Proving Unproved Euclidean Propositions on a New Foundational Base

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Abstract.-This paper introduces a new foundational basis for Euclidean geometry that includes productive definitions of concepts so far primitive, or formally unproductive, allowing to prove a significant number of axiomatic statements, unproved propositions and hidden postulates, among them the strong form of Euclid's First Postulate, Euclid's Second Postulate, Hilbert's Axioms I.5, II.1, II.2, II.3, II.4 and IV.6, Euclid's Postulate 4, Posidonius-Geminus' Axiom, Proclus' Axiom, Cataldi's Axiom, Tacquet's Axiom 11, Khayyām's Axiom, Playfair's Axiom, Euclid's Postulate 5. The proposed foundation is more formally detailed and productive than other classical and modern alternatives, and at least as accessible as any of them.

1. Introduction

After more than two millennia of discussions on Euclid's original geometry and at a time in which such discussions have been practically abandoned, this article introduces a new foundational basis for Euclidean geometry that includes productive definitions of concepts so far formally unproductive, as sidedness, betweenness, straight line, straightness, angle, or plane, among others (all of them properly legitimized by axioms or by formal proofs). The result is an enriched Euclidean geometry in which it is possible to prove some propositions that were proved to be unprovable on other Euclidean geometry bases. It will be introduced in the next sections. Conventions and general fundamentals are the objectives of Section 2. Section 3 introduces the new foundational basis: 28 definitions, 10 axioms and 40 corollaries (8 of the 10 axioms and most of the 40 corollaries are implicit (hidden) postulates in other Euclidean geometries). Section 5 introduces plane Euclidean geometry through 43 basic propositions and corollaries on triangles, perpendicularity, parallelism, and convergence.

2. Conventions and general fundamentals

The nth axiom, corollary, definition, postulate and proposition will be referred to, respectively, as [Ax. n], [Cr. n], [Df. n], [Ps. n] and [Pr. n]. The same letters, for instance AB or BA, will be used to denote a line of endpoints A and B [Df. 1], as well as its length [Df. 8], and the distance between A and B [Df. 14] if AB is a straight line [Df. 10]. Unless otherwise indicated, different letters will denote different points, including endpoints. When convenient, lines will also be denoted by lower case Latin letters, whether or not indexed. Symbols as $0, +, -, =, \neq, \leq$, etc. will be used conventionally. The expressions 'point in a line,' and 'point of a line' will be used as synonyms. The same goes for 'line in a plane' and 'line of a plane'. Closed lines [Df. 2] will be referred to as such closed lines, or by specific names, as circle [Df. 18]. As in classical Euclidean geometry [5, p. 8], [3, p. 153], in Euclidean geometry a straight line is a particular type of line. So, and in contrast with modern English, in Euclidean geometry 'line' and 'straight line' are not synonyms. Asterisked expressions as 'for instance*', 'for example*', 'assume*' etc., will always indicate that only one of the possible alternatives in a proof will be considered and proved, because the other alternatives can be proved in the same way. Proofs begin with the symbol $\triangleright$ and end with the symbol $\square$. The biconditional logical connective will be shortened by the term 'iff'. And, unless otherwise indicated, the word 'number' will always mean natural number.

The following definitions and postulates are not exclusive to geometry, they have a general use in all sciences. For that reason they have been separated from the very fundamentals of geometry.

Definition 1 *A quantity to which a real number can be assigned is said a numerical quantity. Numerical quantities that can be symbolically represented and operated with one another according to the procedures and laws of algebra, are said operable values.*

Definition 2 *An operable value is said to vary in a continuous way iff for any two different operable values of the corresponding variation, the variation contains any operable value greater than the less and less than the greater of those two operable values.*

Definition 3 *Metric properties and metric transformations: properties (transformations) to which operable values that vary in a continuous way are univocally assigned: to each quantity of the property (transformation) a unique and exclusive operable value, even zero, is assigned.*

Definition 4 *To define an object is to give the properties that unequivocally identifies the object. Objects with the same definition are said of the same class. To draw objects is to make a descriptive representation of them by means of graphics or texts, or by both of them, without the drawing modifies neither their properties nor their relative relations with other objects, if any.*

Postulate A *Of any two operable values, either they are equal to each other, or one of them is greater than the* *other, and the other is less than the one. Symbolic representations of equal operable values, or of equal objects, are interchangeable in any expression where they appear.*

Postulate B *To be less than, equal to, or greater than, are transitive relations of operable values that are preserved when adding to, subtracting from, multiplying or dividing by the same operable value, the operable values so related. Metric properties (transformations) are algebraically operable through their corresponding operable values.*

Postulate C *Belonging to, and not belonging to, are mutually exclusive relations. Belonging to is a reflexive and transitive relation.*

Contrarily to, for instance, fuzzy set theory or non-Boolean logics, this Euclidean geometry assumes [Ps. C], according to which it is not possible for an object to partially belong and partially not to belong to another object.

3. Foundational basis of Euclidean geometry

Fundamentals on lines

Definition 1 *Endedness.- A point at which a line ends is said endpoint. If such a point belongs to the line, the line is said closed at that end; if not, the line is said open at that end. Two endpoints, whether or not in the line, define two opposite directions in the corresponding line, each from an endpoint, said initial, to the other, said final.*

Definition 2 *Collinearity.-A line is said through one or more given points iff such given points are points of the line. A line whose points belong, all of them, to a given line is said a segment of the given line. Points, or segments, or points and segments, are said collinear if they belong to the same line, and non-collinear if they do not belong to the same line.*

Definition 3 *Commonness.-Two lines (surfaces) are said different from each other iff one of them has at least one point that is not in the other. Points and segments belonging to different lines are said common to them, otherwise they are said non-common to them. Non-collinear lines with at least one common segment are said locally collinear. Lines without common segments but with at least one common point are said intersecting lines, and their common points are also said intersection points. Intersecting lines are said to cut or to intersect one another at their intersection points.*

Definition 4 *Adjacency.-Lines whose only common point is a common endpoint are said adjacent at that common endpoint iff no point of any of them is a non-common endpoint of any of the others. Lines containing all points of a given line, and only them, are said to make the given line. If any two points of a line are the common endpoints of only two segments of the line, the line is said closed or figure. Otherwise it is said non-closed. Closed lines that are not self-intersecting are said simply closed.*

Definition 5 *Sidedness.-Adjacent lines containing all points of a given line, and only them, whose common endpoint is a given point of the given line and whose non-common endpoints are the endpoints of the given line, if any, are said sides of the given point in the given line.*

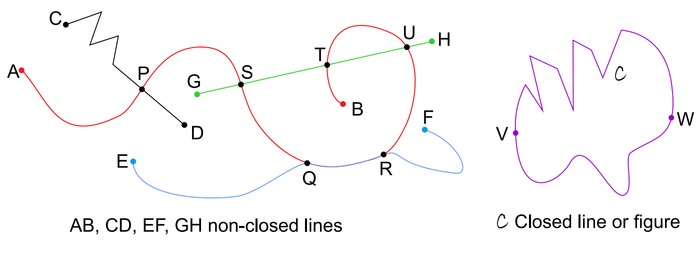


Figure 1 – A, B: endpoints of AB; C, D: endpoints of CD etc. AB, EF: locally collinear lines. AP, PS: lines (segments) adjacent at P. AP, PB: sides of P in AB. QR: common segment of AB y EF. S is between A y Q; between P and R etc.

Definition 6 *Betweenness.-A point is said to be between two given points of a line, iff it is a point of the line and each of the given points is in a different side of the point in the line.*

Definition 7 *Uniformity.-Lines whose segments have the same definition as the whole line are said uniform. Two or more uniform lines are said mutually uniform iff any segment of any of them has the same definition as any segment of any of the others.*

Definition 8 *Metricity.-Length (area) is a metric property of lines (figures) of which arbitrary units can be defined. Lengths (areas) are said equal iff their corresponding operable values are equal. Lines (figures) with a* *finite length (area) are said finite. If the sides of a point of a line have the same length, the point is said to bisect the line.*

Axiom 1 *Any number of points, lines, surfaces and solids can be considered and drawn, being, or not, the points in lines, the lines in surfaces and the surfaces in solids.*

Axiom 2 *A line has at least two points, at least one point between any two of its points, and at most two endpoints, whether or not in the line.*

Axiom 3 *Two adjacent lines make a line, and a point of a line can be common to any number of any other different lines, either collinear, or non-collinear, or locally collinear.*

Axiom 4 *Except endpoints, each point of a non-closed line has two, and only two, sides in the line, whose lengths are greater than zero and sum the length of the whole line.*

Unless otherwise indicated, only non-self-intersecting and non closed lines whose endpoints, if any, belong to the line will be considered.

Corollary 1 *The number of points of a line is greater than any given number.*

▷ It is an immediate consequence of [Axs. 1, 2]. □

Corollary 2 *Each side of a point, except endpoints, of a line is a segment of the line and both sides make the line.*

▷ Except endpoints, a point P of a line l [Ax.1, Cr. 1]

has two, and only two, sides in l [Ax. 4],

which are two lines adjacent at P [Df. 5]

containing all points of l, and only them [Df. 5].

So, each side is a segment of the line [Dfs. 5, 2],

and both sides make the line l [Ax. 3, Df. 4]. □

Corollary 3 *Any point of a line is in one, and only in one, of the two sides of any other point, except endpoints, of the line.*

▷ Except endpoints, a point P of a line l [Ax.1]

has two, and only two, sides in l [Ax. 4].

Any other point of l [Cr. 1]

will be in one of such sides [Cr. 2],

and only in one of them, otherwise both sides would not be adjacent at P [Df. 4],

which is impossible [Dfs. 5, 4]. □

Corollary 4 *A point is in a line with two endpoints iff, being not an endpoint of the line, it is between the endpoints of the line.*

▷ If a point P is between the two endpoints of a line AB [Axs. 1, 2, 4, Df. 6],

it is in AB [Df. 6].

If a point P is in a line AB [Cr. 1],

it has just two sides in AB [Ax. 4],

whose respective non-common endpoints are the endpoints A and B of AB [Dfs. 5, 4].

So, P is between both endpoints A and B [Df. 6]. □

Corollary 5 *Any two points of a line are the endpoints of a segment of the line. And the line has a number of segments and a number of points between any two of its points greater than any given number.*

▷ Let P and Q be any two points of a line l different from its endpoints, if any [Ax.1, Cr. 1].

Q has two sides in l [Ax. 4],

which are two lines adjacent at Q [Df. 5]

that contains all points of l and only them [Cr. 2].

So, in one, and only in one, of such lines will be P [Cr. 3].

In turn, P has two sides in that side of Q [Df. 5, Ax. 4],

the side PQ in which it is Q and the side in which it is not Q [Cr. 3].

PQ is a line [Df. 5]

all of whose points belong to l [Df. 5].

Hence, PQ is a segment of l [Df. 2].

Being P and Q any two of its points, l has a number of segments and a number of points between any two of its points greater than any given number [Crs. 1, 4] □

Corollary 6 *A segment of a segment of a line, it is also a segment of the line.*

▷ Let RS be a segment of a segment PQ of a line l [Ax.1, Cr. 5].

PQ is a line whose points belong to l [Df. 2].

RS is a line whose points belong to PQ [Df. 2],

and then to l [Ps. C].

So, RS is a segment of l [Df. 2]. □

Corollary 7 *If a point is between two given points of a given line, it is also between the given points in any line of which the given line is a segment.*

▷ Let R be a point of a segment PQ of a line l' [Ax.1, Cr. 5],

which is a segment of another line l [Cr. 5].

Since PQ is a segment of l', it is also a segment of l [Cr. 6].

So, R is a point of a segment PQ of l [Df. 2],

and then a point of l [Df. 2]

between P and Q [Cr. 4]. □

Corollary 8 (A variant of Hilbert's Axiom II.2) *At least one of any three points of a line is between the other two.*

▷ Let P, Q and R be any three points of any line l [Ax.1, Cr. 1].

At least one of them, for example* Q, will not be an endpoint of l [Ax. 2].

P can only be in one of the two sides of Q in l [Cr. 3];

R can only be in one of the two sides of Q in l [Cr. 3].

So, either P and R are in different sides of Q in l, or they are in the same side of Q in l. If P and R are in different sides of Q in l, then Q is between P and R in l [Df. 6].

If not, P and R are in the same side of Q in l, which is a segment l' of l [Cr. 2],

one of whose endpoints is Q [Df. 5].

If R is an endpoint of l', P can only be between the endpoints Q and R of l' [Cr. 4],

and then between Q and R in l [Cr. 7].

If R is not an endpoint of l', it has two sides in l' [Ax. 4]:

the side RQ in which it is Q, and the side in which it is not Q [Cr. 3].

If P is in RQ, it is between R and Q in l' [Cr. 4],

and then between R and Q in l [Cr. 7].

If P is in the side of R in l' in which it is not Q, then P and Q are in different sides of R in l', and R is between P and Q in l' [Df. 6]

and then between P and Q in l [Cr. 7].

So, in all possible cases [Ax. 4, Cr. 3]

at least one of the three points is between the other two in l. □

Corollary 9 (Hilbert's Axioms II.3, II.1) *One, and only one, of any three points of a line is between the other two.*

▷ Let P, Q and R be any three points of any line l [Ax.1, Cr. 1].

At least one of them, for example* Q, will be between the other two, P and R, in l [Cr. 8],

in which case Q is a point of PR [Cr. 4].

So, Q has two sides in PR [Ax. 4],

which are two lines, QP and QR, adjacent at Q [Df. 5].

P cannot be between Q and R, otherwise it would be in QR [Cr. 4],

QP would be a segment of QR [Cr. 5],

all points QP [Cr. 1]

would be points of QR [Df. 2],

and QP and QR would not be adjacent at Q [Df. 4],

which is impossible [Df. 5].

For the same reasons R cannot be between P and Q either. Therefore, one [Cr. 8],

and only one, of any three points of a line is between the other two. □

Corollary 10 (a variant of Hilbert's Axiom II.4) *Of any four points of a line, two of them are between the other two.*

▷ Let P, Q, R and S be any four points of a line l [Ax.1, Cr. 1].

Consider any three of them, for instance* P, Q and R. One, and only one, of them, for instance* Q, will be between the other two, P and R [Cr. 9],

and Q will be in PR [Cr. 4].

Of the other three points P, R and S, one, and only one, of them will be between the other two [Cr. 9]:

if P is between R and S, it is in SR [Cr. 4],

so that PR is a segment of SR [Cr. 5],

Therefore Q, which is in PR, is also in SR [Cr. 6].

So, Q and P are between R and S [Cr. 4].

For the same reasons, if R is between P and S then Q and R are between P and S; and if S is between P and R, then Q and S are between P and R. So, in all possible cases [Ax. 4, Cr. 3]

two of the four points are between the other two. □

Corollary 11 *Two segments can only be either collinear or non-collinear. And if a segment of a given line is non-collinear with another segment of another given line, then both given lines are also non-collinear.*

▷ Since belonging to is a reflexive relation [Ps. C]

and segments are lines [Df. 2],

any two segments l_1 and l_2 [Ax. 1]

belong to a line, even if the line is the own segment itself [Df. 2].

So, l_1 and l_2 will be either collinear, or non-collinear, or collinear and non-collinear. If they were collinear and non-collinear they would be segments that belong to the same line l [Df. 2],

and segments that do not belong to the same line l [Df. 2],

which is impossible [Ps. C].

So, l_1 and l_2 can only be either collinear or non-collinear. Let now l'_1 be a segment of a line l_1 and l'_2 another segment of a line l_2 [Cr. 5],

such that l'_1 and l'_2 are non-collinear [Df. 2].

If l_1 and l_2 were collinear, they would be segments of the same line l [Df. 2],

and being their respective segments l'_1 and l'_2 also segments of l [Cr. 6],

l'_1 and l'_2 would also be collinear [Df. 2],

which is not the case. Hence, l_1 and l_2 must also be non-collinear. □

Corollary 12 *If two points of a line have a given property, and all points between any two points with the given property have also the given property, then the line has a unique segment whose points are all points of the line with the given property.*

▷ Let A and B be two points [Ax.1, Cr. 5]

with a given property (gp-points for short) of a line l such that all points of l between any two of its gp-points are also gp-points. So, l has a number of gp-points greater than any given number [Cr. 5].

Let a segment whose points are gp-points, except at most its endpoints, be referred to as gp-segment. Any gp-point C of l is at least in the gp-segment AC of l [Crs. 5, 4].

So, all gp-points of l are in gp-segments. If all gp-points of l were not in a unique gp-segment, they would be in at least two gp-segments DE and FG of l [Cr. 5],

so that, being* E and F between D and G [Cr. 10],

DG is not a gp-segment. If so, there will be at least one point P between D and G that is not a gp-point. P has two sides in DG, namely PD and PG [Ax. 4, Df. 5].

E must be in the side PD of P in DG in which it is D, otherwise it would be in the side PG of P in DG in which it is not D [Cr. 3],

P would be between D and E [Df. 6],

and it would be a gp-point of DE [Cr. 4],

which is not the assumed case. So, DE is a segment of the side PD of P in DG [Crs. 5].

For the same reasons, FG is a segment of the other side PG of P in DG. Hence, P is between any gp-point of DE and any gp-point of FG [Df. 5].

It is then impossible for P not to be a gp-point, and for DG not to be a gp-segment. □

Corollary 13 *The length of a finite line is greater than the length of each of the sides of any of its points, except endpoints, and it is greater than zero. The length of each side is equal to the length of the whole line minus the length of the other side. And the length of a segment of the line is less than the length of the whole line if at least one endpoint of the segment is not an endpoint of the line.*

▷ Let P be a point of a finite line AB [Df. 8, Axs. 1, 2].

Assume the length AP is not less than the length AB. It will be $AP \geq AB$ [Ps. A],

and being $AB = AP + PB$ [Ax. 4],

it would hold $AP \geq AP + PB$ [Ps. A].

Hence, $0 \geq PB$ [Ps. B],

which is impossible [Ax. 4].

So, it must be $AP < AB$ [Ps. A].

And for the same reasons $PB < AB$. Therefore, and being $0 < PB$ [Ax. 4],

it holds $0 < AB$ [Ps. B].

So, the length of any line is greater than zero. And from $AP + PB = AB$ [Ax. 4],

it follows immediately $AP = AB - PB$; $PB = AB - AP$ [Ps. B].

Let now Q be any point of AB different from P [Crs. 1].

It will be in one, and only in one, of the sides of P in AB [Cr. 3],

for instance* in AP. It has just been proved that $AP < AB$. If Q were the endpoint A of AP we would have $QP = AP$ [Ps. A].

If not, and for the same reasons above, it will be $QP < AP$. So that we can write $QP \leq AP$, and then $QP < AB$ [Pss. B, A].

Therefore, the length of a segment of AB is less than AB if at least one if its endpoints P is not an endpoint of AB. □

Fundamentals on straight lines

Definition 9 *Extensible lines.-To produce (extend) a given line by a given length is to define a line, said production (extension) of the given line, so that the production has the given length, is adjacent to the given line and the production and the produced lines are of the same class as the given line. Lines that can be produced from any of its endpoints by any given length are said extensible.*

Definition 10 *Straight lines: extensible and mutually uniform lines that cannot be locally collinear nor have non-common points between common points.*

Definition 11 *Straightness.-Three or more points are said to be in straight line with one another iff they are in the same straight line, whether or not produced. A point is said in straight line with a given straight line iff it is in straight line with any two points of the given straight line, whether or not produced. Collinear straight lines whose common line is a straight line, and only them, are said to be in straight line with one another.*

Axiom 5 *Any two points can be the endpoints of a straight line, and only both points are necessary to draw the straight line.*

Corollary 14 *A segment of a straight line is also a straight line.*

▷ It is an immediate consequence of [Ax. 5, Dfs. 10, 7]. □

Corollary 15 (Strong form of Euclid's First Postulate) *Any two points can be the endpoints of one, and only of one, straight line.*

▷ Assume two different straight lines l_1 and l_2 have the same endpoints A and B. At least one of them will have a point which is not in the other [Df. 3].

And they would have at least one non-common point between the two common points A and B, which is impossible [Df. 10].

So, any two points can be the endpoints of one [Ax. 5],

and only of one, straight line. □

Hereafter, to join two points will always mean to consider and draw the unique straight line whose endpoints are both points.

Corollary 16 (Strong form of Euclid's Second Postulate) *There is one, and only one, way of producing a given straight line by any given length and from any of its endpoints, being the produced line a straight line; and the given straight line and its production, adjacent straight lines in straight line with each other.*

▷ Let AB be any straight line [Ax. 1, Cr. 15].

It can be produced from any of its endpoints, for example* from B, by any given length [Dfs. 10, 9]

to a point C, so that BC and AC are straight lines [Dfs. 10, 9, 4],

and AB and BC are adjacent segments [Dfs. 10, 9].

Assume AB can be produced from B by the same given length to another point C'. The straight lines AC, AC' [Dfs. 10, 9]

would have a common segment AB [Cr. 5];

they would be collinear since they cannot be locally collinear [Dfs. 10, 3, Cr. 11];

and BC and BC' would be two segments of the same line l [Cr. 5],

both adjacent at B to AB [Ax. 5, Df. 9].

Being C and C' different points of the same line l, one of them, for example* C', would be between B and the other in l [Cr. 9],

and we would have $BC' < BC$ [Cr. 13],

which is not the case. So, C' can only be the point C. And being BC a straight line [Dfs. 10, 9, 4],

it is the unique straight line joining B and C [Cr. 15].

So, there is a unique way of producing a straight line by a given length from any of its endpoints. And being segments of the produced straight line AC, the straight lines AB and BC are in straight line with each other [Df. 11] □

Corollary 17 *Through any two points, any number of collinear straight lines of different lengths can be drawn.*

▷ It is an immediate consequence of [Df. 2, Crs. 15, 16]. □

Corollary 18 *Two straight lines with two common points belong to the same straight line.*

▷ All points between any two common points of two straight lines are also common points [Df. 10].

So, if two straight lines AB and CD have two common points, they have a unique segment PQ with all of their common points [Cr. 12].

PQ is a straight line [Cr. 14].

AB and PQ belongs to the same straight line l [Df. 2],

and PQ and CD belong to the same straight line l [Df. 2].

Therefore, AB and CD belong to the same straight line l [Ps. C]. □

Corollary 19 *Any point between the endpoints of a given straight line can be common to any number of intersecting straight lines not in straight line with the given straight line, and that point is the only common point of those straight lines and the given straight line, even arbitrarily producing them and the given straight line.*

▷ Any point P between the endpoints of a straight line l [Ax. 1, Cr. 15]

can be common to any number n of non-collinear straight lines [Ax. 3],

which being non-collinear cannot be in straight line with the given straight line [Df. 11].

Assume there is a second common point Q of l and of any one of those n intersecting straight lines l', whether or not producing l and l' [Cr. 16].

Both straight lines would have a common segment PQ [Df. 10, Cr. 4]

and they would be locally collinear [Df. 3],

which is impossible [Df. 10].

Therefore, P is the only intersection points of l and each of those n intersecting straight lines, even arbitrarily producing l and any of the n intersecting straight lines. □

Corollary 20 *There is a number of points greater than any given number that are not in straight line with any two given points, or with a given straight line.*

▷ Let A and B be any two points [Ax. 1].

Join A and B [Cr. 15],

and let PC be a straight line non-collinear with AB that intersects AB at P [Cr. 19].

P is the only common point of both straight lines even arbitrarily produced [Cr. 19].

So, PC has a number of points greater than any given number [Cr. 1]

none of which, except P, is in straight line with A and B [Df. 11].

On the other hand, if AB is any straight line, it has just been proved there is a number greater than any given number of points that are not in straight line with A and B, and then with AB [Df. 11]. □

Corollary 21 *Each endpoint of a given straight line can be the common endpoint of any number of adjacent straight lines not in straight line with the given straight line.*

▷ Let AB be any straight line [Ax. 1, Cr. 15].

There is a number greater than any given number of points not in straight line with AB [Cr. 20].

Join each of them with, for instance*, the endpoint A of AB [Cr. 15].

Each of these straight lines are adjacent at A to AB [Df. 4].

If any of them, for instance* AP, were in straight line with AB, they would be segments of the same straight line [Df. 11],

and P would be in straight line with AB [Df. 11],

which is not the case. □

Corollary 22 *If two adjacent straight lines are not in straight line, then no point of any of them, except their common endpoint, is in straight line with the other.*

▷ Let AB and AC be two straight lines adjacent at A not in straight line with each other [Cr. 21].

Let P be a point of, for instance*, AB [Cr. 1].

A, P and B belong to AB. So, if P were in straight line with AC, it would be in straight line with A and C [Df. 11],

P, A and C would belong to the same straight line [Df. 11],

and then A, P, B and C would belong to the same straight line [Ps. C],

which is not the case [Df. 11]. □

Fundamentals on planes

Definition 12 *Plane: a surface that contains at least three points not in straight line and any straight line through any two of its points. A line is said in a plane iff all of its points are points of the plane. Lines in a plane are said plane lines. Points, or lines, or points and lines in the same plane are said coplanar.*

Definition 13 *Sides of a given straight line in a plane: parts of the plane that contain all points of the plane, and only them, each part with at least two common points and at least two non-common points, where a point is said common, or common to all parts, if it is in straight line with the given straight line; and non-common if it is not, being said non-common of a part iff it is in that part. Any other straight line is said to be in one of those parts iff all of its points between its endpoints are non-common points of that part.*

Axiom 6 *Any three points lie in a plane, in which any straight line has just two sides. Any other straight line is in one of such sides iff its endpoints are in that side.*

Corollary 23 (A variant of Hilbert's Axiom I.5) *A plane has a number of points greater than any given number, any two of which can be joined by a unique straight line in that plane. And any given straight line is at least in a plane, in which it can be produced by any given length.*

▷ Let P, Q and R be any three points not in straight line [Cr. 20],

and Pl a plane in which they lie [Ax. 6].

Pl has at least the points P, Q and R and all points of any straight line [Cr. 1]

through any two of its points [Dfs. 12, Ax. 5, Cr. 17].

So, Pl has a number of points greater than any given number [Cr. 1].

Let, then, A and B be any two points of Pl. Join A and B [Cr. 15],

and produce AB from A and from B by any given length to the respective points A' and B' [Cr. 16].

Since $A'B'$ is a straight line [Cr. 16]

through two points A and B [Df. 2, Cr. 17]

of Pl, $A'B'$ is in Pl [Df. 12],

so that all points of $A'B'$ are in Pl [Df. 12],

and then all points of its segment AB are in Pl [Df. 2, Cr. 5].

Hence, Pl contains the unique straight line joining any two of its points A and B [Crs. 14, 15].

Let now AB be any straight line [Ax. 1, Cr. 15],

and P and Q any two of its points [Cr. 1].

There is a plane Pl containing A, P and Q [Ax. 6],

and the straight line AB through P and Q is in Pl [Df. 12].

Produce AB from A and from B by any given length to the points A' and B' respectively [Cr. 16].

Since the produced straight line $A'B'$ is a straight line [Cr. 16]

through two points A and B [Cr. 17]

of Pl, it is a straight line of Pl [Df. 12]. □

Corollary 24 *A point of a plane can only be either common to both sides of a straight line in that plane, or* *non-common of one, and only of one, of such sides.*

▷ Let A and B be any two points of a plane Pl [Ax. 6].

Join A and B [Cr. 15].

AB is in Pl [Cr. 23].

Let P be any point of Pl. Either P belongs to AB, whether or not produced [Cr. 16],

or it does not [Ps. C].

If P belongs to AB, whether or not produced [Cr. 16],

P is a point common to both sides of AB [Ax. 6, Df. 13].

If P does not belong to AB [Df. 13],

whether or not produced [Cr. 16],

P cannot be in both sides of AB [Df. 13],

and being a point of Pl, it can only be in one, and only in one, of the two sides of AB [Df. 13, Ax. 6].

So, it is a non-common point of that side, and only of it [Df. 13]. □

Corollary 25 *There is a plane containing any two adjacent straight lines not in straight line, being each of them in the same side of the other. And there is a plane containing any two intersecting and non-adjacent straight lines.*

▷ Let AB and AC be two straight lines adjacent at A and not in straight line with each other [Cr. 21].

A, B and C are not in straight line, otherwise they would be in the same straight line [Df. 11],

which is not the case [Df. 11].

So, there is a plane in which lie A, B and C [Ax. 6]

and the adjacent straight lines AB and AC [Cr. 23].

B is not in straight line with AC [Cr. 22],

so it is a non-common point of one of the sides of AC [Df. 13].

Therefore AB is in that side [Ax. 6].

For the same reasons AC is in one of the sides of AB. Let now l_1 and l_2 be any two non-adjacent straight lines that intersect at a unique point P [Cr. 19],

Q a point of l_1, and R a point of l_2 [Cr. 1].

There is a plane containing P, Q and R [Ax. 6], the straight line l_1 through Q and P [Cr. 17, Df. 12],

and the straight line l_2 though R and P [Cr. 17, Df. 12]. □

Corollary 26 *All points between two points of a straight line in the same side of a given straight line lie in that side of the given straight line, and that side has a number of non-common points greater than any given number.*

▷ Let P and Q be any two non-common points in the same side Pl_1 [Df. 13]

of a straight line l in a plane Pl [Cr. 23].

Join P and Q [Cr. 15].

PQ is in Pl_1 [Ax. 6].

All points between P and Q are non-common points of Pl_1 [Df. 13].

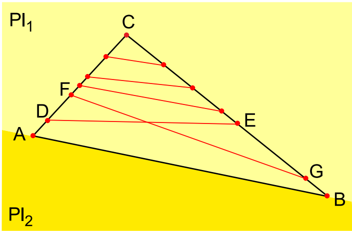
So, Pl_1 has a number of non-common points greater than any given number [Cr. 1]. □

Corollary 27 *In a plane and in each side of a straight line in that plane, it is possible the existence of a number greater than any given number of straight lines, whether or not adjacent, none of which is in straight line with any of the others.*

▷ (Fig. 2, left) Let A, B and C be any three points not in straight line [Cr. 20],

and Pl a plane in which they lie [Ax. 6].

Join A and B [Cr. 15]



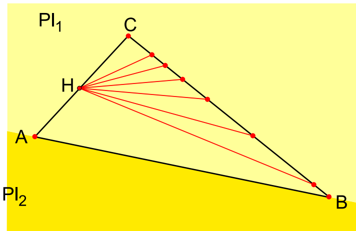


Figure 2 – Corollary 27

and let Pl_1 and Pl_2 be the two sides of AB in Pl [Ax. 6].

C will be a non-common point [Df. 13]

of, for example*, Pl_1 [Cr. 24].

Join C with A and with B [Cr. 15].

CA and CB are not in straight line, otherwise A, C and B would be in straight line [Df. 11],

which is not the case. Join each of any number n of points of CA between C and A with a different point of CB between C and B [Crs. 5, 15],

and let DE and FG be any two of such straight lines, D and F in CA, and E and G in CB. The straight lines DE and FG cannot be in straight line with each other, otherwise they would be segments of the same straight line [Df. 11],

and D, E, F and G would be in that straight line [Df. 2],

so that D would be in straight line with E and G, and then with CB [Df. 11],

which is impossible [Cr. 22].

The same argument applies to the n straight lines joining the same point H of CA between A and C (Fig. 2, right) with n different points of CB between C and B [Crs. 5, 15],

being all of these straight lines adjacent at H [Df. 4].

And being CA and CB in Pl_1 [Ax. 6],

all of these straight lines in Pl, whether or not adjacent, have their respective endpoints on Pl_1 [Df. 13],

so that all of them are in Pl_1 [Ax. 6]. □

Corollary 28 *The intersection point of two intersecting straight lines has its two sides in each of the intersecting straight lines in different sides of the other intersecting straight line in the plane that contains both straight lines.*

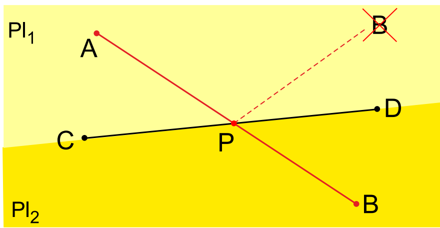


Figure 3 – Corollary 28

▷ (Fig. 3)Let P be the unique intersection point of two straight lines [Cr. 19]

AB and CD in a plane Pl [Cr. 25].

Since the only points of Pl common to both sides of CD in Pl are the points in straight line with CD [Df. 13],

and P is the only common point of AB and CD, even arbitrarily produced [Crs. 16, 19],

P is the only point of AB in straight line with CD [Df. 11],

and then the unique point common to both sides of CD in Pl [Df. 13].

Therefore, the endpoints A and B can only be non-common points of the sides of CD in Pl [Df. 13, Cr. 24].

So, if PA and PB were in the same side of CD in Pl, the endpoints A and B would be non-common points of that side [Ax. 6],

and being P between them [Cr. 4],

P would also be a non-common point of that side [Cr. 26],

which is impossible because it is a common point of both sides [Cr. 24]. ç

Therefore, PA and PB cannot be in the same side of CD in Pl. So, they must be in different sides of CD in Pl. The same argument proves PC and PD can only be in different sides of AB in Pl. □

Corollary 29 *The straight line joining any two non-common points, each in a different side of another given coplanar straight line, intersects the given straight line, or a production of it, at a unique point.*

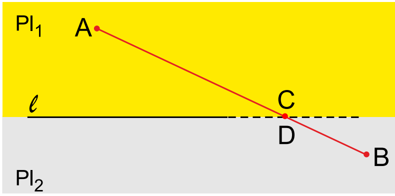


Figure 4 – Corollary 29

▷ (Fig. 4) Join any two non-common points A and B [Cr. 15]

respectively in the sides Pl_1 and Pl_2 of a straight line l in a plane Pl [Cr. 23].

AB is in Pl [Cr. 23].

Except A and B, all points of AB are between A and B [Cr. 4].

If all points of AB between A and B were non-common points of Pl_1, AB, including B, would be in Pl_1 [Df. 11, Ax. 6],

which is not the case. So, AB contains points of Pl_2 other than B and, for the same reason, points of Pl_1 other than A [Cr. 1]

So, AB has at least two points in each side of l. Since all points between two points of a straight line in the same side of another coplanar straight line are also in that side [Crs. 26],

AB has a segment AC whose points are all points of AB in Pl_1 [Cr. 12].

And for the same reasons it also has a segment BD whose points are all points of AB in Pl_2 [Cr. 12].

If C and D were different points, all points of AB between them [Cr. 5]

would be in no side of l in Pl, which is impossible because all points of AB are points of Pl [Df. 12],

and all points of Pl are points either of Pl_1, or of Pl_2, or of both of them [Df. 13].

So, C and D are the same point. Since all points between A and C are in Pl_1, AC is in Pl_1 [Df. 13],

and C is also in Pl_1 [Ax. 6].

For the same reasons D is in Pl_2. Since C and D are the same point, and this point belongs to Pl_1 and to Pl_2, it is a point of l, whether or not produced [Cr. 16, Df. 13].

So, it is an intersection point of AB and l [Df. 3]

whether or not produced [Cr. 16].

And it is the unique intersection point of AB and l, otherwise the non-common point A of Pl_1 would be in straight line with at least two points of l and it would be a common point of Pl_1 and Pl_2 [Dfs. 13, 11],

which is impossible [Cr. 24]. □

Corollary 30 *A plane contains at least two non-intersecting straight lines, which can be intersected by any number of different coplanar straight lines.*

▷ Let l be a straight line in a plane Pl [Cr. 23];

Pl_1 and Pl_2 the two sides of l in Pl [Ax. 6];

A, B any two non-common points of Pl_1; and C, D any two non-common points of Pl_2 [Cr. 26].

Joint A with B; and C with D [Cr. 15].

AB is in Pl_1, and CD in Pl_2 [Ax. 6].

AB and CD cannot intersect with each other because the intersection point would be a common point of Pl_1 and Pl_2 [Df. 13] while all of their points, even endpoints, are non-common points of Pl_1 and of Pl_2 respectively [Df. 13, Ax. 6].

On the other hand, AB and CD can be intersected by any number n of straight lines in Pl, each joining each of any n points of AB with a point of CD [Crs. 1, 15, 23]. □

Fundamentals on distances

Definition 14 *Distance between two points: length of the straight line joining both points. If both points are the same point, whether or not of a common point of different lines, the distance between them is said zero.*

Definition 15 *Distance from a point not in a given line to the given line: the shortest distance between the point and a point of the given line, or of a production of the given line if the given line is a straight line and the point is not in straight line with it.*

Definition 16 *Distancing direction and relative distancing.-Two non-common points in the same side of a given coplanar straight line and at different distances from the given straight line define a distancing direction in the straight line joining both points: from the nearest to the farthest of them. The difference between the distances to the given straight line from the endpoints of a segment of another straight in the same side of the given straight line is called relative distancing of the segment with respect to the given straight line.*

Definition 17 *Parallel straight lines.-A straight line is said parallel to another coplanar straight line, iff all of its points are at the same distance, said equidistance, from the other straight line.*

[Pr. 32] proves the existence of parallel straight lines. According to [Df. 14], the length of a straight line AB and the distance from A to B will be used as synonyms.

Axiom 7 *The distances from the points of a line to a fixed point or to another line vary in a continuous way.*

Corollary 31 *The distance between any two given points is unique.*

▷ It is an immediate consequence of [Cr. 15, Df. 14]. □

Fundamentals on circles

Definition 18 *Circle: a simply closed and plane line whose points are all points of the plane, and only them, at the same given finite distance, said radius, from a fixed point of that plane, said centre of the circle. A straight line joining any point of the circle with its centre is also said a radius of the circle. A segment of a circle is called arc, and the straight line joining its endpoints is a chord, or straight line subtending the arc. If the center of the circle is a point of a chord, the chord is said a diameter, and the corresponding arc a semicircle. Coplanar circles, and their corresponding segments, with the same centre are said concentric. The centre and any coplanar point at a distance from the centre less than its radius are said interior to the circle; if that distance is greater than the radius of the circle, the coplanar point is said exterior to the circle.*

Axiom 8 *Any point of a plane can be the centre of a circle of any finite radius, being all points of any of its arcs in the same side of its chord.*

Corollary 32 *A circle has interior points, other than its centre, and exterior points. And any point coplanar with a circle is either in the circle, or it is interior or exterior to the circle.*

▷ Let O be the centre of a circle c in a plane Pl [Ax. 8],

and A any point of c [Df. 18].

Joint A with O [Cr. 15].

Produce OA from A by any given finite length to a point A' [Cr. 16].

OA' is in Pl [Cr. 23].

Let P be any point of OA [Cr. 5].

Since $OP < OA$ and $OA < OA'$ [Cr. 13],

P is interior and A' is exterior to c [Dfs. 14, 18].

Join now any point R of Pl [Cr. 23] with O [Cr. 15].

It holds $RO \gtreqless OA$ [Ps. A],

and R will be either in c ($RO = OA$), or it will be interior ($RO < OA$) or exterior ($RO > OA$) to c [Dfs. 14, 18]. □

Corollary 33 *A plane line intersects a coplanar circle at a point between its endpoints iff it has points interior and exterior to the circle.*

▷ Let O be the centre and AO the finite radius of a circle c [Ax. 8]

in a plane Pl; BC a plane line in Pl [Ax. 8],

and P and Q two points of BC [Cr. 1]

such that P is interior and Q exterior to c [Cr. 32].

Being P interior to c, its distance to O is less than AO [Df. 18].

Being Q is exterior to c, its distance to O is greater than AO [Df. 18].

Therefore, there will be at least one point R in PQ, and then in BC [Cr. 1, 2],

whose distance to O is just AO [Ax. 7, Df. 2].

And R will also be in c [Df. 18].

So, R is an intersection point of BC and c [Df. 3].

On the other hand, if all points of a plane line BC are interior (exterior) to c, none of its points is at a distance AO from O [Df. 18],

and then no point of BC is in c [Df. 18].

Therefore c and BC have no point in common, and they do not intersect with each other [Df. 3]. □

Corollary 34 *Any point of a circle defines a unique diameter and two unique semicircles, each on a different side of the diameter.*

▷ Let O be the centre and AO the finite radius of a circle c [Ax. 8],

and let P be any of its points [Cr. 1].

Join P with the centre O of c [Cr. 15],

and produce PO from O by any given length greater than OA to a point P' [Cr. 16].

Since $OP' > OA$, PP' is a straight line with points interior, as O, and exterior, as P', to c [Cr. 16, Df. 18],

PP' intersects c at a point Q [Cr. 33].

Since O is a point of PQ and PQ is unique [Cr. 15],

PQ is the unique diameter defined by P [Df. 18].

And being c a simply closed line, P and Q are the common endpoints of two semicircles of c [Dfs. 4, 18, Ax. 8],

each on a different side of its diameter PQ [Ax. 8]. □

Fundamentals on angles

Definition 19 *Rigid transformations of lines: metric and reversible displacements of lines that preserve the definition and the metric properties of the displaced lines, each of whose points moves from an initial to a final position along a fixed finite line called trajectory, in any of the two opposite directions defined by the endpoints of the trajectory. If all points of the displaced line, except at most one, move around a fixed point and their trajectories are segments of concentric and coplanar circles whose centre is the fixed point, the rigid transformation is called rotation.*

Definition 20 *Superpose two adjacent lines: to place them with at least two common points by means of rotations around their common endpoint. Lines with at least two common points are said superposed.*

Definition 21 *Angle.-Two straight lines are said to make an angle greater than zero iff they are adjacent, one of them can be superposed on the other by two opposite rotations around their common endpoint, and the other can be superposed on the one by the same two rotations, though in opposite directions. The least of the rotations, of both if they are equal, is said (convex) angle, the greater one is said concave angle. The angle is said zero iff both straight lines are superposed. The angle is said to be in the side of one of the adjacent straight lines where the other adjacent straight lines lies. The straight lines and their common endpoint are said respectively sides and vertex of the angle. A side is said to make an angle with the other at their common vertex. A line joining a point on each side of the angle is said to subtend the angle, its points are called interior. The non-interior points are called exterior to the angle.*

Definition 22 *Adjacent angles and union angle.-Two angles are said adjacent iff they have the same vertex, a common side, the first angle superposes its non-common side on the common side, and the second angle superposes the common side on its non-common side, both angles in the same directions of rotation. The angle* *that superposes the non-common sides of both angles in the same direction of rotation of both angles is their union angle, which can be concave. If two adjacent angles are equal to each other, they are said to bisect their union angle.*

Definition 23 *Straight angle.-The angle that make the two sides of a point, except endpoints, in a straight line is said straight angle.*

Definition 24 *Acute, obtuse and right angles.-If a straight line cuts another given straight line and makes with it at the intersection point two adjacent angles that are equal to each other, both angles are said right angles, in which case, and only in it, the two sides of each angle are said perpendicular to each other, and the first straight line is also said perpendicular to the given one. Angles less (greater) than a right angle are said acute (obtuse).*

Definition 25 *Interior and exterior points and angles.-If two given coplanar straight lines are intersected by another coplanar straight line, said common transversal, a point of this transversal, different from the intersection points, is said interior to the given straight lines if it is between the intersection points of the transversal with both given straight lines; otherwise it is said exterior to them. Of the angles that the common transversal makes with the two given coplanar straight lines at their intersection points, those whose sides in the transversal have only exterior points are said exterior angles; and those whose sides in the transversal have interior points are said interior angles.*

Definition 26 *Alternate, corresponding and vertical angles.-Of the angles that a common transversal makes with two coplanar straight lines, the angles of a couple of non-adjacent angles are said alternate if they are both interior, or both exterior, and they are in different sides of the transversal; and corresponding if they are in the same side of the transversal, being the one interior and the other exterior. Of the angles that two intersecting straight lines make with each other at their intersection point, the couples of angles with no common side are said vertical angles.*

Axiom 9 *It is possible for two adjacent straight lines to make any angle at their common endpoint.*

Corollary 35 *Two straight lines make an angle greater than zero iff they are adjacent, being equal and unique the angle that each of the straight lines make with the other at their common endpoint, though both rotations are in opposite directions. And the adjacency point is their only common point, even arbitrarily produced*

▷ Each of two coplanar adjacent straight lines [Cr. 25],

and only them, makes with the other the same angle greater than zero at their common endpoint, though in opposite directions [Df. 21, Ax. 9].

And that angle is unique [Dfs. 19, 3].

And the only common point of both sides, even arbitrarily produced, is the vertex of the angle, otherwise both sides would be superposed [Df. 20]

and they would make an angle zero [Df. 21]. □

Corollary 36 *The superposition by rotation of two adjacent straight lines around their common endpoint is a unique straight line.*

▷ It is an immediate consequence of [Df. 20, Cr. 18]. □

Corollary 37 *An angle does not change by producing arbitrarily its sides from their non-common endpoints.*

▷ Let AB and AC be two adjacent straight lines [Cr. 25]

that make an angle $\alpha > 0$ at A [Cr. 35].

Produce AB from B and AC from C by any given length respectively to the points B' and C' [Cr. 16].

The rotation α superposes at least two points of AB' and AC', because it is the least rotation that superposes at least two points of AB and AC [Df. 21]

and all points of AB and AC are points respectively of AB' and AC' [Cr. 16, Df. 2].

On the other hand, if a rotation α' less than α superposes AB' on AC' but not AB on AC, the non-superposed (non-common) points of AB and CD would be between then common point A of AB' and AC' and any other superposed (common) point of AB' and AC' [Df. 20],

which is impossible [Df. 10].

So, AB' and AC' also make at A an angle α. □

Corollary 38 *Three adjacent straight lines define three angles at their common endpoint. And two intersecting straight lines define with each other at most four angles at their intersection point.*

▷ Three coplanar straight lines AB, AC and AD adjacent at the same point A [Cr. 27]

define three couples of coplanar straight lines adjacent at that point: AB, AC; AB, AD; and AC, AD [Df. 4].

So, AB, AC and AD define three angles at that point A [Cr. 35].

For the same reason, two intersecting straight lines define at most four angles whose two sides are not in the same straight line. □

Corollary 39 *(Fig. 5) Three straight lines adjacent at the same point define a couple of adjacent angles at that point.*

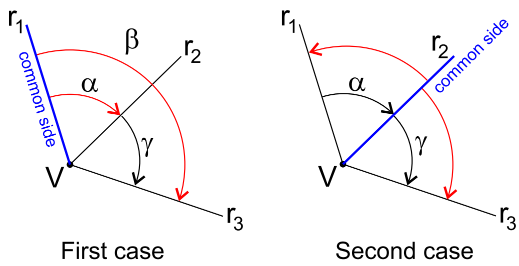


Figure 5 – Corollary 39

▷ (Fig. 5) Three straight lines r_1, r_2, r_3 adjacent at V define three angles α, β and γ at V [Cr. 38],

and then three couples of angles: α and β; α and γ; and β and γ. Being only three sides, the two angles of each of such couples must have a common side [Df. 21].

The angles of such couples that superpose their common side on their respective non-common sides can only be rotations either in the same or in opposite directions [Dfs. 1, 19].

In the second case, both angles are adjacent because any of them superposes its non common side on the common side in the same direction the other superposes the common side on its non-common side [Dfs. 21, 22].

In the first case both rotations, for instance* α and β, will be different [Dfs. 21, 19, 3],

and one of them, for instance* α, will be less than the other [Ps. A].

Assuming* α superposes r_1 on r_2, and β superposes r_1 on r_3 in the same direction of rotation, γ can only be the angle that superposes r_2 on r_3. Hence, α and γ are adjacent [Df. 22]. □

Corollary 40 *Two adjacent straight lines make a straight line iff they make a straight angle at their common endpoint.*

▷ If two adjacent straight lines l_1 and l_2 [Cr. 25]

make at their common endpoint P a straight angle, they are the two sides of the point P in a straight line l [Df. 23],

so that l_1 and l_2 make the straight line l [Cr. 2].

If two straight lines l_1 and l_2 adjacent at P make a straight line l, l_1 and l_2 are the sides in l of their common endpoint P [Df. 5]

in l, so that they make a straight angle at P [Df. 23]. □

Corollary 41 *A non-straight angle has a number of interior points and exterior points greater than any given number, and none of them is in a straight line with the sides of the angle.*

▷ Let AB and AC be two straight lines in a plane Pl, which are adjacent at A, are not in a straight line with each other [Cr. 25].

and make any non-straight angle $\alpha > 0$ [Ax. 9].

Let P be any point of AB between the endpoints A and B. [Ax. 2]

and let Q be any point of AC between the endpoints A and C. [Ax. 2].

Join P and Q [Cr. 15].

Produce PQ, for example* from P, by any length to a point P' [Cr. 16].

PQ has a number of points between P and Q greater than any given number [Cr. 5]

and all of them are interior to the angle α [Df. 21]

and none of them is in a straight line with AB, otherwise AB and $P'Q$ would be in the same straight line. [Cr. 18],

Q would be in straight line with AB, and AC would be in straight line with AB [Df. 11]

which is not the case. For the same reason, no interior point of PQ is in a straight line with AC [Cr. 18].

PP' has a number of points between P and P' greater than any given number [Cr. 5],

all of which are exterior to the angle α Df. 21],

and for the same reason as with the interior points, none of them is in a straight line with AB nor with AC [Cr. 18]. □

Fundamentals on polygons

Definition 27 *Polygon:Three or more finite coplanar straight lines, called sides, each of which is adjacent at each of its two endpoints, called vertexes, to just one of the others, being not in straight line with each other, and being their common endpoints their only intersection points, are said to make a polygon. Two sides of the same or of different polygons are said equal iff they have the same length. Two polygons are said adjacent iff they have a common side; opposite iff they have two opposite angles at a common vertex; similar iff the angles of the one are equal to the angles of the other; and equal if they are similar and the sides of each angle of the one are equal to the sides of the corresponding equal angle of the other. Polygons with at least one concave angle are said concave. The angle each side makes with the production of another adjacent side is said exterior. A straight line joining two points not in the same side of a polygon is a diagonal of the polygon. A diagonal bisects a polygon iff it is the common side of two adjacent polygons with the same area.*

Note the classical definition of diagonal is a particular case of the above general definition of diagonal.

Definition 28 *Triangles and quadrilaterals. A polygon of three (four) sides is a triangle (quadrilateral). A triangle (quadrilateral) is said equilateral if its three (four) sides are equal to one another. A triangle is said isosceles if it has two equal sides; and scalene if the three of them are unequal. If one of its angles is a right angle, it is said a right-angled triangle. A rectangle is a quadrilateral all of whose angles are right angles. An equilateral rectangle is a square. And a parallelogram is a quadrilateral with two couples of equal and parallel sides. Polygons with more than four sides are named pentagons, hexagons, heptagons etc. A polygon is said to lie between two given lines iff its vertexes are in the given straight lines or in straight lines whose endpoints are point of the given straight lines.*

Axiom 10 *The area of a polygon is greater than zero, and is the sum of the areas of the two adjacent polygons defined by any of its diagonals. Equal polygons have equal areas.*

Corollary 42 *Any two adjacent sides of a polygon make an angle greater than zero at their common endpoint, and the polygon has as many angles as sides.*

▷ Being coplanar all sides of a polygon [Df. 27],

each couple of its adjacent sides makes a unique angle greater than zero at their common endpoint [Cr. 35].

So, the polygon has as many angles as couples of adjacent sides. Since each couple of adjacent sides is defined by two adjacent sides, and each side defines two of such couples, one at each of its two endpoints [Df. 27],

the polygon has as many angles as sides. □

On angles and triangles

In this section, all points and lines will be coplanar and, unless otherwise indicated, all angles will be greater than zero. For the sake of brevity, three of the first 16 propositions of Euclid's Elements (Book I) will be used in this section, they will be referred to as [EBI, n], where n is the proposition number [3]. The proofs of these Euclid's propositions make use of other two Euclid's propositions not proved here either. The definitions, common notions and postulates (first, second and third) used in the proofs of these 5 Euclid's propositions are also formal elements of the foundational basis of Euclidean geometry (Section 4). So, they can also be proved within it, even in different and more formally detailed forms. They will be assumed to follow [Cr. 45] and to precede [Pr. 10].

Proposition 1 (Euclid's Proposition 3 extended) *To take a point in a given finite straight line, produced if necessary, at any given finite distance from a given point of the given straight line, and in any given direction* *of the two opposite directions of the given straight line.*

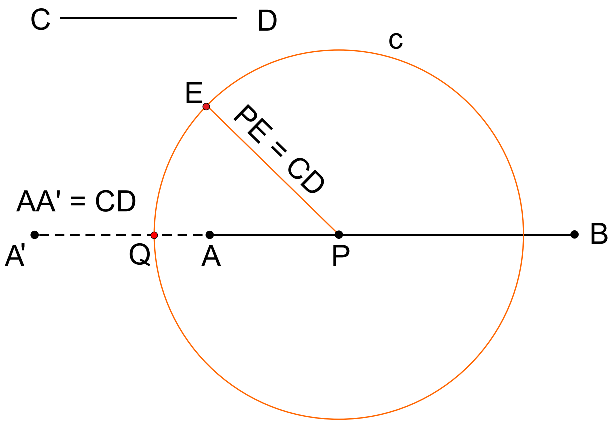


Figure 6 – Proposition 1

▷ Let AB be a finite straight line [Cr. 23];

P the given point of AB [Cr. 5];

CD the given finite distance [Df. 14];

and the give direction in AB, for example,* from B to A [Ax. 2, Df. 1].

Produce AB from A by the given distance CD to a point A' [Cr. 16].

With centre P and radius CD draw the circle c [Ax. 8].

P is interior to c [Df. 18],

and being $PA' > AA'$ [Cr. 13]

and $AA' = CD$, it holds $PA' > CD$ [Ps. B].

Therefore, A' is exterior to c [Df. 18].

Hence, there is an intersection point Q of c and BA' [Cr. 33].

Q is in AB [Df. 3],

whether or not produced, at the given finite distance CD from the point P of AB [Df. 18];

and in the given direction from B to A. $\square$

From now on, to take a point in a straight line at a given finite distance from one of its points will always mean to take the point in that straight line produced if necessary [Pr. 1]. And the distance between two points will always be finite, which, in addition, is founded on the (non-strictly geometric) proposition [Pr. 13] that will be proved in the next chapter.

Proposition 2 *All straight angles are equal to one another.*

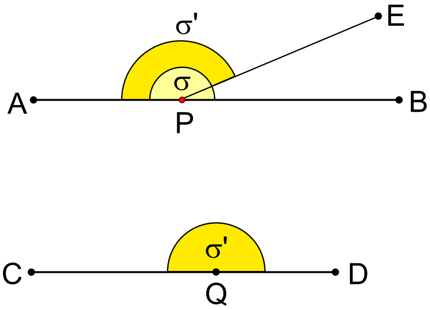


Figure 7 – Proposition 2

▷ Let P and Q be any two points respectively of any two straight lines AB and CD [Crs. 1, 27],

and σ and σ' the respective straight angles that PA makes at P with PB, and QC makes at Q with QD [Cr. 40].

Assume $\sigma' \neq \sigma$. In such a case a straight line PE adjacent at P to PA and making an angle σ' at P with PA is possible [Ax. 9].

Being each angle unique [Cr. 35]

and $\sigma \neq \sigma'$, PE and PB will not be superposed [Df. 21]

and they will be adjacent at P [Dfs. 20, 4].

PA and PB are the two sides of σ; and PA and PE the two sides of σ'. Being σ and σ' straight angles, AB and AE are straight lines [Cr. 40];

AP is a common segment of them [Cr. 5]

and PE and PB are non-common segments of them [Dfs. 4, 3]

Consequently, AB and AE are locally collinear [Dfs. 2, 3],

which is impossible [Df. 10]

So, it is impossible for PB and PE to be adjacent at P. The assumption $\sigma' \neq \sigma$ is, then, impossible. And it can be concluded that all straight angles are equal to one another. □

Proposition 3 *The union angle of two adjacent angles is the sum of both adjacent angles and is greater than each of them.*

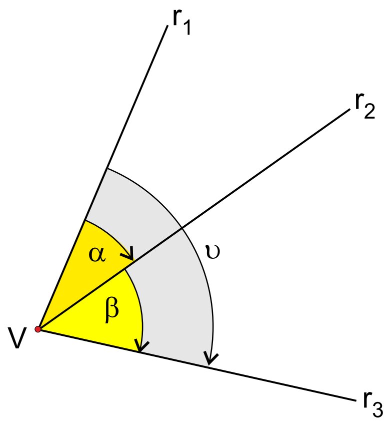


Figure 8 – Proposition 3

▷ Let r_1, r_2 and r_3 be three straight lines adjacent at their common endpoint V [Cr. 27],

where they make a couple of adjacent angles α and β [Cr. 39].

Assume* α superposes r_1 on r_2, and β superposes r_2 on r_3 in the same direction of rotation [Df. 22].

The rotation α around V superposes r_1 on r_2 [Df. 21]

in a unique straight line [Cr. 36],

and then the rotation β around V superposes r_1 on r_3 in a unique straight line [Df. 21, Cr. 36].

So, the rotation $\upsilon = \alpha + \beta$ [Ps. B]

around V in the same direction of rotation of α and β superposes the non-common sides r_1 and r_3 of α and β. It is, then the union angle of α and β [Df. 22].

And being $\alpha > 0$, $\beta > 0$ [Ax. 9],

it holds $\alpha + \beta > \beta$; $\beta + \alpha > \alpha$ [Ps. B],

and then $\upsilon > \beta$; $\upsilon > \alpha$ [Ps. A]. □

Proposition 4 (A variant of Euclid's Proposition 13) *If a straight line makes with another straight line two adjacent angles, these angles can be either equal or unequal to each other, and they always sum a straight angle.*

▷ Let D be the unique intersection point of two straight lines AB and DC [Crs. 19, 25].

DA, DC and DB are straight lines [Cr. 14]

adjacent at D [Df. 4].

So, DA makes at D with DC, and DC at D with DB two adjacent angles α and β [Cr. 39]

of which D is the common vertex and DC the common side [Df. 22];

α and β can be either equal or unequal to each other [Ax. 9, Ps. A],

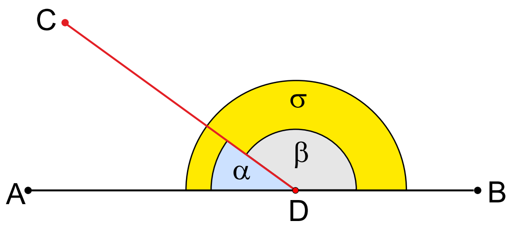


Figure 9 – Pr. 4

and their union angle σ is the rotation $\alpha + \beta$ around D [Pr. 3]

that in the same direction of rotation of α and β superposes the non-common sides DA and DB respectively of α and β [Df. 22],

and being DA and DB the two sides of D in the straight line AB [Df. 5, Ax. 4],

σ is a straight angle [Cr. 40, Df. 23].

Therefore α and β sum a straight angle [Pr. 3]. □

Proposition 5 (Euclid's Proposition 15) *The two angles of any couple of vertical angles are equal to each other.*

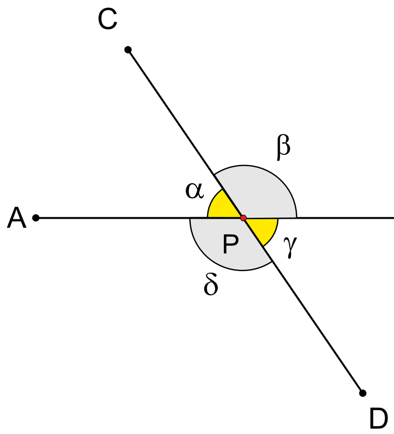


Figure 10 – Pr. 5

▷ Let P be the unique intersection point of two straight lines AB and CD [Crs. 19, 25].

PA, PC, PB and PD are straight lines [Cr. 14]

adjacent at P [Df. 4].

PC makes at P with AB two adjacent angles α and β [Cr. 35]

that sum a straight angle [Pr. 4].

PB makes at P with CD two adjacent angles β and γ [Cr. 39]

that sum a straight angle [Pr. 4].

PD makes at P with AB two adjacent angles γ and δ [Cr. 39]

that sum a straight angle [Pr. 4].

Therefore $\alpha + \beta = \beta + \gamma = \gamma + \delta$ [Pr. 2].

Consequently $\alpha = \gamma$ and $\beta = \delta$ [Ps. B]. □

Proposition 6 *Three given points define a triangle iff they are not in straight line, being the vertexes of the triangle the given points and its sides the straight lines joining them. A point defines a triangle with any two points iff it is a non-common points of on of the sides of the straight line joining the points.*

▷ Let A, B and C be any three points not in straight line [Cr. 20].

There is a plane Pl that contains them [Ax. 6].

Join A with B; B with C; and C with A [Cr. 15].

AB, BC and AC are in Pl [Cr. 23].

And none of them is in straight line with any of the others, otherwise A, B and C would be in straight line [Df. 11],

which is not the case. B is the only common point of AB and BC, otherwise they would be in straight line [Cr. 18]

and A, B and C would be in straight line [Df. 11],

which is not the case. So, AB and BC are adjacent at B [Df. 4].

For the same reason BC is adjacent at C to AC, and AC adjacent at A to AB. So, each of them is adjacent at each of its two endpoints to one, and only to one, of the others [Df. 4].

Therefore, A, B and C define a triangle ABC whose vertexes are A, B and C and whose sides are AB, BC and CA [Dfs. 28, 27].

Alternatively, if ABC is a triangle, its vertexes cannot be in straight line, otherwise they would be in the same straight line and ABC would not be a triangle [Df. 27, 28].

On the other hand, if P is any non-common point of one of the sides of a straight line l in Pl, it cannot be in straight line with any couple of points Q and R of l, even arbitrarily produced [Df. 13],

so P, Q and R define a triangle, as it has just been proved. And if a point defines a triangle with any two points, it cannot be in straight line with these two points [Df. 27, 28]

so, it cannot be a common point of the sides of the straight line through those points [Df. 11],

and it must be a non-common point of one of such sides [Cr. 24]. □

Corollary 43 *The intersection point of two intersecting straight lines defines a triangle with any couple of points, each of one the intersecting straight lines.*

▷ It is an immediate consequence of [Cr. 19, Df. 11, Pr. 6] □

Corollary 44 *A point of a side between two vertexes of a triangle is not in straight line with any of the other sides of the triangle.*

▷ If a point P of a side* AB of a triangle ABC were in straight line with the side* BC, these two sides would have two common points, B and P, and they would belong to the same straight line [Cr. 18],

which is impossible [Pr. 6]. □

Proposition 7 *If the length of a given line joining the centers of two circles is less than the sum of their radii, and each radius is less than the sum of the other radius and the length of the given line, then the circles intersect at two points, each on a different side of the given line and not in a straight line with it.*

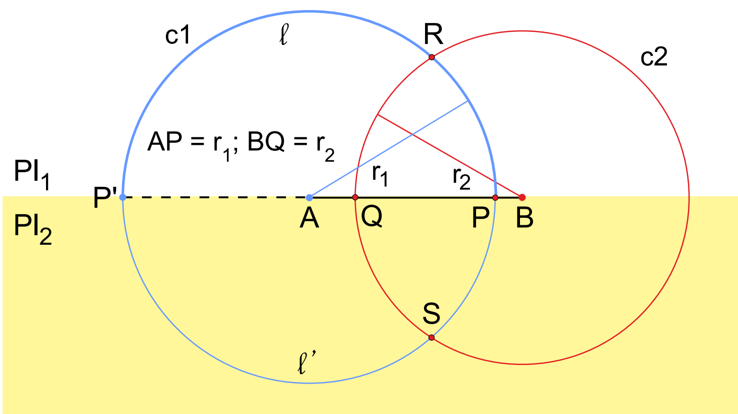


Figure 11 – Proposition 7

▷ Let AB be any straight line [Cr. 23],

P any point of AB different from A, and Q any point between AP different from P [Cr. 5].

Consider first the case $P \neq B$ $Q \neq P$. In this case, A, B,P and Q satisfy $AP < AB$, $BQ < AB$; and also $AB = AQ + QB < AQ + QP + QB = AP + QB$ [Cr. 13, Pss. B, A].

Draw the circle c_1 with centre A and radius AP; and the circle c_2 with centre B and radius BQ [Ax. 8].

Produce AB from A to a point P' such that $AP' = AP$ [Pr. 1].

$P'P$ is a diameter of c_1 [Df. 18]

that define two semicircles l and l' of c_1 each on a different side of AP [Cr. 34].

P and P' are the only points of PP', even arbitrarily produced, at a distance PA from A [Cr. 13].

So, except P and P', no point of l and l' is in straight line with AB [Dfs. 18, 11].

On the other hand, P is interior to c_2 because $BP < BQ$ [Cr. 13, Df. 18],

and P' is exterior to c_2 because $BP' > BQ$ [Cr. 13, Df. 18].

In consequence, l and l' intersect c_2 at two points R and S [Cr. 33],

none of which is in straight line with AB, otherwise P and Q would be the same point of AB and $AP+QB = AB$ [Cr. 13],

which is not the case. The same above argument applies to the cases $P = B$ and/or $Q = A$, i.e the cases in which two, or three, of the straight lines AB, AP and BQ are equal. $\square$

Proposition 8 *If two triangles have equal one of its sides and the two angles whose respective vertexes are the endpoints of that side, then the other two sides of each triangle are also equal to the corresponding two sides of the other.*

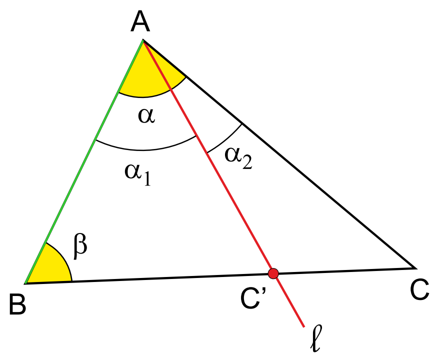


Figure 12 – Pr. 8

▷ Let ABC be a triangle [Pr. 6]

with an angle α at A and an angle β at B [Cr. 42].

Assume it is possible a triangle ABC' with a side AB, an angle α at A, and an angle β at B, and the side BC' of β different from BC, for instance* $BC' < BC$ [Ps. A].

Being ABC a triangle, C is the only intersection point of BC and AC [Dfs. 27, 28],

even arbitrarily produced [Cr. 19]

So, A, C' and C are not in straight line [Df. 11]

and $AC'C$ is a triangle [Pr. 6].

Being ABC, ABC' and $AC'C$ triangles, AB, AC' and AC are adjacent at A [Df. 28]

where they make the adjacent angles α_1 and α_2 [Df. 22, Cr. 39]

whose union angle is the angle α that AB makes at A with AC [Pr. 3],

and $\alpha_1 < \alpha$ [Pr. 3].

So, it is impossible a triangle with a side AB, an angle α at A, an angle β at B and a side $BC' \neq BC$. The same argument applies to the side AC. $\square$

Proposition 9 (Hilbert's Axiom IV.6) *If two triangles have equal one of their angles and the two sides of that angle, then they have also equal their corresponding other two angles.*

▷ Let ABC be a triangle [Pr. 6],

and α, β and γ its corresponding angles respectively at A, B and C [Cr. 42].

A triangle ABC' with an angle α at A, a side AB, a side AC and an angle β' at B different from β is impossible because, being β unique [Cr. 35],

$\beta' \neq \beta$ implies that β' will not superpose BA on BC but on BC', where C' is a point of AC, whether or not produced, different from C [Df. 20],

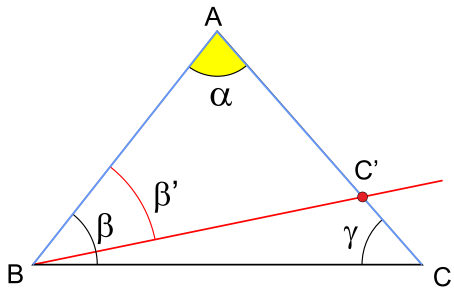


Figure 13 – Pr. 9

otherwise BC and BC' would be the same straight line [Cr. 15].

So, $AC' \neq AC$ [Cr. 13].

For the same reasons it is impossible a triangle with a side AB, a side AC an angle α at A and an angle γ' at C such that $\gamma' \neq \gamma$. □

Corollary 45 (Euclid's Proposition 4) *If two triangles have equal one of their corresponding angles and the two sides of that angle, then they have also equal their corresponding other two angles and their corresponding third side.*

▷ It is an immediate consequence of [Prs. 9, 8]. □

Proposition 10 (Euclid's Proposition 8) *If the three sides of a triangle are equal to the three sides of another triangle, then the three angles of the one are also equal to the corresponding three angles of the other.*

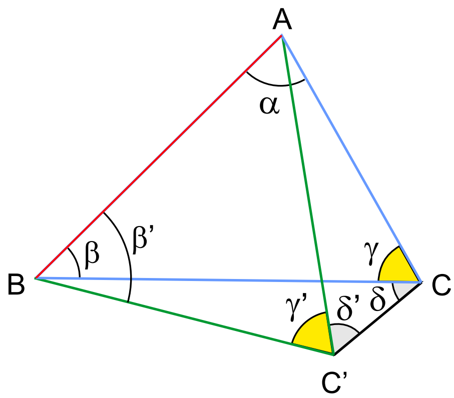


Figure 14 – Pr. 10

▷ Let ABC and ABC' be two triangles [Pr. 6]

with a common side AB and such that $BC = BC'$; $AC = AC'$. Assume $\beta' \neq \beta$. The angle β' will not superpose BA on BC but on BC' [Cr. 35],

where C' can only be different from C [Cr. 35].

Join C and C' [Cr. 15].

C' cannot be in straight line with A and C, otherwise it would be a point of AC, whether or not produced [Df. 11, Cr. 16],

different from C, and then $AC \neq AC'$ [Cr. 13],

which is not the case. So, $AC'C$ is an isosceles triangle [Cr. 6, Df. 28].

For the same reasons, $BC'C$ is an isosceles triangle. Since ABC', $AC'C$ and $BC'C$ are triangles [Pr. 6],

$C'B$, $C'A$ and $C'C$ are adjacent at C' [Dfs. 28, 27].

Since ABC, $AC'C$ and $BC'C$ are triangles [Pr. 6],

CA, CB and CC' are adjacent at C [Dfs. 28, 27].

So, $\gamma' + \delta' > \delta'$ and $\gamma + \delta > \delta$ [[Cr. 39, Pr. 3].

On the other hand, in $BC'C$ it holds $\gamma' + \delta' = \delta$ [EBI, 5], and in $AC'C$: $\gamma + \delta = \delta'$ [EBI, 5].

In consequence, $\delta' > \delta$ and $\delta > \delta'$ [Ps. A], which is impossible [Ps. A].

Therefore, the initial assumption is false, and the same rotation β that superposes AB on BC superposes AB on BC' [Df. 20],

so that $\beta = \beta'$ [Df. 21].

And the other two angles of ABC' are equal to the angles α and γ of ABC [Cr. 45]. □

Proposition 11 (Euclid's Propositions 18 and 19) *A side is the greatest of a triangle iff it subtends its greatest angle.*

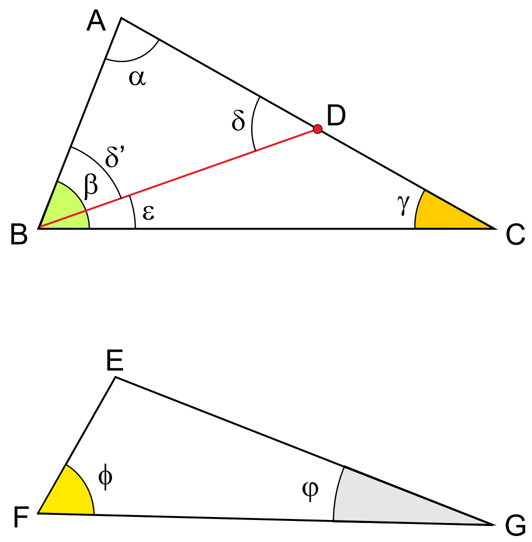


Figure 15 – Pr. 11

▷ (Fig. 15, left) Consider any two sides AB and AC of a triangle ABC [Pr. 6],

and assume* $AB < AC$ [Ps. A].

On AC take a point D such that $AD = AB$ [Pr. 1]

and join D with B [Cr. 15].

D is not in straight line with AB nor with BC [Cr. 44].

So, ABD and DBC are triangles [Pr. 6].

ABD is isosceles [Df. 28],

and then $\delta = \delta'$ [EBI, 5].

It also holds $\delta > \gamma$ [EBI, 16].

Since ABC, ABD and DBC are triangles, BA, BD and BC are adjacent at B [Dfs. 28, 27],

where they make the adjacent angles δ' and ϵ [Cr. 39]

whose union angle is β [Df. 22, Pr. 3].

Therefore, $\beta > \delta'$ [Pr. 3].

From $\beta > \delta'$, $\delta' = \delta$ and $\delta > \gamma$, it follows $\beta > \gamma$ [Ps. B].

Being AB and AC any two sides of ABC, it can be concluded that in a triangle the greatest side subtends the greatest angle [Df. 21].

Let now ϕ and φ (Fig. 15, right) be any two angles of a triangle EFC [Pr. 6, Cr. 42]

and assume $\phi > \varphi$ [Ps. A].

EG cannot be equal to EF, otherwise $\phi = \varphi$ [EBI, 5],

which is not the case. Nor can it be less than EF, because in such a case the least side would subtend the greatest angle, which is impossible, as just proved. Therefore, EG must be greater than EF [Ps. A].

Being ϕ and φ any two angles of a triangle, we conclude that the greatest angle is subtended by the greatest side. □

Proposition 12 (Euclid's Proposition 20) *In a triangle the sum of the lengths of any two of its sides is greater than the length of the remaining one.*

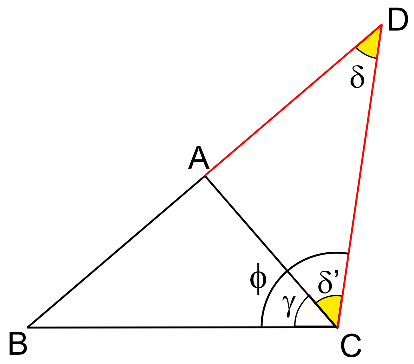


Figure 16 – Pr. 12

▷ Let ABC be a triangle [Pr. 6].

Produce AB from A [Cr. 16]

to a point D such that $AD = AC$ [Pr. 1].

Join D and C [Cr. 15].

Since A, B and C are not in straight line [Pr. 6],

the point D of the production of AB from B cannot be in straight line with B and C [Df. 11],

and DAC and DBC are triangles [Pr. 6].

And being DAC isosceles [Df. 28],

it holds $\delta = \delta'$ [EBI, 5].

Since ABC, DBC and DAC are triangles, CB, CA and CD are straight lines adjacent at C [Dfs. 28, 27];

where they make the adjacent angles γ and δ' [Cr. 39]

whose union angle is ϕ [Df. 22, Pr. 3].

Therefore, $\phi > \delta'$ [Pr. 3].

Hence, $\phi > \delta$ [Ps. A].

In consequence $DB > BC$ [Pr. 11],

and then $AB + AD > BC$; $AB + AC > BC$ [Ps. A].

The same argument proves the sum of the lengths any other couple of sides of ABC is greater than the length of the remaining one. □

On distances and perpendiculars

The next set theoretical proposition is not strictly necessary under the hypothesis that the length of a straight line AB is finite as long as it has two definite endpoints A and B. Here that hypothesis is not assumed but formally demonstrated. Although not developed here, the proposition has important consequences about finitism in geometry [4].

Proposition 13 *In the Euclidean space R^3, the length of a line with two endpoints is always finite. And the distance between any two given points is always finite and unique.*

▷ Let AB be any line in the Euclidean space $\mathbb{R}^3$, and $\lambda > 0$ any finite length. Assume it is possible to define in AB an infinite partition $\mathbf{P} = AP_1, P_1P_2, P_2P_3 \ldots$ whose parts have, all of them, the same length λ. A point x such that $xB < \lambda$ can only belong to a last part $P_\phi B$ of $\mathbf{P}$. A point y of $AP_{1<i<\phi}$ and a point z of $P_{1<i<\phi}B$ can only belong respectively to $P_{i-1}P_{1<i<\phi}$ and $P_{1<i<\phi}P_{i+1}$. So, $\mathbf{P}$ has a first element AP_1, a last element $P_\phi B$, and each element has an immediate predecessor (except AP_1), and an immediate successor (except $P_\phi B$). In addition, any subset **P'** of $\mathbf{P}$ containing for instance the element P_vP_{v+1} will also contain a first element: one of the elements AP_1, $P_1P_2 \ldots$ P_vP_{v+1}. So, P is well ordered and has an infinite ordinal α [2, p. 152]. Since P has a last element $P_\phi B$ and ω-ordered sets do not have last element, if α is infinite it must be greater than ω, in whose case P would have an ωth element $P_\omega P_{\omega+1}$ (Theorem of the ωth Term). But any point u such that $uP_\omega < \lambda$ can only belong to the impossible immediate predecessor of $P_\omega P_{\omega+1}$. In consequence, AB cannot be partitioned in a infinite number of parts of the same finite length, whatsoever it be. Therefore, it can only be partitioned in a finite number of parts of the same finite length. And being finite the sum of any finite number of finite lengths, AB has a finite length. □ □

Note.-From now on, to produce a straight line will always mean to produce a straight line by a finite length. And the distances between points and between points and lines will always be finite.

Proposition 14 *The length of straight line joining any two points interior to a circle is less than the sum of the lengths of two radii of the circle.*

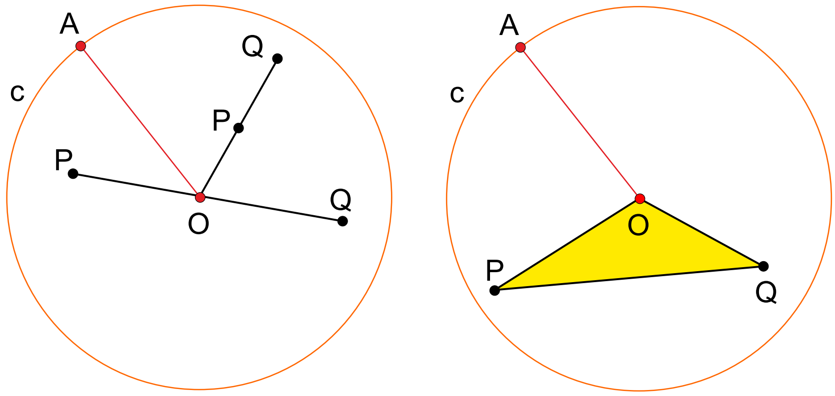


Figure 17 – Pr. 14

▷ Let c be a circle with a centre O and a finite radius OA [Ax. 8, Pr. 13],

and P and Q any two points interior to c [Cr. 32].

It must be $OA < OA + OA$, otherwise $OA \geq OA + OA$ [Ps. A],

and $0 \geq OA$ [Ps. B], which is impossible [Cr. 13].

Join O with P and with Q; and join P with Q. [Cr. 15].

If O, P and Q are in straight line (Fig. 17, left), one of them will be between the other two [Cr. 9].

If O is between P and Q then $PQ = OP + OQ$ [Cr. 13].

And being $OP < OA$, $OQ < OA$, it holds: $OP + OQ < OA + OQ$; $OQ + OA < OA + OA$ [Ps. B].

Therefore, $OP + OQ < OA + OA$, $PQ < OA + OA$ [Pss. B, A].

If P is between O and Q it holds $PQ < OQ$ [Cr. 13]

and being $OQ < OA$ [Df. 18]

and $OA < OA + OA$, it must be $PQ < OA + OA$ [Ps. B].

The same argument applies if Q is between O and P. If O, P and Q are not in straight line (Fig. 17 right) they define a triangle OPQ [Pr. 6]

in which it holds $PQ < OP + OQ$ [Pr. 12].

Being $OP < AO$; $OQ < AO$ [Df. 18],

and for the same reasons above, $PQ < OA + OA$ [Ps. B].

In consequence, the length PQ is always less than the sum of the lengths of two of its radii. □

Proposition 15 (Euclid's Proposition 11) *Through a given point of a given straight line to draw a perpendicular to the given straight line.*

▷ Let AB be any given straight line [Cr. 23]

and P any given point of AB [Cr. 1].

Assume* $PB < PA$. Take a point C in PA such that $PC = PB$ [Pr. 1].

With centers C and B and the same radius CB draw the respective circles c_1 and c_2 [Ax. 8],

which intersect at two points Q and R not in straight line with CP [Pr. 7].

Join Q with C, with P and with B [Cr. 15].

QCP and QPB are triangles because Q is not in straight line with C, P, B [Pr. 6],

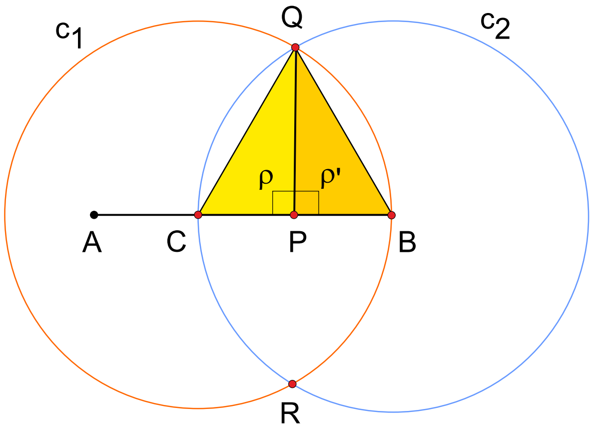


Figure 18 – Pr. 15

and the three sides of QCP are equal to the three sides of QPB.

Therefore $\rho = \rho'$ [Pr. 10].

And being PC, PQ and PB adjacent at P [Dfs. 27, 28],

P is the common vertex and PQ the common side of ρ y ρ'. So, ρ y ρ' are adjacent angles [Cr. 39, Df. 22],

and since they are equal to each other, they are right angles [Df. 24].

And PQ is perpendicular to AB through the given point P [Df. 24]. □

Proposition 16 (A variant of Euclid's Proposition 12) *From a given point not in straight line with a given straight line, to draw a perpendicular to the given straight line, produced if necessary.*

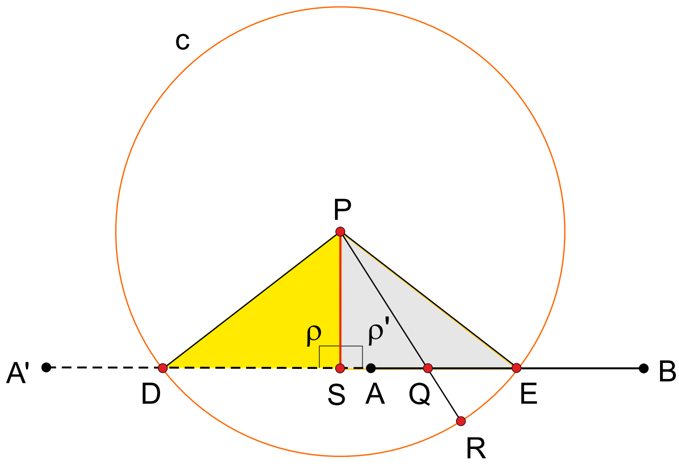


Figure 19 – Pr. 16

▷ Let AB be a straight line [Cr. 23]

and P a point not in straight line with AB [Cr. 20].

Take any point Q in AB [Cr. 1].

Join P and Q [Cr. 15]

and produce PQ from Q by any given length to a point R [Cr. 16].

With centre P and radius PR draw the circle c [Ax. 8].

Since PQ is less than PR [Cr. 13],

Q is interior to c [Df. 18].

In AB and in the direction from B to A, take a point A' at a distance $PR + PR$ from Q [Pr. 1].

Being $QA' = PR + PR$ and Q interior to c, A' cannot be interior to c [Pr. 14].

So, it will be either a point of c or exterior to c [Cr. 32],

and in both cases there will be an intersection point D of c and QA' [Cr. 33, Df. 3].

The same argument applied to the direction from A to B proves the existence of another intersection point E of AB, produced from B if necessary, and c. Join P with D and with E [Cr. 15].

Bisect DE at S [EBI, 10],

where S could also be the point Q, and join S with P [Cr. 15].

Being not P in straight line with AB, it is not in straight line with any two points of AB [Df. 11],

whether or not produced [Cr. 16].

So, PDS and PSE are triangles [Pr. 6]

with a common side PS, being also $PD = PE$ [Df. 18],

and $SD = SE$ [Df. 8].

So, $\rho = \rho'$ [Pr. 10].

SD, SP and SE are adjacent at S because PDS and PSE are triangles [Dfs. 28, 27].

So, SD, SP y SE make two adjacent angles ρ and ρ' at their common point S [Cr. 39],

which being equal, are right angles [Df. 24],

and SP is the perpendicular from P to AB [Df. 24],

produced if necessary. □

Proposition 17 (Euclid's Postulate 4) *All right angles are equal to one another.*

▷ Let DC be a straight line perpendicular to another straight line AB at any point D of AB [Pr. 15],

and let ρ_1 and ρ_1' be the respective adjacent right angles [Df. 24]

that DC makes at D with DA, and DC at D with DB [Cr. 39].

Since DA, DC and DB are adjacent at D [Cr. 35],

the union angle of ρ_1 and ρ_1' is the angle that, in the same direction of rotation of ρ and ρ_1', superposes the non-common sides DA and DB respectively of ρ_1 and ρ_1' [Df. 22, Pr. 3],

which are the sides of the straight angle σ_1 that DA makes at D with DB [Df. 23],

and then $\sigma_1 = \rho_1 + \rho_1'$ [Pr. 3].

So then, any two adjacent right angles sum a straight angle [Pr. 2].

Let ρ_2, ρ_2' be any other couple of adjacent right angles. As just proved, they sum a straight angle σ_2. Since $\sigma_1 = \sigma_2$ [Pr. 2],

it holds $\rho_1 + \rho_1' = \rho_2 + \rho_2'$ [Ps. B].

Assume $\rho_1 < \rho_2$. We would have $\rho_1 + \rho_1' < \rho_2 + \rho_1'$ [Ps. B],

and being $\rho_1 + \rho_1' = \rho_2 + \rho_2'$ [Pr. 2],

we can write $\rho_2 + \rho_2' < \rho_2 + \rho_1'$ [Ps. A],

and then $\rho_2' < \rho_1'$ [Ps. B].

And being $\rho_1' = \rho_1$ and $\rho_2' = \rho_2$ [Df. 24],

we get $\rho_2 < \rho_1$ [Ps. A],

which contradicts our assumption. So ρ_1 cannot be less than ρ_2. The same argument applied to the assumption $\rho_1 > \rho_2$ proves ρ_1 cannot be greater than ρ_2 either. So it must be equal to ρ_2 [Ps. A].

Hence, all right angles, whether or not adjacent, are equal to one another. □

Corollary 46 *A right angle is greater than zero. And two right angles sum a straight angle.*

▷ It is an immediate consequence of [Cr. 35, Prs. 4, 17] □

Corollary 47 *If one of the four angles that two intersecting straight lines make with each other at their intersection point is a right angle, then the other three angles are also right angles; the two sides of each angle are perpendicular to each other; and each straight line is perpendicular to the other.*

▷ It is an immediate consequence of [Df. 24, Prs. 17, 5]. □

Corollary 48 *The two opposite rotations that superpose the two sides of a straight angle are equal to each other.*

▷ It is an immediate consequence of [Crs. 46, 47]. □

Proposition 18 *If the two adjacent angles that a straight line makes with another intersecting straight line at their unique intersection point are different from each other, then the one is acute and the other is obtuse.*

▷ Let D be the unique intersection point of two straight lines AB and CD [Cr. 19].

DA, DC and DB are straight lines [Cr. 14]

adjacent at D [Df. 4],

where they make two adjacent angles α and β [Cr. 39]

that sum two right angles [Pr. 4].

If $\alpha \neq \beta$ one of them, for instance* α, will be less than the other, β [Ps. B].

The angle α must be less than a right angle, otherwise, and being $\alpha < \beta$, the sum $\alpha + \beta$ would be greater than two right angles, which is impossible [Pr. 4].

So, α must be an acute angle. For the same reason, β must be an obtuse angle [Df. 24]. □

Proposition 19 (Euclid's Proposition 17) *Any two angles of a triangle sum less than two right angles.*

▷ Let ABC be a triangle [Pr. 6].

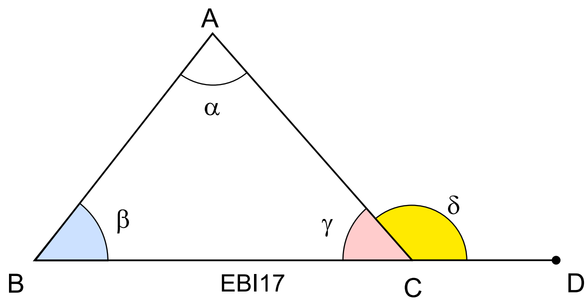


Figure 20 – Pr. 19

Produce the side BC from C by any given length to a point D [Cr. 16].

CB and CA are adjacent at C [Dfs. 27, 28];

CB and CD are adjacent at C [Cr. 16];

C is the only common point of AC and BD, otherwise CB and CA would be in straight line [Cr. 18],

which is not the case [Df. 28].

So, CA and CD are adjacent at C [Df. 4].

Consider the exterior angle δ [Df. 27].

It holds $\beta < \delta$ [EBI, 16].

Hence, $\beta + \gamma < \delta + \gamma$ [Ps. B].

And being $\delta + \gamma$ a straight angle [Pr. 4],

which equals two right angles [Cr. 46],

we conclude that β and γ sum less than two right angles. The same argument proves that any other couple of angles of ABC sum less than two right angles. □

Proposition 20 *From a point, whether or not in straight line with a given straight line, only one perpendicular can be drawn to the given straight line.*

▷ Let AB be a straight line [Cr. 23]

and P any point not in straight line with AB [Cr. 20].

P will be a non-common point of one of the sides, for example Pl_1 of AB [Ax. 6, Df. 13].

A perpendicular PQ from P to AB can be drawn [Pr. 16].

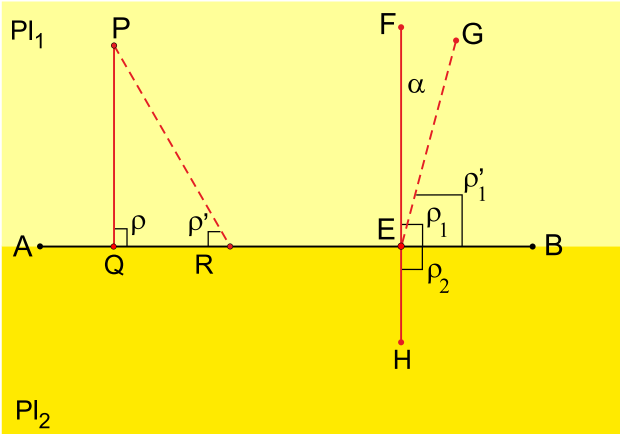


Figure 21 – Pr. 20

Assume a second perpendicular PR from P to AB can be drawn. We would have a triangle PQR [Pr. 6] with two right angles, ρ and ρ', which is impossible [Pr. 19].

PR is then impossible. Let now E be any point of AB, whether or not produced. Draw the perpendicular EF to AB from E [Pr. 15]

and assume a second perpendicular EG from E to AB can be drawn in the same side Pl_1 of AB [Cr. 27].

They will adjacent at E where they make and angle $\alpha > 0$, if not they would be superposed in a unique straight line [Df. 21, Cr. 36].

Being EF, EG and EB straight lines adjacent at E [Cr. 35],

α and ρ'_1 are adjacent angles [Cr. 39]

and ρ_1 is the union angle of them [Df. 22, Pr. 3].

Therefore, $\rho_1 > \rho'_1$ [Pr. 3],

which is impossible [Pr. 17].

So, the second perpendicular EG to AB in Pl_1 is impossible. A perpendicular EH from E to AB in Pl_2 can only be adjacent at E to EF because all points of EF and EH, except E, are non-common points respectively of Pl_1 and Pl_2 [Ax. 6, Df. 13].

So, EF, EB and EH can only be three adjacent straight lines [Cr. 35]

that make at their common endpoint E two adjacent angles ρ_1 and ρ_2 [Cr. 39]

whose union angle is the straight angle $\rho_1 + \rho_2$ [Cr. 46].

So then, EF and EH make a unique straight line [Cr. 40].

Therefore, from a point, whether or not in straight line with a straight line, only one perpendicular to the straight line can be drawn. □

Proposition 21 *The distance from a given point not in straight line with a given straight line to the given straight line is the length of the perpendicular from the given point to the given straight line, produced if necessary. And that distance is unique.*

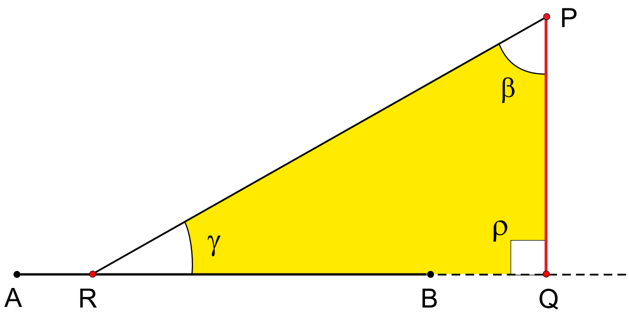


Figure 22 – Pr. 21

▷ Let AB be a straight line [Cr. 23]

and P a point not in straight line with AB [Cr. 20].

From P draw the perpendicular PQ to AB [Pr. 16].

Let R be any point of AB, whether or not produced, different from Q [Cr. 1].

Join P and R [Cr. 15].

P is not in straight line with R and Q [Df. 11].

Therefore, P, R and Q define a triangle PRQ [Pr. 6].

Since ρ is a right angle [Df. 24],

ρ is the greatest angle of PRQ [Pr. 19].

And the side PR is greater than the side PQ [Pr. 11].

Since the distance between two points is unique [Cr. 31],

R is any point of AB, whether or not produced, different from Q, and PQ is less than PR, PQ is the shortest of the distances [Df. 14]

between P and any point in AB, whether or not produced. So, the length of the perpendicular PQ is the distance from the point P to the straight line AB [Df. 15],

and this distance is unique [Pr. 20, Cr. 31]. □

Hereafter, a perpendicular to a straight line drawn from a point that is not in straight line with that straight line, will be drawn by producing the straight line if necessary [Pr. 16]. And, unless otherwise indicated, when considering more than one perpendicular to a given straight line, all of them will be assumed to be in the same side of the given straight line [Ax. 6, 16].

On parallelism and convergence

Proposition 22 *Draw three points on the same side of a given straight line, two of them equidistant and two of them non-equidistant from the given line. Draw a straight line non-parallel to the given straight line.*

▷ Through two points C and D of a given straight line AB [Cr. 1]

draw the perpendiculars CE and DF to AB [Pr. 15].

All points of CE and DF are on the same side of AB [Cr, 26].

Take any point P in CE [Cr. 1];

in DF take a point Q such that $DQ = CP$ [Pr. 1];

and in DQ take any point R [Cr. 1].

It holds $DR < DQ$ [Cr. 13].

Join P and R [Cr. 15].

P and Q are equidistant from AB [Pr. 21],

P and R are non-equidistant from AB [Pr. 21],

P, Q and R are in the same side of AB; and PR is not parallel to AB [Df. 17, Pr. 21],

and it is in the same side of AB [Cr, 26]. □

From now on, all of points equidistant from a straight line, whether or not in another straight line, will be assumed to be in the same side of the straight line and at a distance from the straight line greater than zero.

Proposition 23 (Khayyām-Cataldi's Axiom extended) *All segments of a given straight line in the same side of a second straight line have the same distancing direction with respect to the second straight line as the given straight line. And if the endpoints of the given straight line are equidistant from the second straight line then the given straight line is parallel to the second straight line, being all points of the given straight line non-common points of the same side of the second straight line.*

▷ (Fig. 23, left.) Let l be a straight line in a plane Pl [Cr. 23],

and A and B any two non-common points in the same side of l in Pl [Cr. 26],

so that A and B are non-equidistant from l [Pr. 22].

Draw the perpendiculars AP and BQ respectively from A and B to l [Pr. 16],

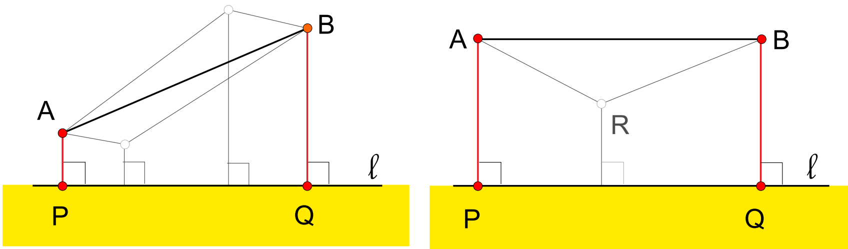


Figure 23 – Pr. 23

and assume* $AP < BQ$. Join A and B [Cr. 15].

The points A and B define a distancing direction, from A to B, of the straight line AB [Dfs. 1, 16] with respect to the straight line l [Df. 15].

All segments of AB must have the same distancing direction with respect to l as AB, otherwise there would be at least one segment whose distancing direction with respect to l would be opposite to that of AB [Dfs. 16, 1].

And then, either the endpoints of that segment are given before drawing AB, which is not the case [Ax. 5, Cr. 15],

or they are unknown before drawing AB, in which case they could only be a consequence of the operation, as such an operation, of drawing AB, which is impossible [Df. 4, Ax.1, Cr. 15],

or the straight line AB cannot be drawn, which is also impossible [Ax. 5, Cr. 15].

Assume now (Fig. 23, right.) that A and B are equidistant from l [Pr. 22].

Join A with B [Cr. 15].

Let R be any point between A and B [Crs. 5, 4],

and assume its distance to l [Pr. 21]

is different from the equidistance of A and B. The segments AR and RB [Cr. 5]

would have different distancing directions with respect to l [Df. 16, Pr. 21],

So, either the point R and the distancing directions of AR and of RB with respect to l are given before drawing AB, which is not the case [Ax. 5, Cr. 15],

or they are unknown before drawing AB, in which case they could only be a consequence of the operation, as such an operation, of drawing AB, which is impossible [Df. 4, Ax.1, Cr. 15],

or the straight line AB cannot be drawn, which is also impossible [Ax. 5, Cr. 15].

Therefore, R can only be at the same distance from CD as A and B. Consequently, AB is parallel to l [Df. 17].

And being A and B non common points in the same side of l, all points of AB are non common points of the same side of l [Ax. 6, Df. 13]. □

Though a straight line could be considered parallel to itself by a zero equidistance, hereafter only parallel straight lines whose equidistance is greater than zero will be considered.

Proposition 24 (A variant of Tacquet's Axiom 11) *If a straight line is parallel to another straight line, then the perpendicular from any point of any of the two straight lines to the other straight line is also perpendicular to the first straight line.*

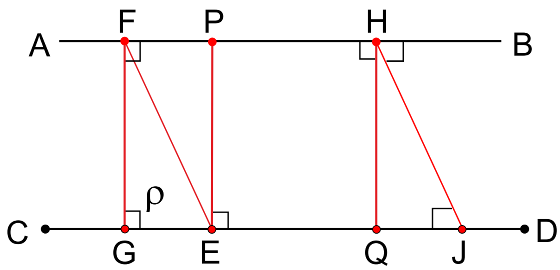


Figure 24 – Pr. 24

▷ Let AB be a straight line parallel to another straight line CD [Pr. 23].

All points of AB are at the same distance greater than zero from CD [Df. 17].

From a point P of AB draw the perpendicular PE to CD [Pr. 16].

Draw the perpendicular from E to AB [Pr. 16]

and assume it is not EP but EF. From F draw the perpendicular FG to CD [Pr. 16].

It will be different from FE, otherwise there would be two perpendiculars to CD from the same point E, namely PE and FE, which is impossible [Pr. 20].

Consider the triangle FGE [Prs. 23, 6].

The right angle ρ [Df. 24]

is the greatest angle of FGE [Pr. 19].

So EF is greater than FG [Pr. 11],

and FG is equal to PE because AB is parallel to CD [Df. 17].

In consequence, the shortest distance from E to AB would not be the length of the perpendicular EF, but that of EP [Ps. B],

which is impossible [Pr. 21].

So, EP is also perpendicular to AB. Let now Q be any point in CD. Draw the perpendicular QH to AB [Pr. 16].

Assume the perpendicular from H to CD is not HQ but HJ. It has just been proved that HJ is also perpendicular to AB. So, there would be two different perpendiculars, HJ and HQ, to AB from the same point H, which is impossible [Pr. 20].

Hence, the perpendicular QH is also perpendicular to CD. □

Proposition 25 *A straight line parallel to another given straight line can only be produced as a straight line parallel to the given straight line.*

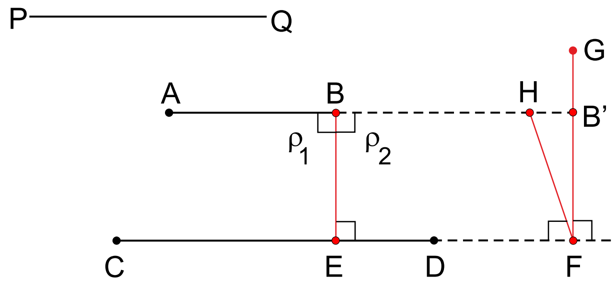


Figure 25 – Pr. 25

▷ Let AB be a straight line parallel to another straight line CD [Pr. 23],

and PQ any given finite distance [Df. 14, Pr. 13].

Draw the perpendicular BE from B to CD [Pr. 16].

In CD and in the direction from C to D take a point F such that $EF = PQ$ [Pr. 1].

From F draw the perpendicular FG to CD [Pr. 15].

Take a point B' in FG such that $B'F = BE$ [Pr. 1].

Join B and B' [Cr. 15].

BB' is parallel to CD [Pr. 23].

And BE is perpendicular to AB and to BB' through their common endpoint B [P. 24].

AB and BB' are, then, the two sides of a straight angle [Cr. 46]

and they make the straight line AB' [Cr. 40],

which is parallel to CD [Pr. 23].

Assume now $BB' \neq EF$, for instance* $BB' > EF$ [Ps. A].

Take a point H in BB' such that $BH = EF$ [[Pr. 1].

Join H and F [Cr. 15].

BE is parallel to HF [Pr. 23];

CF is perpendicular to HF [Pr. 24];

and HF is perpendicular to CF [Cr. 47].

So, if $BB' \neq EF$ there would be two different perpendiculars, $B'F$ and HF, to CF from the same point F, which is impossible [Pr. 20].

In consequence $BB' = EF$. So, BB' is the only production of AB from B by the given length $EF = PQ$ [Cr. 16],

and it is parallel to CD [Pr. 23].

The same argument applies to the endpoint A. □

Proposition 26 (Posidonius-Geminus' Axiom) *If two points of a given straight line are equidistant from a second straight line, then the given straight line is parallel to the second straight line.*

▷ Let AB and CD be two straight lines such that two points P and Q of AB are equidistant from CD [Pr. 23].

The segment PQ, which is the only straight line joining P and Q [Cr. 15],

is parallel to CD [Pr. 23].

If PA were not parallel to CD, the straight line PQ [Cr. 14]

could be produced from Q by a length PA as a straight line PA [Cr. 16]

non parallel to CD, which is impossible [Pr. 25].

The same applies to QB. AB is then parallel to CD. □

Proposition 27 *If a straight line is parallel to another straight line, this second straight line is also parallel, and by the same equidistance, to the first straight line.*

▷ Let AB be a straight line parallel to another straight line CD [Cr. 26].

Let E and F be any two points of CD [Cr. 1].

From E and from F draw the respective perpendiculars EG and FH to AB [Pr. 16].

These perpendiculars are also perpendicular to CD [Pr. 24].

So, the distance from E to AB is the same as the distance from G to CD [Pr. 21];

and the distance from F to AB is the same as the distance from H to CD [Pr. 21].

Since AB is parallel to CD, the distances to CD from G and H are equal to each other [Df. 17].

Hence, the distances to AB from E and F are also equal to each other [Ps. B].

E and F are, then, two points in CD at the same distance from AB. Therefore, CD is parallel to AB [Cr. 26],

and by the same equidistance GE. □

Proposition 28 *To draw a straight line parallel to a given straight line through a given point not in straight line with the given straight line.*

▷ Let CD be a straight line [Cr. 23],

and P a point not in straight line with CD [Cr. 20].

From P draw the perpendicular PQ to CD [Pr. 16].

Take any point R in CD different from Q [Cr. 1].

From R draw the perpendicular RS to CD [Pr. 15].

And in RS take a point T such that $RT = QP$ [Pr. 1].

Join P and T [Cr. 15]

and produce PT respectively from P and from T to any two points A and B [Cr. 16].

The straight line AB has two points, P and T, equidistant from CD. Therefore, AB is a parallel to CD [Cr. 26]

through the point P. □

Proposition 29 (Playfair's Axiom 11) *Through a given point not in straight line with a given straight line, one, and only one, parallel to the given straight line can be drawn.*

▷ Let AB be a straight line [Cr. 23]

and P a point not in straight line with AB [Cr. 20].

Through P a parallel CD to AB can be drawn [Pr. 28].

Assume that through P more than one parallel to AB can be drawn. From P draw the perpendicular PQ to AB [Pr. 16].

PQ is also perpendicular from P to each of the assumed parallels to AB [Pr. 24].

And each of these assumed parallels would be a different perpendicular to PQ through the same point P [Cr. 47],

which is impossible [Pr. 20].

Therefore, through a given point not in straight line with a given straight line, one [Pr. 28],

and only one, parallel to a given straight line can be drawn. □

Proposition 30 *For any given straight line and through different points, a number of parallels to the given straight line greater than any given number can be drawn.*

▷ Let AB be a straight line [Cr. 23]

and CD another straight line that intersects AB at any point P of AB [Crs. 19, 25].

CD has a number of points greater than any given number n [Cr. 1],

none of which, except P, is in straight line with AB, even arbitrarily producing AB and CD [Cr. 19].

Through each of those n points of CD one, and only one, parallel to AB can be drawn [Prs. 28, 29]. Therefore, it is possible to draw a number greater than any given number of parallels to a given straight line. □

Proposition 31 *If two straight lines have a common perpendicular, then they are parallel to each other.*

▷ Let AB be a straight line in the same side of another straight line CD [Cr. 27].

From a point P of AB draw the perpendicular PQ to CD [Pr. 16].

If PQ is also perpendicular to AB, then AB must be parallel to CD, otherwise through P a parallel EF to CD could be drawn [Pr. 28],

PQ would be perpendicular to EF [Pr. 24],

and EF would be perpendicular to PQ [Cr. 47],

and there would be two perpendicular to PQ, namely AB and EF, through the same point P, which is impossible [Pr. 20]. □

Proposition 32 *Two parallel straight lines cannot intersect.*

▷ Assume two parallel straight lines AB and CD [Cr. 30]

intersect at a point P. From a point Q of AB different from P [Cr. 1]

draw the perpendicular QR to CD [Pr. 16].

QR is also perpendicular to AB [Pr. 24].

PQ and PR would be two perpendiculars to QR [Cr. 47]

through the same point P, which is impossible [Pr. 20]. □

Note. The fact that two parallel straight lines cannot intersect with each other, does not imply that non parallel straight lines have to intersect, as Posidonius defended. His pupil Geminus of Rhodes discovered the flaw [3, p. 40, 190], [1, pp. 58-59].

Proposition 33 *If a common transversal cuts two straight lines and makes with them equal the angles of a couple of alternate angles, or of corresponding angles, then the two angles of each couple of alternate angles, and of corresponding angles, are also equal. And the interior angles of the same side of the transversal sum two right angles. If the interior angles of the same side of the transversal sum two right angles, the two angles of each couple of alternate angles, and of corresponding angles, are equal to each other.*

▷ Let AB and CD be two straight lines that are intersected by a common transversal EF [Df. 25, Cr. 30]

at P and Q respectively. On the one hand we have: $\alpha_i = \alpha_e$; $\beta_i = \beta_e$; $\gamma_i = \gamma_e$; $\delta_i = \delta_e$ [Pr. 5].

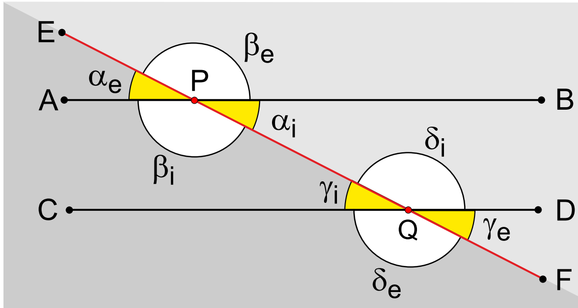


Figure 26 – Pr. 33

On the other, and being ρ a right angle: $\rho + \rho = \alpha_e + \beta_e = \alpha_i + \beta_i = \gamma_i + \delta_i = \gamma_e + \delta_e = \alpha_e + \beta_i = \beta_e + \alpha_i = \gamma_i + \delta_e = \delta_i + \gamma_e$ [Pr. 4].

So, if $\alpha_i = \gamma_i$, and being $\alpha_i = \alpha_e$ and $\gamma_i = \gamma_e$, we immediately get $\alpha_e = \gamma_e$; $\alpha_i = \gamma_e$; $\alpha_e = \gamma_i$ [Ps. A].

A similar argument proves that the two angles of any other couple of alternate angles, or of corresponding angles [Df. 26],

are equal to each other. In addition, from $\alpha_i + \beta_i = \rho + \rho$ [Pr. 4],

$\alpha_i = \gamma_i$ and $\beta_i = \delta_i$, it follows $\gamma_i + \beta_i = \rho + \rho$; $\alpha_i + \delta_i = \rho + \rho$ [Ps. A].

On the other hand, if $\alpha_i + \delta_i = \rho + \rho$, and being $\rho + \rho = \gamma_i + \delta_i$ [Pr. 4],

we immediately get $\alpha_i + \delta_i = \gamma_i + \delta_i$ [Ps. B].

Therefore, $\alpha_i = \gamma_i$ [Ps. B],

and the same argument above proves that the two angles of each couple of alternate angles, and of corresponding angles, are equal. □

Proposition 34 *A common transversal makes with two parallel straight lines equal the two angles of each couple of alternate angles and of corresponding angles.*

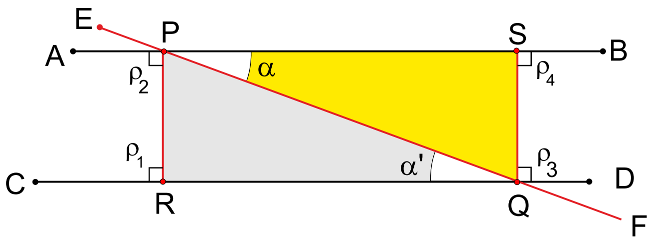


Figure 27 – Pr. 34

▷ Let AB and CD be any two parallel straight lines [Cr. 30].

And EF any common transversal [Df. 25, Cr. 30]

that cuts them at P and Q respectively [Cr. 30].

If EF is perpendicular to AB, it is also perpendicular to CD [Pr. 24],

and the eight angles it makes with AB and CD at its corresponding intersection points are right angles [Cr. 47],

in which case the two angles of each couple of alternate and of each couple of corresponding angles are equal [Pr. 17, Df. 26].

If EF is not perpendicular to AB, draw the perpendicular PR to CD [Pr. 16],

which is also perpendicular to AB [Pr. 24].

And AB and CD are perpendicular to PR [Cr. 47].

Therefore ρ_1 and ρ_2 are right angles [Df. 24].

Take a point S in PB such that $PS = RQ$ [Pr. 1].

Join S and Q [Cr. 15].

SQ and PR are parallel [Cr. 26, Pr. 27].

Hence, AB and CD are perpendicular to SQ [Pr. 24],

and SQ is perpendicular to AB and to CD [Cr. 47].

Therefore, ρ_3 and ρ_4 are right angles [Df. 24, Cr. 47].

And being AB parallel to CD, it holds $PR = SQ$ [Df. 17].

Consider the triangles PRQ and PQS [Prs. 23, 6].

The three sides of PRQ are equal to the three sides of PQS. So, $\alpha = \alpha'$ [Pr. 10].

Therefore, the two angles of any other couple of alternate angles, and of corresponding angles [Df. 26],

are also equal [Pr. 33]. □

Proposition 35 *If a common transversal makes with two straight lines equal the angles of a couple of alternate angles, then both straight lines are parallel to each other.*

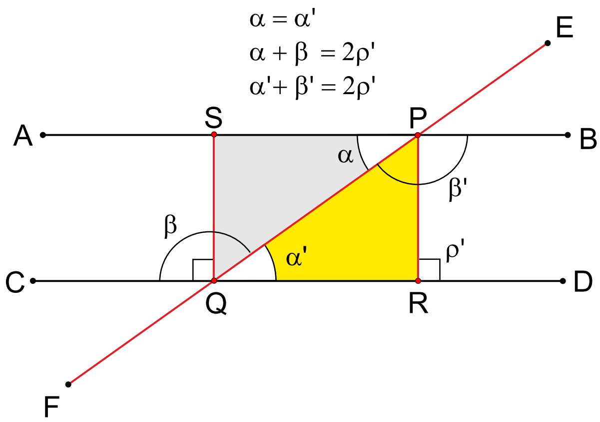


Figure 28 – Pr. 35

▷ Let EF be a common transversal [Df. 25, Cr. 30]

that intersects two straight lines AB and CD respectively at P and Q, where EF makes with AB and CD equal the two angles of a couple of alternate angles α and α' [Df. 26].

The two interior angles of the same side ofEF, α and β in one side, and $\alpha' y \beta'$ in the other, sum two right angles [Pr. 33].

Therefore, AB and CD cannot intersect each other, otherwise there would be a triangle with two angles that sum to two right angles, which is impossible. [Pr. 19].

Therefore, AB and CD are each on the same side of the other. [Cr. 29]

If α and α' are right angles, EF will be perpendicular to AB and to CD [Cr. 47],

and AB and CD will be parallel [Pr. 31].

If α and α' are not right angles, EF is not perpendicular to AB nor to CD [Df. 24].

In this case, draw the perpendicular PR from P to CD [Pr. 16].

On PB take a point S such that $PS = RQ$ [Pr. 1].

Join S and Q [Cr. 15].

PR and SQ are parallel [Cr. 26, Pr. 27],

and then CD is perpendicular to SQ [Pr. 24],

and SQ is perpendicular to CD [Cr. 47].

SQP and PQR are triangles [Prs. 23, 6].

They have a a common side PQ, and also $PS = RQ$, and $\alpha = \alpha'$. Therefore $PR = SQ$ [Cr. 45].

Since PR and SQ are perpendicular to CD, P and S are at the same distance from CD [Pr. 21].

So, AB and CD are parallel to each other [Cr. 26, Pr. 27]. □

Proposition 36 *Two straight lines are parallel to each other if, and only if, a common transversal makes with them two interior angles in the same side of the transversal that sum two right angles.*

▷ If a common transversal EF makes with two straight lines AB and CD [Df. 25, Cr. 30]

two interior angles α and β [Df. 25]

on the same side of the transversal [Df. 21]

that sum two right angles, then the two angles of any couple of alternate angles α and α' are equal to each other [Pr. 33]

and both straight lines are parallel [Pr. 35].

If a transversal cuts two parallel straight lines [Df. 25, Cr. 30],

it makes with them equal the two angles of each couple alternate angles, for instance α and α' [Pr. 34]

and then the two interior angles of the same side of the transversal [Df. 21]

sum two right angles [Pr. 33]. □

Proposition 37 (Proclus' Axiom) *If a first straight line is parallel to a second straight line and the second straight line is parallel to a third straight line, then the first straight line is also parallel to the third straight line.*

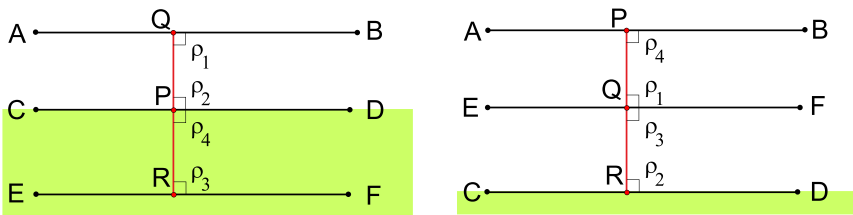


Figure 29 – Pr. 37

▷ (Fig. 29, left) Let AB be a straight line parallel to another straight line CD [Cr. 30],

which is parallel to another straight line EF [Cr. 30].

Assume first that AB and EF are in different sides of CD [Ax. 6] (Fig. 29, left).

From any point P of CD draw the perpendicular PQ to AB and the perpendicular PR to EF [Pr. 16].

PQ and PR are also perpendicular to CD [Pr. 24].

So, ρ_1, ρ_2, ρ_3 and ρ_4 are right angles [Df. 24].

PQ and PR cannot be two different perpendiculars to CD from P [Pr. 20].

So, QR is a unique straight line, which is a common transversal of AB and EF, and makes with them in the same side of QR two interior angles ρ_1 and ρ_3 [Df. 25]

that sum two right angles. Therefore AB is parallel to EF [Pr. 36].

If AB and EF are in the same side of CD [Cr. 27] (Fig. 29, right),

then draw the perpendicular PQ from any point P in AB to EF, and from Q the perpendicular QR to CD [Pr. 16].

So, ρ_1 and ρ_2 are right angles [Df. 24].

Since EF is parallel to CD, QR is also perpendicular to EF [Pr. 24],

and ρ_3 is a right angle [Df. 24].

And, for the same reasons above, PR is a unique straight line, which is perpendicular to EF through Q. And being perpendicular to CD, PR is also perpendicular to AB [Pr. 24],

and then ρ_4 is a right angle [Df. 24].

In consequence, PQ is a transversal of AB and EF that make two interior angles, ρ_1 and ρ_4 [Df. 25],

on the same side of PQ that sum two right angles. So, AB is also parallel to EF [Pr. 36]. □

Proposition 38 *If a common transversal makes with two straight lines two interior angles in the same side of the transversal that sum less (more) than two right angles, the interior angles in the other side of the transversal sum more (less) than two right angles.*

▷ Let EF be a common transversal of two straight lines AB and CD [Df. 25, Cr. 30]

that cuts them respectively at P and Q and makes with them the interior angles α and β [Df. 25]

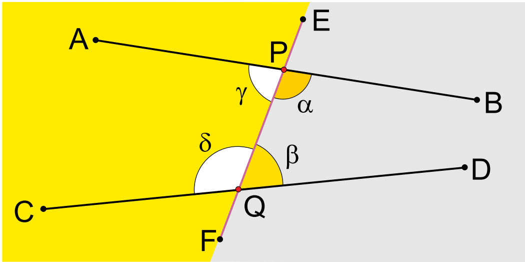


Figure 30 – Pr. 38

on the same side of EF [Ax. 6, Df. 21]

so that $\alpha + \beta < \rho + \rho$, where ρ is a right angle [Pr. 17].

AB and CD are not parallel [Pr. 36].

Let γ and δ be the interior angles that AB and CD make with EF on the other side of EF [Df. 21].

On the one hand we have: $\alpha + \gamma = \beta + \delta = \rho + \rho$ [Pr. 4],

so that $\alpha + \gamma + \beta + \delta = \rho + \rho + \rho + \rho$ [Ps. B].

On the other hand $\gamma + \delta \lessgtr \rho + \rho$, otherwise AB and CD would be parallel [Pr. 36].

But if $\gamma + \delta < \rho + \rho$, we would have $\gamma + \delta + \rho + \rho < \rho + \rho + \rho + \rho$ [Ps. B];

and being $\alpha + \beta < \rho + \rho$, we also have $\alpha + \beta + \gamma + \delta < \rho + \rho + \gamma + \delta$ [Ps. B].

Therefore $\alpha + \beta + \gamma + \delta < \rho + \rho + \rho + \rho$ [Ps. B], which is not the case.

So it must be $\gamma + \delta > \rho + \rho$. A similar argument applies to the case $\alpha + \beta > \rho + \rho$. $\square$

Proposition 39 *All segments with the same length of a given straight line have the same distancing direction and the same relative distancing with respect to any other non-parallel straight line in the same side of the given straight line.*

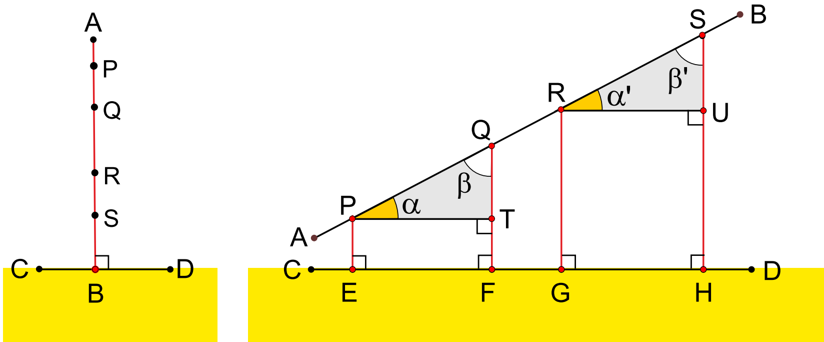


Figure 31 – Pr. 39

▷ Let AB be a straight line in the same side of another straight line CD [Cr. 27]

to which it is not parallel [Pr. 22].

All segments of AB have the same distancing direction, for instance* from A to B with respect to CD [Pr. 23].

Let P, Q and R be any three points of AB [Cr. 1].

Assuming* Q is between P and R [Cr. 9],

take in AB a point S at a distance PQ from R in the direction from A to B [Pr. 1],

so that $PQ = RS$. If AB were perpendicular to CD (Fig. 31, left), the relative distancing of any segment of AB [Df. 16]

with respect to CD would be the length of the segment [Cr. 13, Pr. 21].

So, PQ and RS would have the same relative distancing with respect to CD. Assume AB is not perpendicular to CD (Fig. 31, right). From P, Q, R and S draw the perpendiculars PE, QF, RG and SH to CD [Pr. 16].

And from P and R draw the perpendiculars PT to QF, and RU to SH [Pr. 16].

PT is parallel to CD [Pr. 31]

and Q is not in straight line with PT [Cr. 22].

So, QPT is a triangle [Pr. 6].

For the same reason SRU is also a triangle. PT and RU are parallel to CD [Pr. 36],

and then they are parallel to each other [Pr. 37].

Therefore $\alpha = \alpha'$ [Pr. 34].

QF and SH are parallel to each other [Pr. 31],

and then $\beta = \beta'$ [Pr. 34].

The triangles QPT and SRU verify: $\alpha = \alpha'$; $\beta = \beta'$ [Pr. 34],

and $PQ = RS$. Consequently, $QT = SU$ [Pr. 8].

Being $PE = TF$ [Df. 17]

and $QT = QF - TF$ [Cr. 13],

it will be $QT = QF - PE$ [Ps. A].

QT is, then, the relative distancing of the segment PQ with respect to CD [Df. 16].

For the same reasons SU is the relative distancing of the segment RS with respect to CD. Since $QT = SU$, and PQ and RS are any two segments of AB with the same length, we conclude that all segments of AB with the same length have the same relative distancing with respect to CD [Df. 16]

in the same distancing direction [Pr. 23]. □

Proposition 40 (Khayyām's Axiom) *Two non-parallel straight lines, produced if necessary by a finite length, intersect with each other at a unique point.*

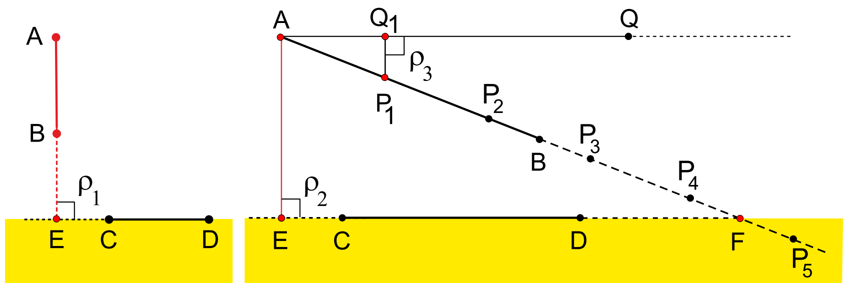


Figure 32 – Pr. 40

▷ Let AB and CD be any two non parallel [Pr. 22]

and non-intersecting straight lines [Cr. 30].

If one of them has its two endpoints in different sides of the other, then it intersects the other or a finite production of the other [Cr. 29, Pr. 13].

Assume, then, the two endpoints of each straight line are non-common points of the same side of the other [Cr. 27].

Each straight line will be in the same side of the other [Ax. 6].

Assume the distancing direction of AB with respect to CD [Df. 16]

is, for instance*, from B to A [Pr. 39].

In this case, draw from A the perpendicular AE to CD [Pr. 16].

If AB were a segment of AE then AB intersects CD at E [Df. 24]

at a finite distance BE [Pr. 13] (Fig. 32, left).

If AB is not a segment of AE (Fig. 32, right), draw though A the parallel AQ to CD [Pr. 28].

AB is not parallel to AQ, otherwise AB would be parallel to CD [Pr. 37],

which is not the case. Take in AB any point P_1, and draw the perpendicular P_1Q_1 to AQ [Pr. 16].

P_1Q_1 is the relative distancing of AP_1 with respect to AQ [Df. 16, Pr. 21].

Being AE and P_1Q_1 finite [Pr. 13], there will be a natural number $n \geq 1$ such that n times P_1Q_1 is greater than AE, otherwise there would be an impossible last natural number. On AB, produced if necessary [Cr. 16],

and from P_1, take just $n-1$ (4 in Figure 32) successive points P_2, P_3, $P_4 \ldots P_n$ [Cr. 1]

each separated from the previous one by the same finite distance AP_1 [Pr. 1].

Since the distances to AQ from the successive P_1, P_2, $P_3 \ldots P_n$ increase in the same direction [Pr. 23]

and by the same distance P_1Q_1 [Pr. 39],

and n times P_1Q_1 is greater than AE, the distance to AQ from P_n is greater than AE, and being zero the distance from A to AQ [Df. 14],

there is a point F in AP_n whose distance to AQ is just AE [Ax. 7].

And F is unique, if not AP_n would be parallel to AQ [Cr. 26],

which is not the case. D is a non-common point of one side of AB and then of AP_n [Dfs. 13, 11].

Join D and F [Cr. 15].

DF is parallel to AQ [Cr. 26],

and it must be a production of CD from D to F [Cr. 16],

otherwise there would be two different parallels, CD and DF, to AQ from the same point D, which is impossible [Pr. 29].

So, AB and CD can be produced respectively from B and from D to the point F. Since A is a point of the perpendicular AE to CD, it is not in straight line with CD [Cr 22].

So, A, E and F define a triangle AEF [Pr. 6],

of which ρ_2 is the greater angle [Pr. 19]

and AF the greater side [Pr. 11].

AF is finite because it is equal or less than n times the finite length AP_1. And $DF < EF$; $BF < AF$ [Cr. 13].

Therefore, the lengths of the productions DF and BF respectively of CD and of AB are both finite, and AB and CD intersect at a point F at a finite distance from their corresponding endpoints B and D [Df. 14].

In consequence, and taking into account that AB and CD are any two non-parallel and non-intersecting straight lines, we conclude that any two non-parallel and non-intersecting straight lines can be produced by a finite length so that they intersect with each other at a unique point. □

Proposition 41 (Euclid's Postulate 5) *If a common transversal makes with two given non-intersecting straight lines two angles in the same side of the transversal that sum less than two right angles, then the given straight lines can be produced in that side of the transversal by a finite length to a unique point where they intersect with each other.*

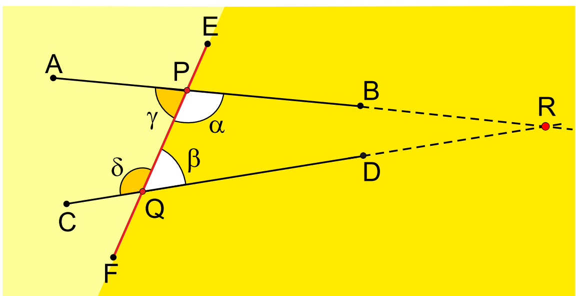


Figure 33 – Pr. 41

▷ Let AB and CD be any two non-intersecting straight lines [Cr. 30].

Each of them will be in the same side of the other [Cr. 28].

Let EF be a common transversal of AB and CD [Df. 25, Cr. 30]

that makes with AB and CD at its respective and unique intersection points P and Q [Cr. 19]

two interior angles α and β [Df. 25]

on the same side of EF [Ax. 6, Df. 21] whose sum is less than two right angles [Df. 24].

AB and CD are not parallel to each other [Pr. 36].

Therefore, they can be produced [Cr. 16]

by a finite length to a unique intersection point R [Pr. 40].

P, R and Q define a triangle PRQ [Pr. 6]

and R can only be a point in the side of EF where EF makes the angles α and β respectively with AB and CD, because in the other side [Ax. 6]

the interior angles sum more than two right angles [Pr. 38],

and PRQ would have two angles whose sum is greater than two right angles, which is impossible [Pr. 19]. $\square$

References

[1] Jason Socrates Bardi, *The Fifth Postulate: How Unraveling a Two Thousand Year Old Mystery Unraveled the Universe*, John Wiley & Sons, Hoboken, New Jersey, 2009.

[2] Georg Cantor, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955.

[3] Thomas Heath, *The Thirteen Books of Euclid's Elements*, second ed., vol. I, Dover Publications Inc, New York, 1956.

[4] Antonio Leon, *Infinity put to the test*, Next publication, 2021.

[5] John Playfair, *Elements of Geometry*, no. London, Forgotten Books, 2015.