Exact Solution of ODEs - Vector Space Transformation Technique - Part 2

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Abstract
Since the number of linearly independent solutions of an HLODE equals the order of the HLODE, the dimension of the solution set of an HLODE is it's order; so the dimension of all HLODEs of the same order is this order. Thus, a first order HLODE may be solved exactly using the single HLODE ; but higher order non-elementary HLODEs require that order number of equations to solve such HLODEs - i.e. usage of linear transformations between linearly independent HLODE solution sets of that same order. The first order HLODE being of dimension 1 may be solved exactly by itself. Higher order HLODEs are subject to condition that the additional dimensional HLODEs must also be satisfied.

Another look at the vector space transformation technique to solving ODEs exactly.

Since the number of linearly independent solutions of an HLODE equals the order of the HLODE, the dimension of the solution set of an HLODE is it's order; so the dimension of all HLODEs of the same order is this order.

Thus, a first order HLODE may be solved exactly using the single HLODE ; but higher order non-elementary HLODEs require that order number of equations to solve such HLODEs - i.e. usage of linear transformations between linearly independent HLODE solution sets of that same order.

i.e.:

$$\sum_{m=0}^{n} P_{n-1-m} y^{(m)} = 0 \Rightarrow \begin{cases} \sum_{m=0}^{n} P_{n-1-m} y_{01}^{(m)} = 0 \\ \vdots \\ \sum_{m=0}^{n} P_{n-1-m} y_{0n}^{(m)} = 0 \end{cases} \quad , \quad \exists y_{01},\ldots,y_{0n}$$

$y_{01},\ldots,y_{0n}$ related by the reduction of order formula.

In particular, for second order HLODEs:

$$y'' + P_0 y' + Q_0 y = 0 \Rightarrow \begin{cases} y_{01}'' + P_0 y_{01}' + Q_0 y_{01} = 0 \\ y_{02}'' + P_0 y_{02}' + Q_0 y_{02} = 0 \end{cases} \quad , \quad \exists y_{01}, y_{02}$$

y_{01} & y_{02} related by the reduction of order formula.

Since, for any HLODE, the middle coefficient may be transformed to any value; another second order HLODEs may be written (without loss of generality):

$$w'' + Pw' + Qw = 0 \quad \& \quad u = we^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left[\left[\left(\tfrac{1}{2}R\right)' - \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)^2\right] + Q\right]u = 0$$

The first order HLODE being of dimension 1 may be solved exactly by itself.
Higher order HLODEs are subject to condition that the additional dimensional HLODEs must also be satisfied.
This leads to:

Theorem I.1: For differentiable functions $u, v, y_1, y_2, r_1, r_2, s_1, s_2, Y_1, Y_2, P, Q, \varphi$;
if y_1 & y_2 are linearly independent solutions to $y_i'' + P_i y_i' + Q_i y_i = 0$
and Y_1 & Y_2 are linearly independent solutions to $Y_i'' + PY_i' + (Q_i + \varphi_i)Y_i = 0$
such that: $\exists u, v$: (u, v linearly independent of y_1, y_2 : $\forall x$)

$$\begin{cases} Y_1 = vy_1 + uy_2 \\ Y_2 = uy_1 - vy_2 \end{cases}$$

and: $\exists r_1, r_2, s_1, s_2$:

$$\begin{cases} y_1' = r_1 y_1 + s_1 y_2 \\ y_2' = r_2 y_1 + s_2 y_2 \end{cases}$$

then:

$$\Rightarrow \begin{cases} 0 = -2(r_1 - s_2)u' + (\varphi_1 - \varphi_2)u + 2(r_2 + s_1)v' \\ 0 = 2(r_1 - s_2)v' + (\varphi_1 - \varphi_2)v + 2(r_2 + s_1)u' \\ 0 = u'' + [2r_1 + P_2]u' + \varphi_2 u - 2r_2 v' \\ 0 = v'' + [2s_2 + P_2]v' + \varphi_2 v - 2s_1 u' \end{cases}$$

Proof:
When y_1 , y_2 satisfy:

$$\begin{cases} y_1' = r_1 y_1 + s_1 y_2 \\ y_2' = r_2 y_1 + s_2 y_2 \end{cases}$$

$$\Rightarrow \begin{cases} y_1'' = (r_1' + r_1^2 + r_2 s_1)y_1 + (s_1' + s_1 s_2 + r_1 s_1)y_2 \\ y_2'' = (r_2' + r_1 r_2 + r_2 s_2)y_1 + (s_2' + s_2^2 + r_2 s_1)y_2 \end{cases}$$

$$\begin{cases} Y_1 = vy_1 + uy_2 \\ Y_2 = uy_1 - vy_2 \end{cases}$$

$$\left.\begin{cases} Y_1 = vy_1 + uy_2 \\ Y_2 = uy_1 - vy_2 \end{cases}\right\} \Rightarrow \begin{cases} Y_1' = vy_1' + v'y_1 + uy_2' + u'y_2 = v(r_1 y_1 + s_1 y_2) + v'y_1 + u(r_2 y_1 + s_2 y_2) + u'y_2 \\ Y_2' = uy_1' + u'y_1 - vy_2' - v'y_2 = u(r_1 y_1 + s_1 y_2) + u'y_1 - v(r_2 y_1 + s_2 y_2) - v'y_2 \end{cases}$$

$$\Rightarrow \begin{cases} Y_1' = (vr_1 + v' + ur_2)y_1 + (vs_1 + us_2 + u')y_2 \\ Y_2' = (ur_1 + u' - vr_2)y_1 + (us_1 - vs_2 - v')y_2 \end{cases}$$

$$\Rightarrow \begin{cases} Y_\mu'' + P_1 Y_\mu' = (vy_1 + uy_2)'' + P_1(vy_1 + uy_2)' \\ Y_\nu'' + P_2 Y_\nu' = (uy_1 - vy_2)'' + P_2(uy_1 - vy_2)' \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = (v'y_1 + vy'_1 + u'y_2 + uy'_2)' + P_1(v'y_1 + vy'_1 + u'y_2 + uy'_2) \\ Y''_\nu + P_2 Y'_\nu = (u'y_1 + uy'_1 - v'y_2 - vy'_2)' + P_2(u'y_1 + uy'_1 - v'y_2 - vy'_2) \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = v''y_1 + 2v'y'_1 + vy''_1 + u''y_2 + 2u'y'_2 + uy''_2 + \\ \qquad + P_1(v'y_1 + vy'_1 + u'y_2 + uy'_2) \\ Y''_\nu + P_2 Y'_\nu = u''y_1 + 2u'y'_1 + uy''_1 - v''y_2 - 2v'y'_2 - vy''_2 + \\ \qquad + P_2(u'y_1 + uy'_1 - v'y_2 - vy'_2) \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = v''y_1 + P_1v'y_1 + 2v'y'_1 + P_1vy'_1 + vy''_1 + u''y_2 + P_1u'y_2 + 2u'y'_2 + P_1uy'_2 + uy''_2 \\ Y''_\nu + P_2 Y'_\nu = u''y_1 + P_2u'y_1 + 2u'y'_1 + P_2uy'_1 + uy''_1 - v''y_2 - P_2v'y_2 - 2v'y'_2 - P_2vy'_2 - vy''_2 \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = [v''y_1 + P_1v'y_1 + 2v'y'_1 + P_1vy'_1 + vy''_1] + \\ \qquad + [u''y_2 + P_1u'y_2 + 2u'y'_2 + P_1uy'_2 + uy''_2] \\ Y''_\nu + P_2 Y'_\nu = [u''y_1 + P_2u'y_1 + 2u'y'_1 + P_2uy'_1 + uy''_1] + \\ \qquad - [v''y_2 + P_2v'y_2 + 2v'y'_2 + P_2vy'_2 + vy''_2] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = [(v'' + P_1v')y_1 + (2v' + P_1v)y'_1 + vy''_1] + \\ \qquad + [(u'' + P_1u')y_2 + (2u' + P_1u)y'_2 + uy''_2] \\ Y''_\nu + P_2 Y'_\nu = [(u'' + P_2u')y_1 + (2u' + P_2u)y'_1 + uy''_1] + \\ \qquad - [(v'' + P_2v')y_2 + (2v' + P_2v)y'_2 + vy''_2] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = [(v'' + P_1v')y_1 + 2v'(r_1y_1 + s_1y_2) + v(y''_1 + P_1y'_1)] + \\ \qquad + [(u'' + P_1u')y_2 + 2u'(r_2y_1 + s_2y_2) + u(y''_2 + P_1y'_2)] \\ Y''_\nu + P_2 Y'_\nu = [(u'' + P_2u')y_1 + 2u'(r_1y_1 + s_1y_2) + u(y''_1 + P_2y'_1)] + \\ \qquad - [(v'' + P_2v')y_2 + 2v'(r_2y_1 + s_2y_2) + v(y''_2 + P_2y'_2)] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = [(v'' + P_1v')y_1 + 2v'r_1y_1 + 2v's_1y_2 + v(y''_1 + P_1y'_1)] + \\ \qquad + [(u'' + P_1u')y_2 + 2u'r_2y_1 + 2u's_2y_2 + u(y''_2 + P_1y'_2)] \\ Y''_\nu + P_2 Y'_\nu = [(u'' + P_2u')y_1 + 2u'r_1y_1 + 2u's_1y_2 + u(y''_1 + P_2y'_1)] + \\ \qquad - [(v'' + P_2v')y_2 + 2v'r_2y_1 + 2v's_2y_2 + v(y''_2 + P_2y'_2)] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = [(v'' + [2r_1 + P_1]v')y_1 + 2v's_1y_2 + v(y''_1 + P_1y'_1)] + \\ \qquad + [(u'' + [2s_2 + P_1]u')y_2 + 2u'r_2y_1 + u(y''_2 + P_1y'_2)] \\ Y''_\nu + P_2 Y'_\nu = [(u'' + [2r_1 + P_2]u')y_1 + 2u's_1y_2 + u(y''_1 + P_2y'_1)] + \\ \qquad - [(v'' + [2s_2 + P_2]v')y_2 + 2v'r_2y_1 + v(y''_2 + P_2y'_2)] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = [(v'' + [2r_1 + P_1]v')y_1 + 2u'r_2y_1 + v(y''_1 + P_1y'_1)] + \\ \qquad + [(u'' + [2s_2 + P_1]u')y_2 + 2v's_1y_2 + u(y''_2 + P_1y'_2)] \\ Y''_\nu + P_2 Y'_\nu = [(u'' + [2r_1 + P_2]u')y_1 - 2v'r_2y_1 + u(y''_1 + P_2y'_1)] + \\ \qquad - [(v'' + [2s_2 + P_2]v')y_2 - 2u's_1y_2 + v(y''_2 + P_2y'_2)] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu = [(v'' + [2r_1 + P_1]v' + 2r_2u')y_1 + v(y''_1 + P_1y'_1)] + \\ \qquad + [(u'' + [2s_2 + P_1]u' + 2s_1v')y_2 + u(y''_2 + P_1y'_2)] \\ Y''_\nu + P_2 Y'_\nu = [(u'' + [2r_1 + P_2]u' - 2r_2v')y_1 + u(y''_1 + P_2y'_1)] + \\ \qquad - [(v'' + [2s_2 + P_2]v' - 2s_1u')y_2 + v(y''_2 + P_2y'_2)] \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu + (Q_1 + \varphi_1)Y_\mu = [(v'' + [2r_1 + P_1]v' + 2r_2u')y_1 + v(y''_1 + P_1y'_1)] + \\ \qquad + [(u'' + [2s_2 + P_1]u' + 2s_1v')y_2 + u(y''_2 + P_1y'_2)] \\ \qquad + (Q_1 + \varphi_1)(vy_1 + uy_2) \\ Y''_\nu + P_2 Y'_\nu + (Q_2 + \varphi_2)Y_\mu = [(u'' + [2r_1 + P_2]u' - 2r_2v')y_1 + u(y''_1 + P_2y'_1)] + \\ \qquad - [(v'' + [2s_2 + P_2]v' - 2s_1u')y_2 + v(y''_2 + P_2y'_2)] \\ \qquad + (Q_2 + \varphi_2)(uy_1 - vy_2) \end{cases}$$

$$\Rightarrow \begin{cases} Y''_\mu + P_1 Y'_\mu + (Q_1 + \varphi_1)Y_\mu = [(v'' + [2r_1 + P_1]v' + \varphi_1v + 2r_2u')y_1 + v(y''_1 + P_1y'_1 + Q_1y_1)] + \\ \qquad + [(u'' + [2s_2 + P_1]u' + \varphi_1u + 2s_1v')y_2 + u(y''_2 + P_1y'_2 + Q_1y_2)] \\ Y''_\nu + P_2 Y'_\nu + (Q_2 + \varphi_2)Y_\mu = [(u'' + [2r_1 + P_2]u' + \varphi_2u - 2r_2v')y_1 + u(y''_1 + P_2y'_1 + Q_2y_1)] + \\ \qquad - [(v'' + [2s_2 + P_2]v' + \varphi_2v - 2s_1u')y_2 + v(y''_2 + P_2y'_2 + Q_2y_2)] \end{cases}$$

So, for:

$$\begin{aligned} & Y''_1 + P_1Y'_1 + (Q_1 + \varphi_1)Y_1 = 0 = Y''_2 + P_2Y'_2 + (Q_2 + \varphi_2)Y_2 \\ & y''_1 + P_1y'_1 + Q_1y_1 = 0 = y''_2 + P_2y'_2 + Q_2y_2 \end{aligned}$$

$$\Rightarrow \begin{cases} 0 = [(v'' + [2r_1 + P_1]v' + \varphi_1v + 2r_2u')y_1] + [(u'' + [2s_2 + P_1]u' + \varphi_1u + 2s_1v')y_2] \\ 0 = [(u'' + [2r_1 + P_2]u' + \varphi_2u - 2r_2v')y_1] - [(v'' + [2s_2 + P_2]v' + \varphi_2v - 2s_1u')y_2] \end{cases}$$

and if: y_1 & y_2 are linearly independant:

$$\Rightarrow \begin{cases} 0 = v'' + [2r_1 + P_1]v' + \varphi_1v + 2r_2u' & 0 = u'' + [2s_2 + P_1]u' + \varphi_1u + 2s_1v' \\ 0 = u'' + [2r_1 + P_2]u' + \varphi_2u - 2r_2v' & 0 = v'' + [2s_2 + P_2]v' + \varphi_2v - 2s_1u' \end{cases}$$

$$\Rightarrow \left\{ \begin{array}{|l|l|} \hline 0 = u'' + [2s_2 + P_1]u' + \varphi_1 u + 2s_1 v' & 0 = v'' + [2r_1 + P_1]v' + \varphi_1 v + 2r_2 u' \\ \hline 0 = u'' + [2r_1 + P_2]u' + \varphi_2 u - 2r_2 v' & 0 = v'' + [2s_2 + P_2]v' + \varphi_2 v - 2s_1 u' \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|l|} \hline 0 = 2(s_2 - r_1)u' + (\varphi_1 - \varphi_2)u + 2(s_1 + r_2)v' & 0 = 2(r_1 - s_2)v' + (\varphi_1 - \varphi_2)v + 2(r_2 + s_1)u' \\ \hline 0 = u'' + [2r_1 + P_2]u' + \varphi_2 u - 2r_2 v' & 0 = v'' + [2s_2 + P_2]v' + \varphi_2 v - 2s_1 u' \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|l|} \hline 0 = -2(r_1 - s_2)u' + (\varphi_1 - \varphi_2)u + 2(r_2 + s_1)v' & 0 = 2(r_1 - s_2)v' + (\varphi_1 - \varphi_2)v + 2(r_2 + s_1)u' \\ \hline 0 = u'' + [2r_1 + P_2]u' + \varphi_2 u - 2r_2 v' & 0 = v'' + [2s_2 + P_2]v' + \varphi_2 v - 2s_1 u' \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline 0 = -2(r_1 - s_2)u' + (\varphi_1 - \varphi_2)u + 2(r_2 + s_1)v' \\ \hline 0 = 2(r_1 - s_2)v' + (\varphi_1 - \varphi_2)v + 2(r_2 + s_1)u' \\ \hline 0 = u'' + [2r_1 + P_2]u' + \varphi_2 u - 2r_2 v' \\ \hline 0 = v'' + [2s_2 + P_2]v' + \varphi_2 v - 2s_1 u' \\ \hline \end{array} \right.$$

□

Corollary I.1: For differentiable functions $u, v, y_1, y_2, r_1, r_2, s_1, s_2, Y_1, Y_2, P, Q, \varphi$;
if y_1 & y_2 are linearly independent solutions to $y_i'' + P_i y_i' + Q_i y_i = 0$
and Y_1 & Y_2 are linearly independent solutions to $Y_i'' + PY_i' + (Q_i + \varphi_i)Y_i = 0$
such that: $\exists u, v$: (u, v linearly independent of y_1, y_2 : $\forall x$)

$$\left\{ \begin{array}{|l|} \hline Y_1 = vy_1 + uy_2 \\ \hline Y_2 = uy_1 - vy_2 \\ \hline \end{array} \right.$$

and: $\exists r_1, r_2, s_1, s_2$:

$$\left\{ \begin{array}{|l|} \hline y_1' = r_1 y_1 + s_1 y_2 \\ \hline y_2' = r_2 y_1 + s_2 y_2 \\ \hline \end{array} \right.$$

then:

if: $(r_2 + s_1) = 0$ & $(r_1 - s_2) \neq 0$

for:

$$\varphi_2 = \varphi_1 - (r_1 - s_2)\left(\log\left(\frac{[(r_1 + s_2) + P_2] \pm \sqrt{[(r_1 + s_2) + P_2]^2 + 4r_2^2}}{2r_2} \right) \right)'$$

then:

$$u = e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} = \pm \sqrt{\frac{[(r_1 + s_2) + P_2] \pm \sqrt{[(r_1 + s_2) + P_2]^2 + 4r_2^2}}{2r_2}}$$

$$v = \frac{1}{u}$$

Proof:

Given the main theorem:

if: $(r_2 + s_1) = 0$ & $(r_1 - s_2) \neq 0$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline \frac{(\varphi_1 - \varphi_2)}{2(r_1 - s_2)} = \frac{u'}{u} = (\log u)' \Rightarrow u = e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \\ \hline -\frac{(\varphi_1 - \varphi_2)}{2(r_1 - s_2)} = \frac{v'}{v} = (\log v)' \Rightarrow v = e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \\ \hline 0 = \left(e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \right)'' + [2r_1 + P_2]\left(e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \right)' + \varphi_2 \left(e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \right) - 2r_2 \left(e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \right) \\ \hline 0 = \left(e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \right)'' + [2s_2 + P_2]\left(e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \right)' + \varphi_2 \left(e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \right) + 2r_2' \left(e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \right) \\ \hline \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|r|} \hline 0 = \left(\left(\frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} \right)' + \frac{1}{4}\left(\frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} \right)^2 \right) e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} + \\ \hline + [2r_1 + P_2]\frac{1}{2}\frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} + \varphi_2 e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} + \\ \hline + 2r_2 \frac{1}{2}\frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \\ \hline 0 = \left(-\left(\frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} \right)' + \frac{1}{4}\left(-\frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} \right)^2 \right) e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} + \\ \hline - [2s_2 + P_2]\frac{1}{2}\frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} + \varphi_2 e^{-\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} + \\ \hline + 2r_2 \frac{1}{2}\frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} e^{\frac{1}{2}\int \frac{(\varphi_1 - \varphi_2)}{(r_1 - s_2)} dx} \\ \hline \end{array} \right.$$

$$\Rightarrow \begin{cases} 0 = \left(\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}\right)' + \frac{1}{4}\left(\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}\right)^2 + [2r_1+P_2]\frac{1}{2}\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)} + \varphi_2 + \\ \quad + 2r_2\frac{1}{2}\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)} e^{-\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx} \\ 0 = -\left(\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}\right)' + \frac{1}{4}\left(-\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}\right)^2 - [2s_2+P_2]\frac{1}{2}\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)} + \varphi_2 + \\ \quad + 2r_2\frac{1}{2}\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)} e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx} \end{cases}$$

$$\Rightarrow 0 = [2r_1+2s_2+2P_2]\frac{1}{2}\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)} + 2r_2\frac{1}{2}\frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}\left(e^{-\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx} - e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right)$$

$$\Rightarrow 0 = [(r_1+s_2)+P_2] + r_2\left(e^{-\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx} - e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right)$$

$$\Rightarrow 0 = [(r_1+s_2)+P_2]\left(e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right) + r_2\left(1 - \left(e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right)^2\right)$$

$$\Rightarrow 0 = [(r_1+s_2)+P_2]\left(e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right) + r_2 - r_2\left(e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right)^2$$

$$\Rightarrow r_2\left(e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right)^2 - [(r_1+s_2)+P_2]\left(e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right) - r_2 = 0$$

$$\Rightarrow \left(e^{\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx}\right) = \frac{[(r_1+s_2)+P_2] \pm \sqrt{[(r_1+s_2)+P_2]^2 + 4r_2^2}}{2r_2}$$

$$\Rightarrow \int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx = \log\left(\frac{[(r_1+s_2)+P_2] \pm \sqrt{[(r_1+s_2)+P_2]^2 + 4r_2^2}}{2r_2}\right)$$

$$\Rightarrow \varphi_1 = \varphi_2 + (r_1-s_2)\left(\log\left(\frac{[(r_1+s_2)+P_2] \pm \sqrt{[(r_1+s_2)+P_2]^2 + 4r_2^2}}{2r_2}\right)\right)'$$

$$\Rightarrow u = e^{\frac{1}{2}\int \frac{(\varphi_1-\varphi_2)}{(r_1-s_2)}dx} = \pm\sqrt{\frac{[(r_1+s_2)+P_2] \pm \sqrt{[(r_1+s_2)+P_2]^2 + 4r_2^2}}{2r_2}}$$

□

Corollary I.2: For differentiable functions $u, v, y_1, y_2, r_1, r_2, s_1, s_2, Y_1, Y_2, P, Q, \varphi$;
if y_1 & y_2 are linearly independent solutions to $y_i'' + P_i y_i' + Q_i y_i = 0$
and Y_1 & Y_2 are linearly independent solutions to $Y_i'' + PY_i' + (Q_i + \varphi_i)Y_i = 0$
such that: $\exists u, v$: (u, v linearly independent of y_1, y_2 : $\forall x$)

$$\begin{cases} Y_1 = vy_1 + uy_2 \\ Y_2 = uy_1 - vy_2 \end{cases}$$

and: $\exists r_1, r_2, s_1, s_2$:

$$\begin{cases} y_1' = r_1y_1 + s_1y_2 \\ y_2' = r_2y_1 + s_2y_2 \end{cases}$$

then:

if: $(r_2 + s_1) \neq 0$ & $(r_1 - s_2) = 0$

$$\begin{cases} u = \cosh\left(k\int e^{-\int(2r_1+P_2)dx}dx\right) \\ v = -\sinh\left(k\int e^{-\int(2r_1+P_2)dx}dx\right) \\ \varphi_1 = -\varphi_2 = k(r_2+s_1)e^{-\int(2r_1+P_2)dx} \end{cases}$$

Proof:

Given the main theorem:

if: $(r_2 + s_1) \neq 0$ & $(r_1 - s_2) = 0$

$$\Rightarrow \begin{cases} 0 = (\varphi_1-\varphi_2)u + 2(r_2+s_1)v' \\ 0 = (\varphi_1-\varphi_2)v + 2(r_2+s_1)u' \\ 0 = u'' + [2r_1+P_2]u' + \varphi_2 u - 2r_2 v' \\ 0 = v'' + [2s_2+P_2]v' + \varphi_2 v - 2s_1 u' \end{cases}$$

$$\Rightarrow \begin{cases} 0 = (\varphi_1-\varphi_2)(u+v) + 2(r_2+s_1)(v+u)' \\ 0 = (\varphi_1-\varphi_2)(v-u) + 2(r_2+s_1)(u-v)' \\ 0 = u'' + [2r_1+P_2]u' + \varphi_2 u - 2r_2 v' \\ 0 = v'' + [2s_2+P_2]v' + \varphi_2 v - 2s_1 u' \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} = \dfrac{(v+u)'}{(u+v)} = (\log(u+v))' \\ \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} = \dfrac{(u-v)'}{(u-v)} = (\log(u-v))' \\ 0 = u'' + [2r_1+P_2]u' + \varphi_2 u - 2r_2 v' \\ 0 = v'' + [2s_2+P_2]v' + \varphi_2 v - 2s_1 u' \end{cases}$$

$$\Rightarrow \begin{cases} e^{-\int \frac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx} = u+v \\ e^{\int \frac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx} = u-v \\ 0 = u'' + [2r_1+P_2]u' + \varphi_2 u - 2r_2 v' \\ 0 = v'' + [2s_2+P_2]v' + \varphi_2 v - 2s_1 u' \end{cases}$$

$$\Rightarrow \begin{cases} e^{-\int \frac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx} + e^{\int \frac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx} = 2u \\ e^{-\int \frac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx} - e^{\int \frac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx} = 2v \\ 0 = u'' + [2r_1+P_2]u' + \varphi_2 u - 2r_2 v' \\ 0 = v'' + [2s_2+P_2]v' + \varphi_2 v - 2s_1 u' \end{cases}$$

$$\Rightarrow \begin{cases} u = \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ v = -\sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ 0 = u'' + [2r_1+P_2]u' + \varphi_2 u - 2r_2 v' \\ 0 = v'' + [2s_2+P_2]v' + \varphi_2 v - 2s_1 u' \end{cases}$$

$$\Rightarrow \begin{cases} u = \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ v = -\sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ 0 = \left(\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}\right)' \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \left(\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}\right)^2 \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \\ \qquad + [2r_1+P_2]\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \varphi_2 \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \\ \qquad + 2r_2 \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ 0 = \left(\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}\right)' \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \left(\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}\right)^2 \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \\ \qquad + [2s_2+P_2]\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) - \varphi_2 \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \\ \qquad - 2s_1 \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \end{cases}$$

$$\Rightarrow \begin{cases} u = \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ v = -\sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ 0 = \left(\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}\right)' \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + [2r_1+P_2]\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \\ \quad + \left(\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}\right)^2 \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + \varphi_2 \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + 2r_2 \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ 0 = \left(\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}\right)' \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) + [2s_2+P_2]\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} \cosh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \\ \quad + \left(\dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}\right)^2 \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) - \varphi_2 \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) - 2s_1 \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)} \sinh\left(\int \dfrac{(\varphi_1-\varphi_2)}{2(r_2+s_1)}dx\right) \end{cases}$$

$$\Rightarrow \left\{ \begin{array}{l} u = \cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ v = -\sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ 0 = \left[\left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)' + [2r_1 + P_2] \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right] \sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) + \\ \quad + \left[\left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)^2 + \varphi_2 + 2r_2 \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right] \cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ 0 = \left[\left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)' + [2s_2 + P_2] \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right] \cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) + \\ \quad + \left[\left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)^2 - \varphi_2 - 2s_1 \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right] \sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \end{array} \right.$$

So, restricting $\varphi_1, \varphi_2, P_2, r_1, r_2, s_1$ such that the coefficients are independent of $\sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right)$ & $\cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right)$:

$$\Rightarrow \left\{ \begin{array}{l} u = \cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ v = -\sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ 0 = \left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)' + [2r_1 + P_2] \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \\ 0 = \left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)^2 + \varphi_2 + 2r_2 \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \\ 0 = \left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)' + [2s_2 + P_2] \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \\ 0 = \left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)^2 - \varphi_2 - 2s_1 \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} u = \cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ v = -\sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ 0 = \left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right)' + [2r_1 + P_2] \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \\ \varphi_2 = -\varphi_1 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} u = \cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ v = -\sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ 0 = \left(\left(\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} \right) e^{\int (2r_1 + P_2) dx} \right)' e^{-\int (2r_1 + P_2) dx} \\ \varphi_2 = -\varphi_1 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} u = \cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ v = -\sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ \varphi_1 = \varphi_2 + 2k(r_2 + s_1) e^{-\int (2r_1 + P_2) dx} \\ \varphi_2 = -\varphi_1 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} u = \cosh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ v = -\sinh\left(\int \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)} dx \right) \\ \varphi_1 = \varphi_2 + 2k(r_2 + s_1) e^{-\int (2r_1 + P_2) dx} \\ \varphi_2 = -k(r_2 + s_1) e^{-\int (2r_1 + P_2) dx} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} u = \cosh\left(k \int e^{-\int (2r_1 + P_2) dx} dx \right) \\ v = -\sinh\left(k \int e^{-\int (2r_1 + P_2) dx} dx \right) \\ \varphi_1 = -\varphi_2 = k(r_2 + s_1) e^{-\int (2r_1 + P_2) dx} \end{array} \right.$$

□

Corollary I.3: For differentiable functions $u, v, y_1, y_2, r_1, r_2, s_1, s_2, Y_1, Y_2, P, Q, \varphi$;

if y_1 & y_2 are linearly independent solutions to $y_i'' + P_i y_i' + Q_i y_i = 0$
and Y_1 & Y_2 are linearly independent solutions to $Y_i'' + PY_i' + (Q_i + \varphi_i)Y_i = 0$
such that: $\exists u, v$: (u, v linearly independent of y_1, y_2 : $\forall x$)

$$\begin{cases} \boxed{Y_1 = vy_1 + uy_2} \\ \boxed{Y_2 = uy_1 - vy_2} \end{cases}$$

and: $\exists r_1, r_2, s_1, s_2$:

$$\begin{cases} \boxed{y_1' = r_1 y_1 + s_1 y_2} \\ \boxed{y_2' = r_2 y_1 + s_2 y_2} \end{cases}$$

then:

if: $(r_2 + s_1) \neq 0$ & $(r_1 - s_2) \neq 0$

Proof:

Given the main theorem:

if: $(r_2 + s_1) \neq 0$ & $(r_1 - s_2) \neq 0$

$$\Rightarrow \begin{cases} \frac{(r_1 - s_2)}{(r_2 + s_1)}u' - \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}u = v' \\ -\frac{(r_1 - s_2)}{(r_2 + s_1)}v' - \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}v = u' \\ 0 = u'' + [2r_1 + P_2]u' + \varphi_2 u - 2r_2 v' \\ 0 = v'' + [2s_2 + P_2]v' + \varphi_2 v - 2s_1 u' \end{cases}$$

$$\Rightarrow \begin{cases} \frac{(r_1 - s_2)}{(r_2 + s_1)}u' - \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}u = v' \\ -\frac{(r_1 - s_2)}{(r_2 + s_1)}v' - \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}v = u' \\ 0 = u'' + [2r_1 + P_2]u' + \varphi_2 u - 2r_2\left[\frac{(r_1 - s_2)}{(r_2 + s_1)}u' - \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}u\right] \\ 0 = v'' + [2s_2 + P_2]v' + \varphi_2 v - 2s_1\left[-\frac{(r_1 - s_2)}{(r_2 + s_1)}v' - \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}v\right] \end{cases}$$

$$\Rightarrow \begin{cases} \frac{(r_1 - s_2)}{(r_2 + s_1)}u' - \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}u = v' \\ -\frac{(r_1 - s_2)}{(r_2 + s_1)}v' - \frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}v = u' \\ 0 = u'' + \left[2r_1 + P_2 - 2r_2\frac{(r_1 - s_2)}{(r_2 + s_1)}\right]u' + \left[\varphi_2 + 2r_2\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}\right]u \\ 0 = v'' + \left[2s_2 + P_2 + 2s_1\frac{(r_1 - s_2)}{(r_2 + s_1)}\right]v' + \left[\varphi_2 + 2s_1\frac{(\varphi_1 - \varphi_2)}{2(r_2 + s_1)}\right]v \end{cases}$$

□

Although these theorems, corollaries, and equations may seem pointless/purposeless, further research examination and investigation yield worthwhile results.