The Mathematical Underpinnings of Utility Functions and their Broader Implications on Game Theory

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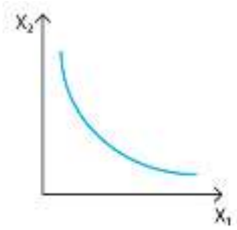
Abstract:
The predominating objective of this short paper is to lay forth a series of results derived from the optimization of canonical utility and indifference functions (insofar as they are used conventionally), and then analogizing them in the context of payoffs. It concerns the categorizations of utility functions, indifference curves and other analytical constraints.

To commence, it draws a generalization as to how utilities change with regards to marginal rates of substitution. Following this, it invokes a series of arguments as to how they can be maximized in both univariate and multivariate states.

Marginal Rates of Substitution in the context of opportunity cost

Firstly, consider an indifference curve associated with any two commodities/goods.



Notice that this curve is the equivalent production possibilities frontier in an economy with decreasing opportunity cost. As the obtainment/production of any one good grows, the equivalent sacrifice made with regards to the other continually decreases over the range of both axes. A convex curve implies precisely this.

A consumer (or agent) will necessarily be indifferent to any combination on the curve relative to any other. The tangent to an indifference curve derives its marginal rate of substitution at any point.

$$MRS_{x1x2} = -\frac{d[x2]}{d[x1]}$$

$$MRS_{x_2x_1} = -\frac{d[x1]}{d[x2]}$$

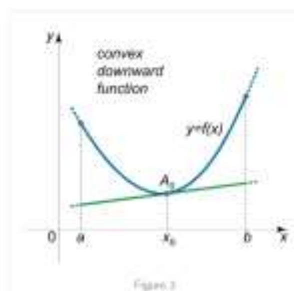
wherein MRS_{ab} is representative of the degree to which one is willing to interchange a quantity associated with commodity a in exchange for commodity b.

Both signs are reversed in order to obtain absolute values of MRS_{ab}. In either case, either $d[x2]$ or $d[x1]$ lies in the negative.

In the event that the overlying indifference curve is concave, it will be naturally representative of an increasing opportunity cost amongst its distributed commodities.

While all formulations for marginal rates of substitutions are invariant to their associated opportunity cost, one must recognize that their relative size with regards to the individual goods they are connected to are not.

For any function that is convex downward, the tangent that is drawn in association with any point on its curve necessarily lies below the curve.



Consequently, the value of the related tangent function (whose gradient equals MRS_{ab}) will be inferior to or equal to the value of its indifference curve/PPF (herein thought of as a second commodity).

Habitually, one describes indifference curves for two commodities using

$$f(x,y) = c$$

wherein c remains a constant.

Note that the first convex curve (for two goods) described above assumes the form of a decreasing exponential function. Subsequently, if one were to derive its derivative MRS_{ab}, one must first reconstitute it to obtain the form

$$f(a) = b$$

or

$$b = \gamma k^a$$

wherein $$k < 1$$

and $$\gamma > 0$$

Herein, one must presuppose the above constraints to ensure that the resultant function assumes the convex curve necessitated by a decreasing opportunity cost.

Note: $f(a) = b$ does not imply that the extent of one's obtainment or utility drawn from commodity a unconditionally determines one's consumption of commodity b. Instead, it is solely a precondition to discerning a general form of MRS_{ab}, and a few other conditions associated with a convex PPF.

$$MRS_{ab} = -\frac{d[b]}{d[a]} = -\frac{d}{d[a]}\gamma k^a$$

$$-\frac{d}{d[a]}\gamma k^a = -\gamma \frac{d}{d[a]} k^a = -\gamma \ln[k]\, k^a$$

Parametrically, this implies that the marginal rate of substitution associated with any indifference curve is a function of the curve's initial value γ, decay constant k (which is rendered by a decreasing opportunity cost to be sub-zero), and an independent consumption variable (a).

Notice that the initial value of the curve is merely the point on the indifference curve that corresponds to quantity b when a equals 0.

Should all other prerequisites remain unaltered, γ and MRS_{ab} will maintain a directly proportionality. This is precisely what one would gauge intuitively. When the principal size associated with a second quantity is elevated, its favoring MRS will be accordingly tractable.

Now that we've developed a formulation associated with a marginal rate of substitution, we can unravel the constraints associated with a convex indifference.

As stated prior, with any such indifference;

$$the\ function\ of\ the\ tangent\ line\ to\ f(a) \leq f(a)\ for\ any\ point\ \boldsymbol{a}\ on\ the\ curve$$

The general equation of a tangent line is delineated using:

$$y - f(a) = f'(a)(x - a)$$

If we attach all the general forms we have thus far found,

$$f(a) = \gamma k^a$$

$$f'(a) = -MRS_{ab} = \gamma \ln[k]\, k^a$$

$$y - \gamma k^a = \gamma \ln[k]\, k^a (x - a)$$

$$y = \gamma \ln[k]\, k^a (x - a) + \gamma k^a$$

$$y = \gamma k^a + \gamma k^a \ln[k]\,(x - a)$$

$$y = \gamma k^a (1 + \ln[k]\,(x - a))$$

Now that we've materially constructed the function associated with the marginal rate of substitution amongst any two commodities, we can impose the inequality constraint associated with a decreasing OC/convex downward indifference between them.

$$\gamma k^a (1 + \ln[k]\,(x - a)) \leq f(a)\ for\ any\ point\ \boldsymbol{a}\ on\ the\ curve$$

$$\gamma k^a (1 + \ln[k]\,(x - a)) \leq \gamma k^a$$

Inferring from the inequality reveals:

$$\left(1 + \ln[k]\,(x - a)\right) \leq 1$$

Or, equivalently, $\ln[k]\,(x-a) \leq 0$.

Note, that

$$k < 1$$

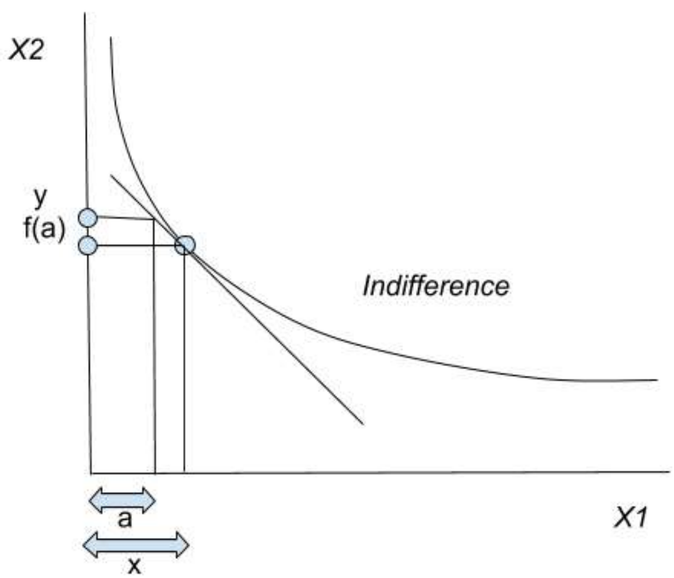
For the decreasing exponential function we've assumed.

$$\ln[k] < 0$$

$$\text{Using } \ln[k]\,(x-a) \leq 0, (x-a) \geq 0.$$

$$Since\ (x-a) \geq 0,\ x \geq a.$$

In order to graphically illustrate what the conclusion above suggests, consider the curve below:



<u>Utility Functions and analogies</u>

For ordinal utility, the axioms of transitivity, completeness and preference warrant that utility functions be subject to reconstruction with unchanged implications. As far as the indifference curves above are concerned, every continuous point captured by them preserves consumer utility, ordinal and cardinal.

The commonly cited Cobb-Douglas utility function assumes the form:

$$U(x,y) = x^a y^b\ for\ two\ constants\ A\ and\ B.$$

For functions that are dependent on both variables, neither one of A or B is zero.

If one were to maximize this concept using strictly a multivariate scheme, partial derivatives may be concurrently employed.

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0$$

$$\frac{\partial U}{\partial x} = y^b a x^{a-1} = 0$$

$$\frac{\partial U}{\partial y} = x^a b y^{b-1} = 0$$

$$In\ terms\ of\ consequent\ possibilities,$$

$$either\ x\ or\ y = 0$$

While the bounds restated above reveal both maximal and minimal utility extremums, they entail a number of significant inferences. If one's risk aversion and/or decision-making proclivities were commensurate with a function of the Cobb-Douglas variant $U(x, y) = x^a y^b$, they'd invariably elect to maximize their utility consistently with a decisive, non-mixed set of variables (either X or Y).

Similarly, if one were to axiomatise risk-indifference (with a linear utility function over an expected payoff):

$$U(x) = A + Bx\ for\ two\ predetermined\ constants\ A\ and\ B\ and\ an\ expected\ payoff\ x,$$

$$\frac{dU}{dx} = B = 0$$

$$U(x) = A$$

One can reiterate the above schematic for a number of utility functions, consistent with prospect theory, Bernoulli's hypothesis on risk aversion or otherwise.

Marginal rates of substitution, however, are of simultaneous relevance in this area. Since they are representative of bundles (certainty equivalents or probability distributions) of commodities that agents or players are internally indifferent to, they are vital determinants in a multitude of applications associated with game theory and behavioral economics.