Non-Euclidean Geometry on the Plane
and Lines of Zero Internal Curvature

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Abstract

An immutable fact: the Gaussian curvature of a non-Euclidean plane is the product of two differently interpreted principal curvatures, radial and tangential. The first reflects the change in the scale of distances relative to the center, and the second reflects the curvature of the trajectory of rectilinear motion in the tangential direction. Curvature of lines includes apparent (visible at scale) and internal (true) curvature. The apparent curvature of the line is considered positive if the line is oriented with concavity to the center, and negative if from the center. The internal curvature of the line is defined as the difference between the apparent curvature and the product of the tangential curvature times the sine of the angle between tangent and radius vector. On a hyperbolic, or rather, a pseudo-circular plane, straight lines look like hypotrochoids enveloping the center.

1 Introduction

As you know, in the Bolyai-Lobachevskian geometry, based on an extended interpretation of the postulate of parallel straight lines, it is not indicated which lines are considered straight lines. Geodesics? OK, then let's leave aside the postulates of Euclid and look at the non-Euclidean plane (NP) in terms of curvature. We will immediately see that both types of the NP are characterized by one general construction principle: one of two principal curvatures of the NP is interpreted by a change in the local scale of distances relative to the center at the origin. On the elliptic plane, this scale increases, and on the hyperbolic plane, it decreases from center to periphery. The scale-related principal curvature can be referred to as radial; its radius vector is normal to the NP. The other principal curvature is tangential; its radius vector lies in the NP and is directed from the center. And if the Gaussian curvature determines the overall landscape of the NP, then the principal curvatures determine the local behavior of the lines. Including straight lines, which should be considered as lines of zero internal curvature.

2 Circular (elliptic) plane and disc model

Consider an elliptic, or rather, a circular plane (CP) with a Gaussian curvature R. Let's imagine that the CP lies on the Euclidean plane (EP). In the center of the CP, the scale of distances coincides with the scale of EP up to a certain coefficient, and as you move away to π, it increases to infinity. For clarity, one can consider such an interpretation of the CP, on which the increase in scale, moving the mark π to infinity on the EP, reaches infinity at some finite distance on the EP. That is, the CP can be represented as a circle or disk, similar to the Poincaré disk model for the hyperbolic plane. And we will conditionally take the radius of the CP disk equal to $3R_L$, where R_L is the radius of linear curvature.

In the center of the CP, the distance scale on the CP and EP is related by the ratio:

$$a_C = \arcsin(1)\, a_E \tag{1}$$

where a_C is the length of an infinitesimal segment on the CP scale,
a_E is the length of an infinitesimal segment on the EP scale.

The circumference of a circle of arbitrary radius on the CP distance scale is:

$$L_{\mathrm{C}} = 2\,\pi\,R_{\mathrm{L}}\sin\!\left(r_{\mathrm{C}}\,R_{\mathrm{L}}^{-1}\right)\tag{2}$$

where r_{C} is radius of the circle on the CP scale,
R_{L} is radius of linear curvature on the EP scale, $R_{\mathrm{L}}=\sqrt{R}$.

Let us find out how the curvature of a non-Euclidean plane affects the curvature of lines. Let's introduce the concept of apparent and internal curvature of lines. The apparent curvature will be considered the curvature of the line visible from the EP, taking into account the scale of the NP distances, and taking into account the sign depending on the orientation of the convexity and concavity relative to the center of the NP. The apparent curvature of the line is assumed to be positive at where the line is oriented by the concavity to the center of the NP, and negative where it is oriented by the convexity to the center of the NP.

Take, for example, on a CP with a Gaussian curvature R=1, a central circle with a radius of π/2. The length of the circle will be 2π and this will be the circle with the maximum length. The apparent curvature of this circle will be exactly equal to the tangential curvature of the CP.

The apparent curvature of any central circle is determined by the formula:

$$\widetilde{k} = 2\,\pi\,L_{\mathrm{C}}^{-1}\tag{3}$$

From the viewpoint of the observer on the EP, the length of the central circle will continue to increase with an increase in its radius from π/2 to π, and from the viewpoint of the observer on the CP, its length will decrease from 2π to 0.

The internal curvature of a line is the curvature of this line from the viewpoint of the observer on the NP, and this curvature is determined by the formula:

$$\overset{\circ}{k} = \widetilde{k} - R_{\mathrm{T}}^{-1}\sin(\alpha)\tag{4}$$

where α is the angle between tangent and radius vector of tangential curvature,
R_{T} is radius of tangential curvature on the EP scale.

In Fig. 1, on the right, the tangent at point A to the off-center circle of the maximum length is conditionally shown as a dashed straight line. In fact, this tangent line is a circle of maximum length and coincides with the circle in question.
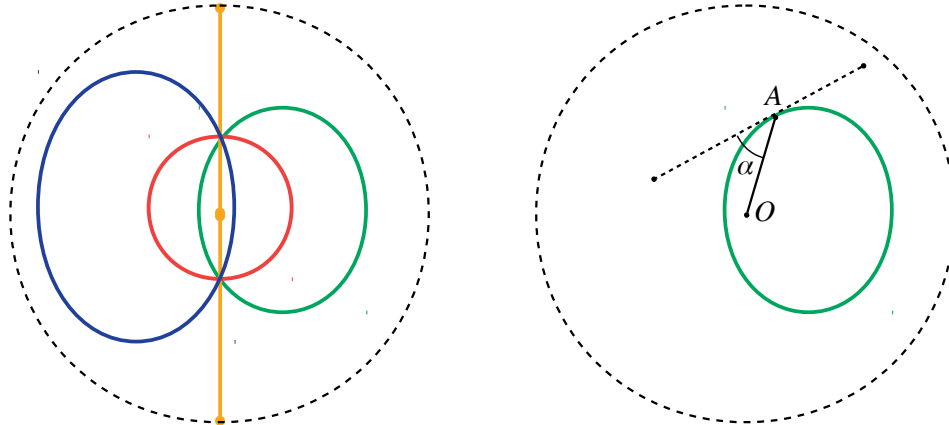

Figure 1. Disk model of a circular plane. Left: straight lines
(circles of maximum length) with zero internal curvature.
Right: angle between line tangent and radius vector

Circles of maximum length, including non-central ones, have zero internal curvature. Accordingly, these circles are considered to be straight lines. A characteristic feature: these circles either go around the center of the CP, or simultaneously intersect the center and boundary of the CP disk.

Together with the group of rotations around the center, the CP has a group of motions of the entire plane, shifting the center in any arbitrary direction. In this, the circular plane is completely equivalent to the sphere.

3 Pseudo-circular (hyperbolic) plane

The Gaussian curvature of the hyperbolic plane is negative, which means that one of two principal curvatures must be negative. This is a radial principal curvature, reflecting the decrease in the scale of distances from the center.

Consider a hyperbolic, or rather a pseudo-circular plane (PP) centered at point O, with a radius of Gaussian curvature R=1, and a radius of linear (tangential) curvature R_L=1. In the center of the PP, the distance scale of the PP and the EP is related by the ratio:

$$a_\mathrm{P} = \operatorname{arsinh}(1)\, a_\mathrm{E} \tag{5}$$

where a_P is the length of an infinitesimal segment on the PP scale,
a_E is the length of an infinitesimal segment on the EP scale.

No special interpretation or model is required for a pseudo-circular plane. The PP serves as a model for itself.

The length of the central circle of an arbitrary radius is determined by the formula:

$$L_\mathrm{P} = 2\pi R_\mathrm{L} \sinh\left(r_\mathrm{P} R_\mathrm{L}^{-1}\right) \tag{6}$$

where r_P is radius of the circle on the PP scale,
R_L is radius of linear (tangential) curvature on the EP scale.

The apparent curvature of any central circle is determined by the formula:

$$\widetilde{k} = 2\pi L_\mathrm{P}^{-1} \tag{7}$$

The apparent curvature of a line is an intermediate value, while the true curvature of a line should be considered its internal curvature. The internal curvature of the lines on the PP is determined in exactly the same way as on the CP, namely, as the difference between the apparent curvature of the line and the product of the tangential curvature by the sine of the angle between tangent and radius vector (see the formula 4).

On the PP there is no group of movements that shift the center. Individual points can move in any direction, but only rotation around the center is available for the entire plane.

If on the CP the curved lines with constant internal curvature are circles, then on the PP such lines are generally hypotrochoids.

The central circle with radius r=arsinh(1) is characterized by zero internal curvature, and thus is a straight line (in Fig. 2, 3, 4, it is shown everywhere by a dotted line).

An interesting feature: central circles with a radius r>arsinh(1) are characterized by negative internal curvature. These are pseudo-circles.
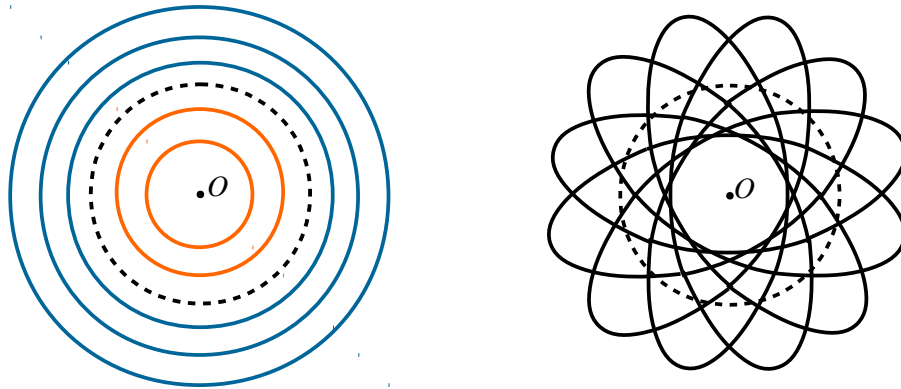
Figure 2. Pseudo-circular plane. Left: circles (orange) and
pseudo-circles (blue). Right: closed 12-turn hypotrochoid,
an example of a line of constant internal curvature.

If movement along geodesic lines is considered as a way of measuring the distances between individual points, then it is easy to see that the shortest paths will always run in some proximity to the center of the PP. But at the same time, the concept of proximity or remoteness is somewhat complicated. Take, for example, a closed 12-turn zero-curvature hypotrochoid similar to the one shown in Fig. 2, right. If you measure the distance between points on the aphelion of the turns, moving along this hypotrochoid itself, then the distance between the vertices of adjacent turns (even and odd) will be greater than the distance between the vertices of the nearest even or odd turns. Nevertheless, we admit that this is an apparent paradox, because we have associated the shortest distance between points with a specific hypotrochoid. In fact, the algorithm for finding the shortest path between two points should be to find a suitable hypotrochoid of zero internal curvature, that is, a geodesic.
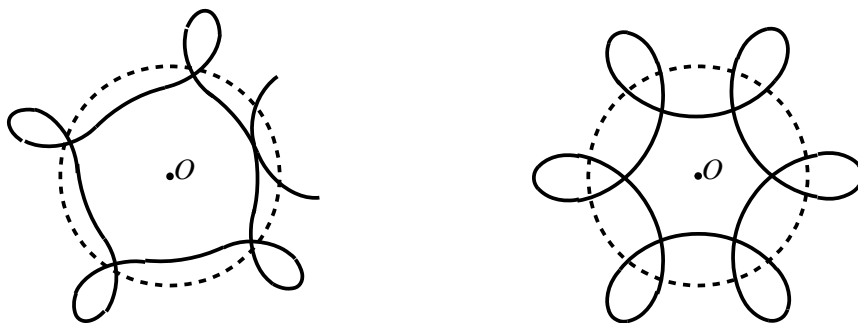

Figure 3. Lines of constant nonzero internal curvature.
Left: Unclosed hypotrochoid with a bend around the center of the inner part.
Right: Closed 6-turn hypotrochoid with a deviation from the center.

While hypotrochoids of zero internal curvature always bend around the center, other hypotrochoids of constant nonzero internal curvature can either bend around or deviate from the center at different parts of the curve (see Fig. 3).

The closer the hypotrochoid of zero internal curvature passes to the center of the PP, the smaller the angle of deflection of its turns near the center. Conversely, the closer the hypotrochoid is to the zero circle, the greater the angle of rotation of its next turn. And it may turn

out that the next loop will overlap the previous one and completely coincide with it. In this case, we get a closed 1-turn hypotrochoid or quasi-circle (see Fig. 4).
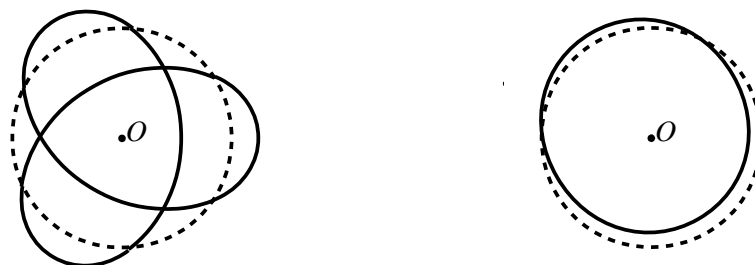

Figure 4. Lines of constant (including zero) internal curvature.
Left: closed 3-turn hypotrochoid. Right: quasi-circle.

Quasi-circles have constant internal curvature, which can be either zero or non-zero, and they also do not have a fixed center.

If we want to draw any arbitrary circle other than those described above, then we get a closed curved line of variable curvature, or a fuzzy circle.

4 Conclusion

On a non-Euclidean plane with negative Gaussian curvature, straight lines look like hypotrochoids enveloping the center. The Bolyai-Lobachevskian geometry with straight lines in the form of hyperbolas deviating from the center is incorrect.

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