Computation of Cup i- product and Steenrod operations on the classifying space of finite groups

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ABSTRACT
The aim of this paper is to give a computational treatment to compute the cup i-product and Steerod operations on cohomology rings of as many groups as possible. We find a new method that could be faster than the methods of Rusin, Vo, and Guillot. There are some available approaches for computing Steenrod operations on these cohomology rings (see [11, 20, 24]). The computation of all Steenrod squares on the Mod 2 cohomology of all groups of order dividing 32 and all but 58 groups of order 64; partial information on Steenrod square is obtained for all but two groups of order 64. For groups of order 32 this paper completes the partial results due to Rusin[20], Thanh Tung Vo [24] and Guillot [11].

KEYWORDS
cohomology operation; cup product; Bockstein homomorphism; cup i-product, Steenrod square.

1. Introduction

For an explicit description of the homomorphism Sq^i we follow [3, 18] and [21], but as in [6] specialize to the case $B = B_G$. This special case has the advantage that we can resort to the fact that the universal covering space $\tilde{B} = E_G$ is contractible, and that the chain complex $C_*(\tilde{B})$ is thus exact. In this case, we can avoid having to implement the Acyclic Carrier Theorem. Additionally, we focus our concerns on the classifying space $B = B_G$ of a finite p-group. For any group G there are various ways to construct CW-space E_G on which G acts freely, with cell permuted by the action. We say that E_G is the total space for G. The quotient space $B_G = E_G/G$ obtained by killing the action is a classifying space for G. The cellular chain complex $C_*(E_G)$ is a free $\mathbb{Z}G$-resolution and $C_*(B_G) = C_*(E_G) \otimes_{\mathbb{Z}G} \mathbb{Z}$. Thus $H^*(B_G, \mathbb{Z}_p) = H^*(G, \mathbb{Z}_p)$. The cellular chain complex $C_*(E_G)$ is the chain complex with $C_k(E_G)$ the free abelian group generated by all k-cells in E_G

$$C_k(E_G) = \sum_{i=1}^{b_k} \mathbb{Z}e_i^k$$

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where b_k is the number of k-cells; and where the boundary map is defined by using singular homology (see [2, 12] for details). In our work, we use a different approach to compute the Steenrod square, namely the cup i-product that is constructed in Mosher and Tangora's book [18] and Steenrod's paper [21]. We are interested in finding a new method that could be faster than the methods of Rusin, Vo, and Guillot. For instance, the running time for Small Group(32,32) is approximatly 15 minutes.Vo [24] mentioned that the running time for Small Group (32, 32) is approximately 2 months.

Definition 1.1. [15, 17]The homology groups of a CW-space E_G are defined to be the homology groups of the cellular chain complex $C_*(E_G)$.

Definition 1.2. [22] A cohomology operation of type (A, n, B, m) is a natural transformation, $\phi : H^n(-, A) \longrightarrow H^m(-, B)$, that for any spaces, X, Y and for any map $f : X \longrightarrow Y$ there are functions ϕ_X, ϕ_Y satisfying the naturality condition $f^*\phi_Y = \phi_X f^*$ (i.e., the following diagram commutes).

$$\begin{tikzcd} H^n(X,A) \arrow[r, "\phi_X"] & H^m(X,B) \\ H^n(Y,A) \arrow[u, "f^*"] \arrow[r, "\phi_Y"] & H^m(Y,B) \arrow[u, "f^*"'] \end{tikzcd}$$

We know the Bockstein homomorphism as an example of a cohomology operation of type $(\mathbb{Z}_p, n, \mathbb{Z}_p, n + 1)$.

Definition 1.3. Consider the cohomology of a space B with coefficients in the field of p elements, p a prime number. The Steenrod squares are cohomology operations of type $(\mathbb{Z}_p, n, \mathbb{Z}_p, n + i)$ for $p = 2$,

$$Sq^i : H^n(B, \mathbb{Z}_2) \longrightarrow H^{n+i}(B, \mathbb{Z}_2),\ i \geq 0 \tag{1}$$

and the Steenrod powers are cohomology operations of type $(\mathbb{Z}_p, n, \mathbb{Z}_p, n + i(p - 1))$ for $p > 2$,

$$P^i : H^n(B, \mathbb{Z}_p) \longrightarrow H^{n+i(p-1)}(B, \mathbb{Z}_p),\ i \geq 0. \tag{2}$$

The Steenrod squares Sq^i of 1, defined for $i \geq 0$ satisfy the following properties:

1. Sq^1 is the Bockstein homomorphism (denoted β in the previous chapter).
2. Sq^0 is the identity homomorphism.
3. if $deg(x) = i$ then $Sq^i(x) = x^2$.
4. if $deg(x) < i$ then $Sq^i(x) = 0$.
5. (Cartan formula) $Sq^n(xy) = \sum_{i+j=n} Sq^i(x) \smile Sq^j(y)$.
6. $Sq^i(x + y) = Sq^i(x) + Sq^i(y)$.
7. Naturality: means that for any map $f : B \longrightarrow B'$, $Sq^i(f^*) = f^*(Sq^i)$ for the cohomology homomorphism f^* induced by the map f.
8. (Adem relations) $Sq^aSq^b = \sum_{c=0}^{a/2} \binom{b-c-1}{a-2c} Sq^{a+b-c}Sq^c$, for $a < 2b$, where Sq^aSq^b denotes the composition of the Steenrod squares and the binomial coefficient is taken modulo 2.

Proposition 1.4. *[1] Let $G = L \times K$ be the direct product of groups L and K, for*

$l \in H^*(L,\mathbb{Z}_2)$, $k \in H^*(K,\mathbb{Z}_2)$ and $(l \times k) \in H^*(L \times K,\mathbb{Z}_2)$ we have $Sq^n(l \times k) = \sum_{i+j=n} Sq^i(l) \times Sq^j(k)$

Proof. The first projection $p_1 : L \times K \twoheadrightarrow L$ and the second projection $p_2 : L \times K \twoheadrightarrow K$ induce ring inclusions $p_1^* : H^*(L,\mathbb{Z}_2) \rightarrowtail H^*(L \times K,\mathbb{Z}_2)$ and $p_2^* : H^*(K,\mathbb{Z}_2) \rightarrowtail H^*(L \times K,\mathbb{Z}_2)$ from property 5 we have $Sq^n(l \times k) = Sq^n((l \times 1) \cup (1 \times k)) = \sum_i Sq^i(l \times 1)Sq^{n-i}(1 \times k)$, then $Sq^i(l \times 1) = Sq^i(p_1^*(l)) = p_1^* Sq^i(l) = Sq^i(l) \times 1$.

$$\begin{tikzcd}
H^*(L,\mathbb{Z}_2) \arrow[r, "p_1^*"] \arrow[d, "Sq^i"'] \arrow[dr, "Sq^i \times 1"] & H^*(L \times K,\mathbb{Z}_2) \arrow[d, "Sq^i"] & H^*(K,\mathbb{Z}_2) \arrow[l, "p_2^*"'] \arrow[d, "Sq^i"] \arrow[dl, "1 \times Sq^i"'] \\
H^{*+i}(L,\mathbb{Z}_2) \arrow[r, "p_1^*"'] & H^{*+i}(L \times K,\mathbb{Z}_2) & H^{*+i}(K,\mathbb{Z}_2) \arrow[l, "p_2^*"]
\end{tikzcd}$$

Accordingly $Sq^n(l \times k) = \sum_{i+j=n} Sq^i(l \times 1)Sq^j(1 \times k) = \sum_{i+j=n}(Sq^i(l) \times 1) \cup (1 \times Sq^j(k)) = \sum_{i+j=n} Sq^i(l) \times Sq^j(k)$. $\square$

Example 1.5. To compute all Steenrod squares on $H^*(G,\mathbb{Z}_2)$, we use the formulas provided by properties in definition 1.3 as well as the cup product and Bockstein homomorphism. For instance, the following commands show that for the small group $G = G_{32,4}$ of order 32 and number 4 in the small group library of the computer algebra system GAP, in cases where this cohomology algebra is generated by elements of degrees 1 and 2, the image of the homomorphism

$$Sq^3 : H^4(G_{32,4},\mathbb{Z}_2) \longrightarrow H^7(G_{32,4},\mathbb{Z}_2)$$

is generated by xv^3; here the algebra $H^*(G_{32,4},\mathbb{Z}_2)$ is generated by four elements x, y of degree 1 and z, v of degree 2.

GAP session

```
gap> G:=SmallGroup(32,4);;
gap> A:=ModPSteenrodAlgebra(G,7);;
gap> gens:=ModPRingGenerators(A);
[ v.1, v.2, v.3, v.5, v.6 ]
gap> List(gens,A!.degree);
[ 0, 1, 1, 2, 2 ]
gap>  H4:=Filtered(Basis(A),x->A!.degree(x)=4);;
gap>  Sq3H4:=List(H4,x->Sq(A,3,x));
[ 0*v.1, 0*v.1, 0*v.1, v.33+v.35, 0*v.1 ]
gap> PrintAlgebraWordAsPolynomial(A, Sq3H4[4]);
v.6*v.6*v.6*v.2
```

We use the HAP fuction $Sq(A,n,u)$ which makes use of the Cartan relation and the properties in Definition 1.3, but it does not make any use of the Adem relations.

Algorithm 1.1 Steenrod squares for Mod-2 cohomology rings

Input:

- ModPSteenrodAlgebra, an integer $n \geq 0$ and a homogeneous element $u \in H^*(G, \mathbb{Z}_2)$.

Output: The information for the Steenrod squares Sq^{2^k}.

Procedure:

if $i = 0$ **then**
    $Sq^i(u) = u$
end if
if $i >$ degrees of all homogeneous elements **then**
    $Sq^i = 0$
    if degree u=i **then**
        $Sq^i(u) = u^2$
        if Length $u > 1$ **then**
            $Sq^i(u_1 + u_2 + ... + u_n) = Sq^i(u_1) + Sq^i(u_2) + ... + Sq^i(u_n)$
            if u is product of generators and Length $u > 1$ **then**
                for $k \in \{1, ..., i\}$ **do**
                    $a := Sq^k(u_1)$
                    $b := Sq^{k-i}(u_{\{2..Length(u)\}})$
                    $Sq^i(u_1 u_{n-1}) = \sum a * b$
                end for
            end if
        end if
    end if
end if
return the information of Steenrod squares.
EndProcedure:

2. Implementation of the cup i-product

For an explicit description of the homomorphism Sq^i we follow [3, 18] and [21], but as in [6] specialize to the case $B = B_G$. This special case has the advantage that we can resort to the fact that the universal covering space $\tilde{B} = E_G$ is contractible, and that the chain complex $C_*(\tilde{B})$ is thus exact. In this case, we can avoid having to implement the Acyclic Carrier Theorem.

2.1. *Construction of cup i-product*

Let S^∞ denote the infinite-dimensional sphere. There exists a cell structure on S^∞ as a CW-complex, with two cells in each dimension. Let $R_*^{C_2}$ be the free $mathbbZC_2$-resolution of $\mathbb{Z}$ obtained from the cellular chain complex for S^∞ with one free $mathbbZC_2$-generator k^n in each degree n.

$$R_*^{C_2} : \cdots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{t-1} \mathbb{Z}[C_2] \xrightarrow{t+1} \mathbb{Z}[C_2] \xrightarrow{t-1} \mathbb{Z}[C_2]$$

Here $C_2 = \langle t : t^2 = 1\rangle$ is the group of order 2 generated by t. Let $B = B_G$ for some group G and set $R_*^G = C_*(\tilde{B})$. Let $e_1^n, e_2^n, ...$ denote free generators for the free $\mathbb{Z}G$-module R_n^G. The group C_2 acts on $R_p^G \otimes_{\mathbb{Z}} R_q^G = R_n^{G\times G}$ by the interchange map

$$\tau : R_*^G \otimes R_*^G \longrightarrow R_*^G \otimes R_*^G,$$

$$t \cdot (g' e_i^p \otimes g'' e_j^q) = (-1)^{pq} g'' e_j^q \otimes g' e_i^p$$

The tensor product $R_*^{G\times G} = R_*^G \otimes_{\mathbb{Z}} R_*^G$ is a free $\mathbb{Z}[G \times G]$-resolution of $\mathbb{Z}$ with free $\mathbb{Z}[G \times G]$-generators $e_i^p \otimes e_j^q$ in degree $n = p + q$. With a free abelian group $R_n^{G\times G}$ is freely generated via $g' e_i^p \otimes g'' e_j^q$, such that $(g', g'') \in G \times G$. The action extends to an action of $C_2 \times G$ via the formula

$$(t, g) \cdot (g' e_i^p \otimes g'' e_j^q) = (-1)^{pq} gg'' e_j^q \otimes gg' e_i^p.$$

We view $R_*^{G\times G}$ as an exact chain complex of $\mathbb{Z}[G \times G]$-modules. The tensor product $R_*^{C_2\times G} = R_*^{C_2} \otimes_{\mathbb{Z}} R_*^G$ is a free $\mathbb{Z}[C_2 \times G]$-resolution of $\mathbb{Z}$.
We will consider the $\mathbb{Z}[C_2 \times G]$-equivariant homomorphism

$$\phi_0 : R_0^{C_2} \otimes R_0^G \longrightarrow R_0^G \otimes R_0^G,$$

defined by $\phi_0(k^0 \otimes e_i^0) = e_i^0 \otimes e_i^0$. The map ϕ_0 extends, using the freeness of $R_*^{C_2} \otimes R_*^G$ and the exactness of $R_*^G \otimes R_*^G$ to a $\mathbb{Z}[C_2 \times G]$-equivariant chain map

$$\phi_* : R_*^{C_2} \otimes R_*^G \longrightarrow R_*^G \otimes R_*^G \tag{3}$$

and the diagram below describes the ϕ_n,

$$\begin{tikzcd}
R_n^{C_2\times G} \arrow[r, "\partial_n"] \arrow[d, "\phi_n"] & R_{n-1}^{C_2\times G} \cdots \arrow[r] \arrow[d, "\phi_{n-1}"] & R_1^{C_2\times G} \arrow[r, "\partial_1"] \arrow[d, "\phi_1"] & R_0^{C_2\times G} \arrow[d, "\phi_0"] \\
R_n^{G\times G} \arrow[r, "\partial_n"] & R_{n-1}^{G\times G} \cdots \arrow[r] & R_1^{G\times G} \arrow[r, "\partial_1"] & R_0^{G\times G}
\end{tikzcd}$$

and ϕ_* is unique up to chain homotopy. The chain map ϕ_* is computed from a contracting homotopy (or discrete vector field) on $R_*^{G\times G}$. Let $\Delta : G \longrightarrow G \times G$ be the diagonal map $\Delta(x) = (x, x)$, $\pi_1 : C_2 \times G \longrightarrow C_2$ be the first projection given by $\pi_1(t, g) = t$, and $\pi_2 : C_2 \times G \longrightarrow G$ be the second projection given by $\pi_2(t, g) = g$. Let $i_1 : G \longrightarrow G\times G$, $i_1(g) = (g, 1)$ and $i_2 : G \longrightarrow G\times G$, $i_2(g) = (1, g)$ the specified embeddings to $G\times G$. We now consider the cochain complex $C^*(R_*^G) = Hom_{\mathbb{Z}G}(R_*^G, \mathbb{Z})$. The group $C^n(R_*^G)$ is a free abelian group with free abelian generators $\bar{e_i^n}$ corresponding to the free $\mathbb{Z}G$-generators e_i^n of R_n^G. More precisely, $\bar{e_i^n} : R_n^G \longrightarrow \mathbb{Z}$ is the $\mathbb{Z}G$-equivariant homomorphism sending $e_i^n \mapsto 1$, $e_j^n \mapsto 0$ for $j \neq i$. This notation describes a homomorphism

$$R_n^G \longrightarrow C^n(R_*^G),\ u \mapsto \bar{u}.$$

For each integer $i \geq 0$ define a $\mathbb{Z}$-linear **cup-i product**

$$C^p(R_*^G) \otimes_{\mathbb{Z}} C^q(R_*^G) \longrightarrow C^{p+q-i}(R_*^G), \ \bar{u} \otimes \bar{v} \mapsto \bar{u} \smile_i \bar{v} \tag{4}$$

by the formula

$$(\bar{u} \smile_i \bar{v})(c) = (\bar{u} \otimes \bar{v})\phi_{p+q}(k^i \otimes c) \tag{5}$$

for $c \in R_{p+q-i}^G$.

Algorithm 2.1 The function HAP-PHI

Input:

- G finite group and
- an integer $n \geq 0$.

Output: A list $[\phi_*, R_*^{C2\times G}, R_*^{G\times G}]$.

Procedure:

1: Construct diagonal function $G \longrightarrow G \times G$.
2: Construct interchange map $\tau : R_*^G \otimes R_*^G \longrightarrow R_*^G \otimes R_*^G, \ t \cdot (g'e_i^p \otimes g''e_j^q) = (-1)^{pq} g''e_j^q \otimes g'e_i^p$.
3: Construct $\phi_0 : R^{C_2} \otimes R_0^G \longrightarrow R_0^G \otimes R_0^G$, $\phi_0(k^0 \otimes e_i^0) = e_i^0 \otimes e_i^0$.
4: The output $\{\phi_1, \phi_2, ..., \phi_n\}$, $R_n^{C_2\times G}$ and $R_n^{G\times G}$.
5: **EndProcedure:**

3. The computation of Steenrod squares

In this section, we give a computational treatment to compute Steenrod squares on cohomology rings of finite 2-groups. There are alternative approaches for computing Steenrod square by Rusin, Guillot and Vo. Steenrod squares for all 2-groups of order 32 have been computed by Rusin, except for two of them, the groups of numbers 8 and 44 in the GAP's library of small groups; and most of the computations were done by hand. For more details, see [20]. Guillot calculated for the five groups of order 8, for 28 of the 51 groups of order 32, for 13 of the 14 groups of order 16, and for 61 of the 267 groups of order 64. The Steenrod squares for all 2-groups of order less than or equal 16, all 2-groups of order 32, except one group of number 8, (He mentions to compute 210 groups of order 64 but gives no details) were calculated by Thanh Tung Vo.

Theorem 3.1. *[18] The operation*

$$C^n(R_*^G) \longrightarrow C^{2n-i}(R_*^G), \ \bar{u} \mapsto \bar{u} \smile_i \bar{u}$$

induces a homomorphism

$$Sq_i : H^n(G, \mathbb{Z}_2) \longrightarrow H^{2n-i}(G, \mathbb{Z}_2).$$

The homomorphism

$$Sq^i = Sq_{n-i} : H^n(G, \mathbb{Z}_2) \longrightarrow H^{n+i}(G, \mathbb{Z}_2) \tag{6}$$

is independent of the choices in ϕ_* made in 3 and satisfies the properties of Definition 1.3.

We use the HAP function Mod2Steenrodalgebra(G,n) which is an implementation of Sq^i defined in 6 that inputs a finite 2-group G and a non-negative integer and returns the first nth degree of Steenrod squares.

Example 3.2. To compute the Steenrod square Sq^k for each generator and each positive 2-power $k = 2^i < degree(x)$, $x \in H^*(G, \mathbb{Z}_2)$ for $G_{32,10}$ the small group of order 32 and number 10 in GAP's library, see the following GAP session.

GAP session

```
gap> G:=SmallGroup(32,10);;
gap> A:=Mod2SteenrodAlgebra(G,8);;
gap> gens:=ModPRingGenerators(A);
[ v.1, v.2, v.3, v.4, v.6, v.9, v.15 ]
gap> List(gens,A!.degree);
[ 0, 1, 1, 2, 2, 3, 4 ]
gap> List(gens,x->Sq(A,2,x));
[ 0*v.1, 0*v.1, 0*v.1, v.13, v.11, v.21, v.23+v.24+v.25 ]
gap> PrintAlgebraWordAsPolynomial(A, List(gens,x->Sq(A,2,x))[4]);
v.4*v.4
gap> PrintAlgebraWordAsPolynomial(A, List(gens,x->Sq(A,2,x))[5]);
v.4*v.3*v.3
gap> PrintAlgebraWordAsPolynomial(A, List(gens,x->Sq(A,2,x))[6]);
v.15*v.2
gap> PrintAlgebraWordAsPolynomial(A, List(gens,x->Sq(A,2,x))[7]);
v.4*v.4*v.6
gap> List(gens,x->Sq(A,4,x));
[ 0*v.1,0*v.1,0*v.1,0*v.1,0*v.1,0*v.1, v.37+v.38+v.39+v.40+v.45 ]
gap> PrintAlgebraWordAsPolynomial(A, List(gens,x->Sq(A,4,x))[7]);
v.15*v.15
```

Also we using the HAP command CohomologicalData(G,n) to determine and print details of the group order, group number, cohomology ring generators with degree and relations and the Steenrod square $Sq^k(x)$ for each generator x and each positive 2-power $k = 2^i < degree(x)$. If we want the cohomology ring details printed to a file then this file name is included as an optional third input to the command. Also the command CohomologicalData(G,n) returns the following information for $n = 6$ and $G_{32,30}$ the small group of order 32 and number 30 in GAP's library. It prints correct information for the cohomology ring $H^*(G, \mathbb{Z}_2)$ of a 2-group G provided that the integer n is at least the maximal degree of a relator in a minimal set of relators for the ring, moreover n trems of a free $\mathbb{Z}G$-resolution is enough to compute the whole mod-2 cohomology ring by the tables of King and Green [10]. When Steenrod squares are composed, the composetion satisfy certain relations known the Adem relation 1.3,

$$Sq^a Sq^b = \sum_{j=0}^{a/2} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j,$$

for $a < 2b$, where $Sq^a Sq^b$ denotes the composition of the Steenrod squares and the binomail coefficient is taken modulo 2. A detailed proof the Adem relation can found in [1, 18].

Suppose that $i = a+b$ where $b = 2^k$ and $0 < a < 2^k$. Then we can rewritten the Adem relations in the form

$$\binom{b-1}{a} Sq^i = Sq^a Sq^b + \sum_{j=1}^{a/2} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j,$$

if $a \leq b-1$ that $\binom{b-1}{a} \equiv 1(mod 2)$ which proof in [Proposition 15.6 [1]], we can used recursively to express Sq^i in terms of Sq^{2k}. For instance,there are relations $Sq^1 Sq^1 = 0$, $Sq^1 Sq^3 = 0$,...;$Sq^1 Sq^{2n+1} = 0$ and $Sq^3 = Sq^1 Sq^2$, $Sq^5 = Sq^1 Sq^4$,... $Sq^{2n+1} = Sq^1 Sq^{2n}$. The expression of Sq^6 in terms of squares of the form Sq^{2^k} as $Sq^6 = Sq^2 Sq^4 + Sq^5 Sq^1$,...,$Sq^{4n+2} = Sq^2 Sq^{4n} + Sq^{4n+1} Sq^1$. Also, $Sq^3 Sq^{4n+2} = 0$, $Sq^{2n-1} Sq^n = 0$, and more details see [18].

Group order: 32

Group number: 30

Group description: (C4 x C2 x C2) : C2

Cohomology generators

Degree 1: a, b, c

Degree 2: d

Degree 3: e, f

Degree 4: g

Cohomology relations

$1 : a * f$

$2 : a * c$

$3 : a * b + c^2$

$4 : c^2 * d + c * e$

$5 : b * c * d + b * e + c * e + c * f$

$6 : b * c^2$

$7 : b^3 * e + b^2 * c * f + c * d * f + e * f$

$8 : b^4 * d + b^2 * c * f + c^2 * g + c * d * e + f^2$

$9 : a^4 * d + a^3 * e + a^2 * g + c * d * e + e^2$

Poincare series

$(x^3 + x + 1)/(-x^6 + 2 * x^5 - x^4 + x^2 - 2 * x + 1)$

Steenrod squares

$Sq^1(d) = d * a + d * c$

$Sq^1(e) = d * a * b + e * b$

$Sq^2(e) = d * a * a * a + d * b * b * c + d * d * c + g * a$

$Sq^1(f) = d * b * b + d * b * c + e * b$

$Sq^2(f) = d * d * c + f * b * b + g * c$

$Sq^1(g) = d * a * a * a + d * b * b * c + e * a * a$

$Sq^2(g) = d * a * a * a * a + d * b * b * b * b + d * d * a * a + d * d * a * b + d * d * b * b + d *$

$d*b*c+e*a*a*a+e*b*b*b+f*f+g*b*b$

4. Detection Method

Definition 4.1. [2] Let G be a finite group, $K \leq G$ a proper subgroup and $i : K \longrightarrow G$ the inclusion map. The induced cohomology homomorphism

$$i_K^G : H^*(G, \mathbb{Z}_2) \longrightarrow H^*(K, \mathbb{Z}_2)$$

is called restriction map.

Definition 4.2. [16] Let G be a finite group, and κ a collection of proper subgroups of G. We say that κ *detects* the cohomology $H^*(G, \mathbb{Z}_2)$ if the product of the restriction maps

$$\prod_{K\in\kappa} i_K^G : H^*(G, \mathbb{Z}_2) \longrightarrow \prod_{K\in\kappa} H^*(K, \mathbb{Z}_2)$$

is an injection.

The HAP function Mod2Steenrodalgebra(G,n) and ModPSteenrodAlgebra(G,n) become less practical as the size of the finite 2-group G increases. For large G it can be useful using Definition 4.2. We are using the HAP function InducedSteenrodHomomorphism(f,n) which input a homomorphism $f : K \longrightarrow G$ of finite 2-groups and a positive integer n. It returns a triple $[HG, HK, l]$, where $HG = H^{\leq n}(G, \mathbb{Z}_2)$, $HK = H^{\leq n}(K, \mathbb{Z}_2)$ and l is a list $[l_1, l_2, ..., l_n]$ with $l_i : H^i(G, \mathbb{Z}_2) \longrightarrow H^i(K, \mathbb{Z}_2)$ the linear homomorphism induced by f. For each Steenrod square Sq^{2^k} and each element v in the generating set of $H^*(G, \mathbb{Z}_2)$, we have

$$\prod_{K\in\kappa} i_K^G(Sq^{2^k}(v)) = \prod_{K\in\kappa} Sq^{2^k}(i_K^G(v)).$$

If $i_K^G(Sq^{2^k}(v)) = Sq^{2^k}(i_K^G(v)) = 0$, then the Steenrod square $Sq^{2^k}(v) \in Kernel(l_1) \cap Kernel(l_2) \cap ... \cap Kernel(l_n)$.

Example 4.3. Consider the group G=$G_{32,10}$ namely the small group of order 32 and number 10 in GAP's library. The six generators of $H^*(G, \mathbb{Z}_2)$ can be denoted $a_1, b_1, c_2, d_2, e_3, f_4$, where the index of each generator indicates the degree of the element. One can check that the cohomology of G is not detected by any family of proper subgroups as in example 3.2. However it is possible to determine some information about Steenrod squares for G using Steenrod square computations in a proper subgroup K. The group $G_{32,10}$ has three subgroups of order 16; we denote them by $[K_{16,4},\ K_{16,12},\ K_{16,5}]$. Let $K_{16,12} < G$ denote the subgroup of order 16 and number 12. The following GAP session uses a Steenrod square computation in $H^*(K, \mathbb{Z}_2)$ in order to determine that either $Sq^2(f_4) = c*d*b*b+c*c*d$ or $Sq^2(f_4) = c*c*d$ [In Example 3.2, we know that $Sq^2(f_4) = c*c*d$]

GAP session

```
gap> G:=SmallGroup(32,10);;
gap> K:=MaximalSubgroups(G)[2];;
gap> f:=GroupHomomorphismByFunction(K,G,x->x);;
gap> L:=InducedSteenrodHomomorphisms(f,8);;
gap> HG:=L[1];;
gap> HK:=L[2];;
gap> iota:=L[3];;
gap> gens:=ModPRingGenerators(HG);
[ v.1, v.2, v.3, v.4, v.6, v.9, v.15 ]
gap> List(gens,HG!.degree);
[ 0, 1, 1, 2, 2, 3, 4 ]
gap> w:=Sq(HK,2,Image(iota[5],gens[7]));
v.34
gap> P:=List(PreImagesElm(iota[7],w),x->x);
[ v.22+v.24+v.25, v.23+v.24+v.25 ]
gap> PrintAlgebraWordAsPolynomial(HG,P[1]);
v.4*v.6*v.3*v.3 + v.4*v.4*v.6
gap> PrintAlgebraWordAsPolynomial(HG,P[2]);
v.4*v.4*v.6
```

Also, we are using the HAP command CohomologicalDetected(G,K,n) which inputs a finite 2-group, K are maximal subgroups and positive integer n to determine and print details of the group order, group number, a list of maximal subgroups, cohomology ring generators with their degree, and Steenrod squares Sq^k for each generator. If a file name is included as an optional fourth input to the command then the details are printed to this file.

Algorithm 4.1 Detects the cohomology $H^*(G, \mathbb{Z}_2)$

Input:

- Finite 2-group, maximal subgroups, an nonnegative integer n.
- The information for the Steenrod squares Sq^{2^k}.

Procedure:

```
for i ∈ gens of H*(G, Z_2) do
    for k ∈ degree gens in H*(G, Z_2) do
        for j ∈ Maximal subgroup L do
            I_i := H^i(G, Z_2) ⟶ H^i(L_j, Z_2)
            if k = 2^{log(k,2)} then
                w = Sq^k(I_{i+1}(gens_i))
                if w = 0 then
                    P = Kernel(I_{i+k+1})
                    else
                    P = I^{-1}_{i+k+1}(w)
                end if
            end if
        end for
    end for
end for
return the information of Steenrod squares.
EndProcedure:
```

Example 4.4. From the previous example we can use the command CohomologicalDetected(G,K,n), which returns the following information for $G_{32,10}$, $n = 8$ and the three maximal subgroups of order 16.
Group order: 32
Group number: 10
Group description: Q8 : C4
Subgroup List:[[16, 4], [16, 12], [16, 5]]
Cohomology generators
Degree 1: a, b
Degree 2: c, d
Degree 3: e
Degree 4: f
Steenrod squares $Sq^1(c) = [0, d*b, c*b, c*b + d*b]$
$Sq^1(d) = [0, d*b, c*b, c*b + d*b]$
$Sq^1(e) = [0, c*b*b + d*b*b, c*b*b, d*b*b]$
$Sq^2(e) = [f*a + f*b, f*a, c*c*b + f*a + f*b, c*c*b + f*a, c*d*b + f*a + f*b, c*d*b + f*a, c*c*b + c*d*b + f*a + f*b, c*c*b + c*d*b + f*a]$
$Sq^1(f) = [0, f*b, c*c*b, c*c*b + f*b, c*d*b, c*d*b + f*b, c*c*b + c*d*b, c*c*b + c*d*b + f*b]$
$Sq^2(f) = [c*c*b*b + c*c*d, c*c*b*b + c*c*d + f*b*b, c*d*b*b + c*c*d, c*d*b*b + c*c*d + f*b*b, c*c*b*b + c*d*b*b + c*c*d, c*c*b*b + c*d*b*b + c*c*d + f*b*b, c*c*d, c*c*d + f*b*b]$
Cohomology generators
Degree 1: a, b
Degree 2: c, d
Degree 3: e

Degree 4: f
Steenrod squares
$Sq^1(c) = [0, c*a]$
$Sq^1(d) = [c*b+d*b, c*a+c*b+d*b]$
$Sq^1(e) = [0, d*b*b]$
$Sq^2(e) = [0, f*a, c*c*a, c*c*a+f*a]$
$Sq^1(f) = [c*c*b+c*d*b, c*c*b+c*d*b+f*a, c*c*a+c*c*b+c*d*b, c*c*a+c*c*b+c*d*b+f*a]$
$Sq^2(f) = [c*d*b*b+c*c*d, c*c*d]$
Cohomology generators
Degree 1: a, b
Degree 2: c, d
Degree 3: e
Degree 4: f
Steenrod squares
$Sq^1(c) = [0, d*b, c*a+c*b, c*a+c*b+d*b]$
$Sq^1(d) = [c*b, c*b+d*b, c*a, c*a+d*b]$
$Sq^1(e) = [0, c*b*b+d*b*b, c*b*b, d*b*b]$
$Sq^2(e) = [f*a, f*b, c*c*a+c*c*b+c*d*b+f*a, c*c*a+c*c*b+c*d*b+f*b, c*d*b+f*a, c*d*b+f*b, c*c*a+c*c*b+f*a, c*c*a+c*c*b+f*b]$
$Sq^1(f) = [c*c*b, c*c*b+f*a+f*b, c*c*a+c*d*b, c*c*a+c*d*b+f*a+f*b, c*c*b+c*d*b, c*c*b+c*d*b+f*a+f*b, c*c*a, c*c*a+f*a+f*b]$
$Sq^2(f) = [c*c*b*b+c*c*d, c*c*b*b+c*c*d+f*b*b, c*d*b*b+c*c*d, c*d*b*b+c*c*d+f*b*b, c*c*b*b+c*d*b*b+c*c*d, c*c*b*b+c*d*b*b+c*c*d+f*b*b, c*c*d, c*c*d+f*b*b]$

In the example above, we can find the Steenrod square taking the intersection of the Steenrod squares over all maximal subgroups. For instance $Sq^1(c) = 0$ by taking the intersection between the list of $Sq^1(c)$ over $K_{16,4}$, $Sq^1(c)$ over $K_{16,12}$ and $Sq^1(c)$ over $K_{16,5}$. We proceed analogously with the other generators.
Occasionally, the HAP command CohomologicalDetected(G,K,n) gives us a huge information therefore we use the HAP commend CohomologicalDetectedIntersection(G,K,n) which input a finite 2-group, K a maximal subgroups and positive integer n, and output the information of Steenrod square see the example below.

Example 4.5. Let the small group $G_{64,175}$ of order 64 and number 175 in GAP's library. The five generators of $H^*(G, \mathbb{Z}_2)$ can be denoted a_1, b_1, c_1, d_2, e_4, where the index of each generator indicates the degree of the element. We can use the commend CohomologicalDetectedIntersection(G,K,n) returns the following information for $G_{64,175}$, $n = 8$ and maximal subgroups and we can see the Steenrod square $Sq^2(e)$ for generator e, which have two result we can choose one of them.
Group order: 64
Group number: 175
Group description:$C4 : Q16$
Maximal Subgroups:[[32, 41], [32, 35], [32, 41], [32, 3], [32, 41], [32, 35], [32, 41]]
Cohomology generators
Degree 1: a, b, c
Degree 2: d
Degree 4: e
Steenrod squares

$Sq^1(d) = d * a$
$Sq^1(e) = d * a * a * b + d * a * a * c$
$Sq^2(e) = d{*}a{*}a{*}b{*}c + d{*}d{*}a{*}a + d{*}d{*}a{*}b + d{*}d{*}a{*}c, d{*}d{*}a{*}a + d{*}d{*}a{*}b + d{*}d{*}a{*}c$

In special cases, we can use the direct products to compute the Steenrod square. Let $G = L \times K$ be the direct product of groups L and K. Then it is known that the projections $G \twoheadrightarrow L$ and $G \twoheadrightarrow K$ induce ring inclusions $H^*(L, \mathbb{Z}_2) \rightarrowtail H^*(G, \mathbb{Z}_2)$ and $H^*(K, \mathbb{Z}_2) \rightarrowtail H^*(G, \mathbb{Z}_2)$ see Proposesion 1.4, and so we can think of $H^*(L, \mathbb{Z}_2)$ and $H^*(K, \mathbb{Z}_2)$ as subrings of $H^*(G, \mathbb{Z}_2)$. It is known that the ring $H^*(G, \mathbb{Z}_2)$ is generated by the generators of two subrings $H^*(L, \mathbb{Z}_2)$ and $H^*(K, \mathbb{Z}_2)$. In fact we have an isomorphism $H^*(G, \mathbb{Z}_2) \cong H^*(L \times K, \mathbf{Z}_2) \cong H^*(L, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(K, \mathbb{Z}_2)$. All of this means that the cohomology ring and the Steenrod squares for $H^*(G, \mathbb{Z}_2)$ are completly determined by the ring and operations for $H^*(L, \mathbb{Z}_2)$ and $H^*(K, \mathbb{Z}_2)$. In other words, if a group G of order 64 is a direct product then we already have the Steenrod operations since we have computed the rings and operations for groups of order less than 64. The Steenrod operations on $H^*(G, \mathbb{Z}_2)$ are determined by the Cartan formula $Sq^n(x \times y) = \sum_{i+j=n} Sq^i(x) \times Sq^j(y)$. For example the group $G_{32,48}$ is the group of order 32 and number 48, and we can compute the Steenrod Squares over it by use of the Cartan formula; in fact, $G_{32,48} \cong G_{16,13} \times C_2$.

5. Experimental results

We present a sample of results from our implementation of Steenrod operations on finite 2-groups.

The implementation is able to compute the Steenrod squares of the following finite 2-groups:

- All groups of order 2,4,8,16 and 32 .
- all but 58 groups of order 64. Partial information on Steenrod squares is obtained for all but two groups of order 64.

We use different methods to compute the above.

Method (1)- For all groups of order less than or equal 32, except the groups $G_{32,8}$, $G_{32,44}$, $G_{32,47}$, $G_{32,48}$, $G_{32,49}$ and $G_{32,50}$ and some groups of order 64 which have cohomology generators in degree one and two, we used only the HAP command `CohomologicalData(G,n)` .

Method (2)-For the following groups of order 64 we used in addition to the HAP command `CohomologicalData(G,n)`, the HAP command `CohomologicalDetected(G,K,n)` and the command `CohomologicalDetectedIntersection(G,K,n)` on the specified maximal subgroups.

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