A Simple Proof for Almost Everywhere Convergence to Imply Weak Convergence of Induced Measures

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Abstract

Establishing on a finite measure space the implication for almost everywhere convergence to imply weak convergence of the corresponding induced measures (in particular to imply convergence in distribution) is usually indirect, convergence in measure being the transition. We give a simple, pedagogically informative proof for the implication.

Keywords: almost everywhere convergence; convergence in distribution; weak convergence of measures

MSC 2020: 28A20; 60B10

1 Introduction

Let $(\Omega, \mathscr{F}, \mathbb{M})$ be a finite measure space; let S be a metric space; and let $f, f_1, f_2, \cdots : \Omega \to S$ be $\mathscr{F}$-measurable. It is a classical result that $f_n \to_{a.e.} f$ implies $\mathbb{M}_{f_n} \rightsquigarrow \mathbb{M}_f$, which is usually obtained as a corollary jointly from the convergence $f_n \to_{a.e.} f$ and that $f_n \to f$ in $\mathbb{M}$-measure. Here (and throughout), the relation $\rightsquigarrow$ denotes the weak convergence relation, and $\mathbb{M}_h$ denotes, whenever legitimate, the induced measure of $\mathbb{M}$ by h. As a special case of the convergence theorem, for $\mathbb{M}$ a probability measure, that $f_n \to_{a.e.} f$ implies that (f_n) converges in distribution to f.

We intend to give a simple proof for the classical convergence theorem, which, we would believe, is also informative from the teaching viewpoint. Moreover, our proof is not more "difficult" nor more "advanced" than the typical one.

2 Proof of the Convergence Theorem

We should like to prove

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Theorem. *Let* $(\Omega, \mathscr{F}, \mathbb{M})$ *be a finite measure space; let* S *be a metric space; and let* $f, f_1, f_2, \dots : \Omega \to S$ *be* $\mathscr{F}$*-measurable. If* $f_n \to_{a.e.} f$*, then* $\mathbb{M}_{f_n} \rightsquigarrow \mathbb{M}_f$.

Proof. Let $g : S \to \mathbb{R}$ be a bounded continuous function. Then

$$g \circ f_n \to_{a.e.} g \circ f$$

by the convergence assumption. If $|g|_{L^\infty}$ is the L^∞-norm of g, then

$$\sup_n |g \circ f_n| \leq |g|_{L^\infty} \quad a.e.{-}\mathbb{M}.$$

Since Ω is by assumption a finite measure space, the constant function $\omega \mapsto |g|_{L^\infty}$ on Ω is contained in $L^1(\mathbb{M})$; the Lebesgue dominated convergence theorem then implies

$$\int_\Omega g \circ f_n \,\mathrm{d}\mathbb{M} \to \int_\Omega g \circ f \,\mathrm{d}\mathbb{M}.$$

But $\int_\Omega g \circ f_n \,\mathrm{d}\mathbb{M} = \int_S g \,\mathrm{d}\mathbb{M}_{f_n}$ for all $n \in \mathbb{N}$, and the same applies to f; the desired weak convergence follows. □