# About the "Addition" of Scalars and Bivectors…

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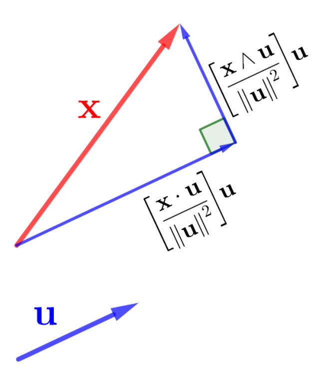
**Abstract**

Unfortunately, some students, teachers, and even physicists object to the "addition" of scalars and bivectors, on the basis that we cannot add things that are not of the same type. Perhaps the objection is not so much to the operation itself, as to the use of the name "addition" for this operation. Still, the operation interacts with others in unfamiliar ways that might cause discomfort to the student. This document explores those potential sources of discomfort, and notes that no problems arise from this unusual "addition" because the developers of GA were careful in choosing the objects (e.g. vectors and bivectors) employed in this algebra, and also in defining not only the operations themselves, but their interactions with each other. The document finishes with an example of how this "addition" proves useful.




Left-multiplying the vector $\mathbf{u}^{-1}$ by the "sum" $\mathbf{x}\cdot\mathbf{u}+\mathbf{x}\wedge\mathbf{u}$ transforms $\mathbf{u}^{-1}$ into a vector sum whose resultant is $\mathbf{x}$.

# 1 Why Do Some People Object to the "Addition" of Scalars and Bivectors?

For centuries, people from all walks of life have invented ways to capture pieces of information about a given situation in symbolic forms, and then to combine those pieces of information via mathematical operations. Said operations tend to be given the name "addition". For example, we have the addition of numbers; a quite-different graphical operation that its inventors chose to call the "addition" of vectors; and yet another operation that it inventors chose to call the "addition" of matrices. Although these operations differ dramatically in their inputs and procedures, they share two features: the inputs to a given "addition" are all of the same type, and the output is of that type as well. That is, the sum of two real numbers is a real number; the "sum" of two vectors is a vector; and the "sum" of two matrices is a matrix.

Such is not the case for the operation that its inventors called the "addition" of a scalar and a bivector. For our present purposes, we may be well advised to put that name aside for the moment, and to concentrate instead upon the operation's purpose, which is to combine two different sorts of information (e.g., about the relative orientations of a pair of vectors) in a single, convenient mathematical expression. The result is something called a "multivector". This feature of the operation proves beneficial in ways that have made the operation a key part of Geometric Algebra, which defines the "addition" well, and employs it consistently with great success. An example is given in Section 2. Still, the operation interacts with others in unfamiliar ways that might cause discomfort to the student.

This brief document explores some of those unfamiliar interactions, recognizing that the student is the ultimate judge of whether an operation with these characteristics is sufficiently useful as to be worth learning. We'll begin with the "sum"

$$\alpha + \mathbf{B},$$

in which $\alpha$ is a scalar and $\mathbf{B}$ is a bivector. Now, let's right-multiply the resulting multivector $(\alpha + \mathbf{B})$ by a vector $\mathbf{v}$ that is parallel to $\mathbf{B}$. (Our conclusions would be the same if we multiplied on the left.)

$$(\alpha + \mathbf{B})\,\mathbf{v} = \alpha\mathbf{v} \;+\; \mathbf{B}\mathbf{v}\;.$$

Because $\mathbf{v}$ is parallel to $\mathbf{B}$, the two terms on the right-hand side are vectors. Therefore, the multiplication by $\mathbf{v}$ did not merely distribute over the "sum" of $\alpha$ and $\mathbf{B}$. Instead, that sum became the familiar "addition of vectors". GA handles such changes consistently, so there's no problem. However, we might want to call students' attention to what just happened. So, let's write the "+" on the right-hand side in red to denote the addition of vectors:

$$(\alpha + \mathbf{B})\,\mathbf{v} = \alpha\mathbf{v} \color{red}{+} \mathbf{B}\mathbf{v}\;. \tag{1}$$

Now, let's suppose that we're working in 3-dimensional GA. We'll express $\mathbf{B}$ and $\mathbf{v}$ according to the basis used by Macdonald ([1], p. 82). The blue "+" signs indicate another type of "sum": the "addition" of bivectors. (Note that bivectors themselves can be "added" geometrically via a well defined operation ([1], p. 74).)

$$\begin{aligned}\mathbf{v} &= \nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2 + \nu_3\mathbf{e}_3\ ,\\ \mathbf{B} &= \beta_1\mathbf{e}_1\mathbf{e}_2 + \beta_2\mathbf{e}_1\mathbf{e}_3 + \beta_3\mathbf{e}_2\mathbf{e}_3\ .\end{aligned}$$

If we substitute these expressions in the right-hand side of Eq. (1), and then expand that side, we'll find ourselves employing yet another type of sum: the familiar addition/subtraction (green symbols, below) of real numbers.

$$\begin{aligned}(\alpha+\mathbf{B})\,\mathbf{v} = &\ (\alpha\nu_1 + \beta_1\nu_2 + \beta_2\nu_3)\,\mathbf{e}_1\\ &+ (\alpha\nu_2 + \beta_3\nu_3 - \beta_1\nu_1)\,\mathbf{e}_2\\ &+ (\alpha\nu_3 - \beta_2\nu_1 - \beta_3\nu_2)\,\mathbf{e}_3.\end{aligned}$$

In that equation, the single symbol "+" represents three distinct operations—or at least, operations that have been taught to students as being distinct. If $\mathbf{v}$ were not parallel to $\mathbf{B}$, we would have yet another "+" on the right-hand side: the "sum" of a vector (the first term on the right-hand side) and a trivector:

$$\begin{aligned}(\alpha+\mathbf{B})\,\mathbf{v} = &\ [(\alpha\nu_1 + \beta_1\nu_2 + \beta_2\nu_3)\,\mathbf{e}_1 + (\alpha\nu_2 + \beta_3\nu_3 - \beta_1\nu_1)\,\mathbf{e}_2\\ &+ (\alpha\nu_3 - \beta_2\nu_1 - \beta_3\nu_2)\,\mathbf{e}_3]\\ &+ \underbrace{(\beta_1\nu_3 - \beta_2\nu_2 + \beta_3\nu_1)\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}_{trivector}.\end{aligned}$$

Like the "sum" with which we started—that of a scalar and a bivector—the "addition" of a vector and a trivector produces a multivector.

Again, and in conclusion, none of this is a problem. Nevertheless, we might help students by pointing out what is going on, and by explaining *why* there is no problem: because the developers of GA were careful in choosing the objects (e.g. vectors and bivectors) employed in this algebra, and also in defining not only the operations themselves, but their interactions with each other.

## 2 An Example of How GA Makes Use of the Two Types of Information that are Combined in the "Sum" of a Scalar and a Bivector

We saw above that the multiplication of $\alpha + \mathbf{B}$ by the vector $\mathbf{v}$ produced a sum of two vectors. Now, let's consider another sum of a scalar and a bivector: $\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$. We've seen that sum many times, of course: it's often used as the definition of the geometric product $\mathbf{ab}$:

$$\mathbf{ab} \equiv \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}.$$

However, we'll use that sum here in a different way.

We begin by recalling that in GA, every non-zero vector $\mathbf{b}$ has a multiplicative inverse: $\mathbf{b}^{-1} \equiv \mathbf{b}/\|\mathbf{b}\|^2$. We make use of that aspect of GA when we solve for an unknown vector $\mathbf{x}$ by first finding that vector's inner and outer products with some known vector $\mathbf{u}$, then adding those products to form $\mathbf{xu}$:

$$\mathbf{xu} = \mathbf{x} \cdot \mathbf{u} + \mathbf{x} \wedge \mathbf{u}.$$

Next, we multiply both sides by $\mathbf{u}^{-1}$:

$$\begin{aligned} \mathbf{xuu}^{-1} &= [\mathbf{x} \cdot \mathbf{u}]\, \mathbf{u}^{-1} + [\mathbf{x} \wedge \mathbf{u}]\, \mathbf{u}^{-1} \\ \mathbf{x} &= \left[\frac{\mathbf{x} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\right] \mathbf{u} + \left[\frac{\mathbf{x} \wedge \mathbf{u}}{\|\mathbf{u}\|^2}\right] \mathbf{u}. \end{aligned}$$

How do we interpret the right-hand side? It's a sum of two vectors (Fig. 1). The first term on that side is a scalar multiple of $\mathbf{u}$ itself. As for the second term, the vector $\mathbf{u}$ is parallel to the bivector $\mathbf{x} \wedge \mathbf{u}$, so $\left[\frac{\mathbf{x} \wedge \mathbf{u}}{\|\mathbf{u}\|^2}\right] \mathbf{u}$ is a scalar multiple of a 90° rotation of $\mathbf{u}$.

# References


[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).

$$\mathbf{x}\mathbf{u} = \mathbf{x}\cdot\mathbf{u} + \mathbf{x}\wedge\mathbf{u}$$

$$\mathbf{x}\mathbf{u}\mathbf{u}^{-1} = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}^{-1} + (\mathbf{x}\wedge\mathbf{u})\mathbf{u}^{-1}$$

$$\mathbf{x} = \left[\frac{\mathbf{x}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\right]\mathbf{u} + \left[\frac{\mathbf{x}\wedge\mathbf{u}}{\|\mathbf{u}\|^2}\right]\mathbf{u}$$

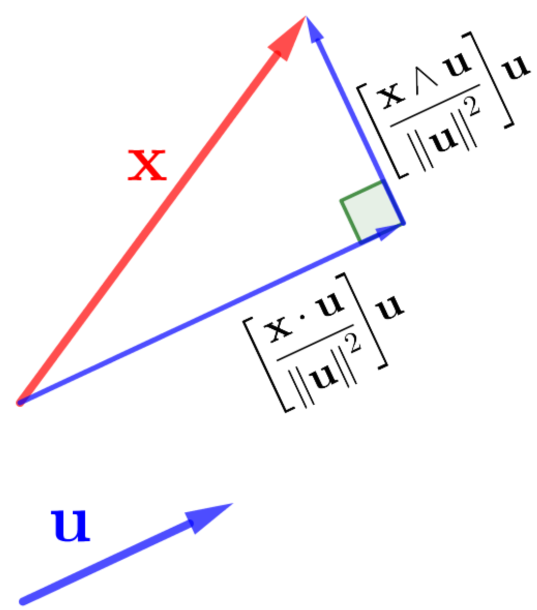



Figure 1: Left-multiplying the vector $\mathbf{u}^{-1}$ by the "sum" $\mathbf{x}\cdot\mathbf{u}+\mathbf{x}\wedge\mathbf{u}$ transforms $\mathbf{u}^{-1}$ into a vector sum whose resultant is $\mathbf{x}$.