The Cover Theorem

Eliahu Shaikevich

Sept. 2020

Abstract
There is such a historical number of trades N0 of both parities and such a way of placing them on the market that for any N > N0 the following inequality always holds: M+(i) ≥ M−(i) + V−(i). we call it the Cover Theorem.

1 Basic Concepts

1.1

Parity of trades. All trades in the market are divided into even and odd. We define *"buy"* trades as ***even*** trades and *"sell"* trades as ***odd*** trades. We define N – the historical number of trades, both even and odd, placed on the market, from the first trade (of any parity) to N.

1.2 $M_+ - Measure$

Let's define the construction M_+. M_+ is a positive measure that in market theory functionally depends on the way trades are placed in the market. In the simplest case, M_+ is a linear function of the argument associated with the historical number of trades placed on the market.

$$M_+ = k \cdot N = k \cdot (N_b + N_s) \tag{1}$$

Where N is the total historical number of trades of both parities (even and odd) placed on the market. Certainly, we can consider any, in "some" way, allowed functions of N. This "some" meaning will be clarified in future works.

1.3

Here we introduce the construction M_- – a negative measure, which in market theory has a sum structure. To form this construction, we use sums of this type:

$$i + (i - 1) + .. + 1 \tag{2}$$

where i is the serial number of the trade, taking into account the parity of the trade. Obviously this is an arithmetic progression. We define the following:

$$M_{-}(i) = k_1 \cdot \sum_{j=1}^{i-1} j = k_1 \cdot \frac{1 + (i-1)}{2} \cdot (i-1) = \frac{k_1}{2} \cdot (i^2 - i) \tag{3}$$

In this construction i numbers trades of only one type of parity, the number of which is greater (compared to other parity) in the market for the current state.

1.4 $V_{-} - Value$

Here we define the construction v_{-} – a negative measure of the value of a single trade, which indicates the fixed value of a trade when it is placed on the market. v_{-} does not depend on the current market price, after setting (constant value).

$$v_{-}(i) = C(i) \cdot P_{fix}(i) \tag{4}$$

here $P_{fix}(i)$ is the price of a trade on the market at the time of opening this trade.

Now we define

$$C(i) = L \cdot U(i). \tag{5}$$

$C(i)$ is a constant associated with the amount of the purchased asset $U(i)$ in conventional units (unit) and taking into account the brokerage leverage L. Let's summarize for all trades without considering the parity

$$V_{-}(i) = \sum_{j=1}^{i} v_{-}(j) = \sum_{j=1}^{i} C(j) \cdot P_{fix}(j) \tag{6}$$

We note that the sum (6) depends on the current market price, unlike (4), since $P_{fix}(j)$ is different for each trade and depends on the trade price.

1.5 $W - Width$

We call the width (half-width) of the trading channel the next function:

$$W(N) = k_{norm} \cdot \sqrt{N} + P_{fix}(1) \tag{7}$$

here 1 is the first trade of any parity. k_{norm} – normalizing coefficient depending on the traded pair. Let's define function (7) and its analogs as a market development function. It will be interesting to prove in subsequent works

the invariance of the market development function regarding the choice of the initial point of entry into the market. In this paper, we assume this invariance axiomatically.
Note one important property of the first derivative of this function,

$$W'(N) = k_{norm} \cdot \frac{1}{2\sqrt{N}}, \tag{8}$$

tends to 0 with increasing N.

We formulate the **Cover Theorem** for such a class of functions for which the first derivative tends to zero as $N \to \infty$

2 Formulation of the Cover Theorem

There is such a historical number of trades N_0 of both parities and such a way of placing them on the market that for any $N > N_0$ the following inequality always holds:

$$M_+(i) \geq M_-(i) + V_-(i) \tag{9}$$

we call it the "Shaikevich inequality"or the Cover Theorem
First, we prove the theorem under the assumption that $V_-(i) = 0$. We explain in a separate chapter why this can be done. Of course, we cannot neglect $V_-(i)$, but in order to understand the mechanism of the theorem, now we do it and later return this term of the inequality to consideration.
Let us assume as an axiom that the coefficient k from formula (1) and the coefficient k_1 from formula (3) are equal to each other: $k = k_1$
Let us prove the inequality:

$$M_+(i) \geq M_-(i) \tag{10}$$

When the market evolves according to formula (7), the value of the function $W(N)$ for some N_0 will be:

$$W(N_0) = k_{norm} \cdot \sqrt{N_0} + P_{fix}(1) \tag{11}$$

In this case, the positive measure M_+ by the time of the trades N_0 will be equal to

$$M_+(N_0) = k \cdot N_0 \tag{12}$$

Than we normalize the function $W(N_0)$ from formula (11)

$$w(N_0) = \frac{1}{k_{norm}} \cdot \ [W(N_0) - P_{fix}(1)] = \sqrt{N_0} \tag{13}$$

And correspondingly

$$w^2(N_0) = N_0 \tag{14}$$

Let the trade N_1 satisfy the condition $N_1 > N_0$ and necessarily have the same parity as N_0. The trade are made to the function $w(N)$ through a unit step. So we write

$$w(N_1) = w(N_0) + 1 \tag{15}$$

or

$$w^2(N_1) = [w(N_0) + 1]^2 = w^2(N_0) + 2 \cdot w(N_0) + 1. \tag{16}$$

Next, we transform

$$w^2(N_1) - w^2(N_0) = 2 \cdot w(N_0) + 1, \tag{17}$$

replace by the formula (14):

$$N_1 - N_0 = 2 \cdot w(N_0) + 1, \tag{18}$$

We multiply by the coefficient k both sides of the equation

$$k \cdot N_1 - k \cdot N_0 = 2 \cdot k \cdot w(N_0) + k \tag{19}$$

Replacing the definition of a positive measure from formula (12), we obtain the change in the positive measure during the development of the market

$$M_+(N_1) - M_+(N_0) = 2 \cdot k \cdot w(N_0) + k \tag{20}$$

or

$$\Delta M_+(N_0) = 2 \cdot k \cdot w(N_0) + k. \tag{21}$$

Now we calculate how the negative measure M_- changes with the development of the market according to the formula (11).

Suppose that by the moment of placing a trade N_0 the number of unclosed trades of the same parity as N_0 is equal to $n = [1...n]$. Then using the definition of negative measure from formula (3) we obtain

$$M_-(N_0) = k_1 \cdot \sum_{j=1}^{n} j = k_1 \cdot \frac{1+n}{2} \cdot n = \frac{k_1}{2} \cdot (n^2 + n), \tag{22}$$

because the N_0 trade was a $n+1$ trade of the same parity as the n trade. The negative measure when placing a trade N_1 is equal to

$$M_-(N_1) = k_1 \cdot \sum_{j=1}^{n+1} j = k_1 \cdot \frac{1+(n+1)}{2} \cdot (n+1) = \frac{k_1}{2} \cdot ((n+1)^2 + (n+1)), \tag{23}$$

because the N_1 trade was a $n+2$ trade of the same parity as the n trade. Then

$$\Delta M_-(N_0) = M_-(N_1) - M_-(N_0) = \frac{k_1}{2} \cdot (2n+2) = k_1 \cdot (n+1). \tag{24}$$

Now we compare the change in positive and negative measures from formula (10) and substituting formulas (21) and (24):

$$2 \cdot k \cdot w(N_0) + k \geq k_1 \cdot (n+1) \tag{25}$$

or, since axiomatically $k = k_1$, we get:

$$2 \cdot w(N_0) \geq n. \tag{26}$$

As we wrote above, n is the number of open trades of the same parity as N_0. All these trades were placed on the market similarly to the N_0 trade in accordance with the $w(N)$ function.

We can assert that the number of unclosed trades of the same parity as N_0 at the moment of placing the trade N_0 is equal to:

$$n(N_0) = w(N \leq N_0), \tag{27}$$

where N is less than N_0, or

$$n(N_0) = w(N_0) - 1, \tag{28}$$

where 1 is the same step unit as in formula (15). Then from formula (26) we obtain the fundamental market equation showing that the market is always profitable:

$$w(N_0) \geq (-1). \tag{29}$$

Formula (29) is a fundamental market equation and is called the ***"Cover Theorem"***.

An important point should be noted that

$$n = \sqrt{N}. \tag{30}$$

Note that due to dependence (30) the Cover Theorem holds.

Now we prove the complete Cover Theorem for any market development function $f(N)$ with $f(N \to \infty) \to 0$.

We start with the fact that formula (29) actually proves the following: the change in a positive measure is always greater than the change in a negative measure when the market develops according to formula (11) for any $N \geq N_0$. We can rewrite formula (10) in already known terms and substituting that $k = k_1$ and taking into account formulas (27) and (14): $n(N) = w(N) = \sqrt{N}$.

If $n(N) = f(N)$ is any arbitrary function, then we have the rewritten formula (10):

$$N \geq \frac{1}{2} \cdot (n^2 + n). \tag{31}$$

For example, we take for n the function $f(N) = \sqrt{N}$. Then formula (31) takes the form:

$$2N \geq N + \sqrt{N} \tag{32}$$

or

$$\sqrt{N} \geq 1 \tag{33}$$

We can restrict ourselves to formula (33) for proving the Cover Theorem, but we have proved it in an even stronger form: the change in positive measure is greater than or equal to the change in negative measure for any $N \geq N_0$. Let's return to our function $f(N)$ and rewrite formula (31) as follows

$$N \geq \frac{1}{2} \cdot (f^2(N) + f(N)). \tag{34}$$

We have limited $f(N)$ with $f'(N \to \infty) \to 0$. Simple transformations and differentiation give us the correct result after some N:

$$2 \geq 2 \cdot f(N) \cdot f'(N) + f'(N) \tag{35}$$

Letting $N \to \infty$ and $f'(N) \to 0$.
Then

$$2 \geq 0 \tag{36}$$

The complete Cover Theorem is proved!
In a separate chapter, we explain why we could differentiate both sides of inequality (34) and be sure that the sign of inequality does not change. Now we can say that this is due to the fact that $f'(N)$ tends to 0, and $f'(N) > 0$.

Now we remove the constraint $f'(N \to \infty) \to 0$ and simply find the class of functions that satisfies inequality (34). Solving the square inequality

$$f^2(N) + f(N) - 2 \cdot N \leq 0 \tag{37}$$

we obtain a class of market development functions that satisfy the Cover Theorem (34) for an arbitrary function:

$$0 \leq f(N) \leq \frac{-1 + \sqrt{1 + 8 \cdot N}}{2}, \tag{38}$$

or for $N >> 0$

$$0 \leq f(N) \leq \sqrt{2 \cdot N}. \tag{39}$$

An interesting result follows from formula (39) that any market development function that is less than $\sqrt{2 \cdot N}$ satisfies the Cover Theorem (31). The proof of the statement that if some function F is less than a function G for which the first derivative G' tends to 0, then F itself must have the first derivative F' tending to 0 is obvious.

So we considered the "mechanism"of the Cover Theorem and proved that not only

$$M_+(i) - M_-(i) \geq 0, \tag{40}$$

but

$$\Delta M_+(i) - \Delta M_-(i) \geq 0. \tag{41}$$

Now we return the $V_-(i)$ term and prove formula (9) in a rewritten form

$$M_+(i) - M_-(i) \geq V_-(i), \tag{42}$$

for any $N \geq N_0$

3 Term $V_-(i)$

This term means "investing"in the market. Let's look at the meaning of this statement. Suppose you bought an asset at 100 \$ and its price rises to 200 \$. The profit of the asset is 200\$ - 100\$ = 100\$ and it has the same meaning as M_+. So you returned the invested 100 \$ $= v_-(i)$ and got additional 100 \$.

Now let's consider if the asset value has dropped to 70 \$. This means that if you have invested 100 \$ in the market, you can potentially take 70 \$ from the market. But mathematically, these are all the same 100 \$ $= v_-(i)$ investments that you return from the market minus a loss of 30 \$. This loss is the same as M_-.

Thus, we set $V_-(i) = 0$, taking as the model zero investment in the market, and counting only profits and losses. We show above that the profit will always exceed the loss for a certain class of market development functions and after some $N \geq N_0$.
We also showed that the change in profit will exceed the change in loss for the specified class of market development functions.

Nevertheless, a legitimate question arises: what if the investment in the market will not be covered by the difference $M_+(i) - M_-(i)$ as indicated in formula (42).
Now we can rewrite this formula in expanded form, taking into account the fact that there are n trades of one parity on the market without profit after placing N trades of both parities on the market.

$$N - \frac{1}{2} \cdot (n^2 + n) \geq \sum_{j=1}^{n} C(j) \cdot P_{fix}(j), \tag{43}$$

or

$$2 \cdot N - N - \sqrt{N} \geq \sum_{j=1}^{n} 2 \cdot C(j) \cdot P_{fix}(j). \tag{44}$$

On the right side, n in the summation sign does not exceed $\sqrt{N}$, as follows from the conclusions from formula (27) for the market development function $\sim \sqrt{N}$ (see formula (7)).

Let us now consider the right-hand side of the inequality (44).
Based on the formula (5) and taking as an additional condition that the trades are homogeneous, that is, they carry the same number of units, we can say that $2 \cdot C(j) = 2 \cdot C$
The number of terms in the sum (44) will not exceed $\sqrt{N}$ on average and this follows from the fact that $n = \sqrt{N}$. Further, since $P_{fix}(j) > P_{fix}(j-1)$, then we can prove inequality (44) for the maximum values of the price $P_{fix}(n)$, and if it holds, then it will also be true for the sum $\sum_{j=1}^{n} 2 \cdot C \cdot P_{fix}(j)$.

We rewrite the inequality (44) taking into account the above arguments:

$$N - \sqrt{N} \geq \sqrt{N} \cdot 2 \cdot C \cdot P_{fix}(n), \tag{45}$$

or

$$\sqrt{N} \geq 2 \cdot C \cdot P_{fix}(n) + 1. \tag{46}$$

We see that the right-hand side of inequality (46) is a constant, so there is N_0 such that for any $N \geq N_0$, inequality (46) will always be satisfied.
We have proved a very strong statement - any market is profitable as long as it develops on average according to the formula $\sim \sqrt{N}$. Most markets develop precisely according to this dependence, but even if this condition is not met, it is sufficient that the market development function would be less than or equal to $\sqrt{2 \cdot N}$. If this condition is not met, then the condition for the derivative of the market development function $f'(N \to \infty) \to 0$ must be met. In this case, inequality (43) is fulfilled. The proof of this inequality (43) for an arbitrary market development function $f(N)$, such that $f'(N \to \infty) \to 0$, will be considered in a separate work.