Fixed-point property on finite-closed topological spaces

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Abstract

This short note presents a very simple and intuitive counterexample that disproves the statement "If (X, τ) is a topological space with the finite-closed topology, then it has the fixed-point property".

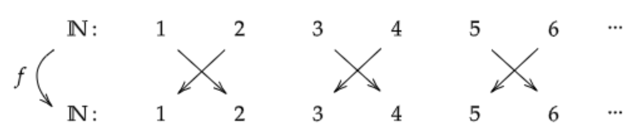
The counter example:

So let's consider the topological space $(\mathbb{N}, \tau)$ where τ is the finite-closed topology. Then let's consider the following function:

$$f : (\mathbb{N}, \tau) \to (\mathbb{N}, \tau)$$

$$f(n) = \begin{cases} n + 1, & \text{if } n \text{ is odd} \\ n - 1, & \text{if } n \text{ is even} \end{cases}$$

So basically this is what our function f does to all elements of $\mathbb{N}$:



Firstly we must prove that f is continuous:

Let $A \in \tau$. Then $\exists A^* \subseteq \mathbb{N}$ such that A^* is finite and $A = \mathbb{N} \setminus A^*$. So we have:

$$f^{-1}(A) = f^{-1}(\mathbb{N} \setminus A^*) = \mathbb{N} \setminus f^{-1}(A^*)$$

Because f is a bijection, we have $f^{-1}(A^*) \sim A^*$, this meaning that $f^{-1}(A^*)$ is closed and therefore $\mathbb{N} \setminus f^{-1}(A^*) = f^{-1}(A)$ is an open set.

So now that we have that f is continuous we note that $\nexists n \in \mathbb{N} : f(n) = n$ and thus the statement "If (X, τ) is a topological space with the finite-closed topology, then it has the fixed-point property" is clearly false.