Distribution of Integrals of Wiener Paths

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Abstract

We show that the normal distribution with mean zero and variance 1/3 is the distribution of the integrals $\int_{[0,1]} W_t \, \mathrm{d}t$ of the sample paths of Wiener process W in $C([0,1], \mathbb{R})$.

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1 Introduction

In contrast with the notion of "martingale (stochastic) integration" associated with Wiener measure, attention is less directed to the integrals of the sample paths of Wiener process W in $C([0,1], \mathbb{R})$. Since every realization of W is a continuous function on a compact interval, it always makes sense to speak of the integral of a Wiener path; investigating the integrals of Wiener paths, in particular the distribution of such integrals (which is evidently possible and is justified in what follows), is then a natural move.

In the present short communication, we prove

Theorem $*$. *If W is Wiener process in $C([0,1], \mathbb{R})$, then*

$$\int_{[0,1]} W_t \, \mathrm{d}t \sim N(0, 1/3). \qquad \square$$

2 Proof

Throughout, let C_w be the metric space $C([0,1], \mathbb{R})$ equipped with the uniform metric; and let W be Wiener process in C_w.

We now give

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Proof (of Theorem $*$). For all $f, g \in C_w$, we have

$$\left| \int (f - g) \right| \leq \sup_t |f(t) - g(t)|;$$

so the integration operator $\int$ is (uniformly) continuous on C_w.

If $X_1, X_2, \ldots$ are independent identically distributed standard normal random variables, let $\widehat{W}^n$ be for each $n \in \mathbb{N}$ the "Donsker process" obtained by linear interpolation between the $\frac{1}{\sqrt{n}}$-scaled cumulative sums of $X_1, \ldots, X_n$ such that the resulting process fixes the origin, so that the sequence $(\widehat{W}^n)_{n \in \mathbb{N}}$ satisfies the assumptions of Donsker's theorem (Theorem 8.2 in Billingsley [1], for concreteness). The continuous mapping theorem and Donsker's theorem then jointly imply the weak convergence

$$\int \widehat{W}^n_t \, \mathrm{d}t \rightsquigarrow \int W_t \, \mathrm{d}t. \tag{1}$$

Let $S_0 \coloneqq 0$; and let $S_j \coloneqq \sum_{i=1}^j X_i$ for all $1 \leq j \leq n$ and all $n \in \mathbb{N}$. If $n \in \mathbb{N}$, then we have $\int \widehat{W}^n_t \, \mathrm{d}t = \sum_{j=1}^n \int_{(j-1)/n}^{j/n} \widehat{W}^n_t \, \mathrm{d}t$, and we have $\widehat{W}^n_{j/n} = S_j/\sqrt{n}$ for each $0 \leq j \leq n$. Given any $1 \leq j \leq n$, we have

$$\begin{aligned}
&\int_{(j-1)/n}^{j/n} \widehat{W}^n_t \, \mathrm{d}t \\
&= \frac{1}{\sqrt{n}} \int_{(j-1)/n}^{j/n} \tau S_j + (1-\tau) S_{j-1} \, \mathrm{d}\tau \\
&= \frac{1}{\sqrt{n}} \left(S_j \frac{\tau^2}{2} \Big|_{(j-1)/n}^{j/n} + \frac{1}{n} S_{j-1} - S_{j-1} \frac{\tau^2}{2} \Big|_{(j-1)/n}^{j/n} \right).
\end{aligned}$$

Summing the last term above over each $1 \leq j \leq n$ gives

$$\int \widehat{W}^n_t \, \mathrm{d}t = \frac{1}{n^{3/2}} \Big(nX_1 + (n-1)X_2 + \cdots + X_n \Big) - \frac{1}{2n^{5/2}} S_n. \tag{2}$$

The last term in (2) vanishes in probability by the continuous mapping theorem and the usual weak law of large numbers.

If $n \in \mathbb{N}$, the sum of the independent normal random variables $(n - j + 1)X_j$ with $1 \leq j \leq n$ in (2) is the normal random variable with mean zero and variance $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$. If $\kappa \coloneqq 2^{3/2}\Gamma(2)/\sqrt{\pi}$, then

$$\sum_{j=1}^n \mathbb{E}|(n-j+1)X_j|^3 = \kappa \sum_{j=1}^n j^3 = \kappa \frac{n^2(n+1)^2}{4},$$

which grows more slowly than $(n(n+1)(2n+1)/6)^{3/2}$ as $n \to \infty$. The classical Lyapunov central limit theorem (e.g. p. 332, Shiryaev [2], for concreteness) and the continuous mapping theorem together imply that

$$
\begin{aligned}
&\frac{1}{n^{3/2}}\Big(nX_1 + (n-1)X_2 + \cdots + X_n\Big) \\
&= \frac{\sqrt{\frac{n(n+1)(2n+1)}{6}}}{n^{3/2}} \left(\sqrt{\frac{n(n+1)(2n+1)}{6}}\right)^{-1} \Big(nX_1 + (n-1)X_2 + \cdots + X_n\Big) \\
&\rightsquigarrow N(0, 1/3).
\end{aligned}
$$

Upon applying the continuous mapping theorem once more, we have

$$\int \widehat{W}_t^n \,\mathrm{d}t \rightsquigarrow N(0, 1/3)$$

from (2). But then from (1) and the uniqueness of weak limit it follows that

$$\int W_t \,\mathrm{d}t \sim N(0, 1/3)$$

as desired. □

References

[1] Billingsley, P. (1999). *Convergence of Probability Measures*, second edition. Wiley.

[2] Shiryaev, A. N. (1996). *Probability*, second edition, translated by R. P. Boas. Springer.