

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 & 0 \\ 24 & -50 & 35 & -10 & 1 & 0 \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}$$

Carrying out the matrix multiplication $B \times A$ results in the identity matrix I_6. We could have included the 00^{th} index term in the matrices and it is trivial to confirm that the result would have remained the same.

References

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