A NOTE ON THE HODGE CONJECTURE

JORMA JORMAKKA

ABSTRACT. The paper presents a counterexample to the Hodge conjecture.

1. INTRODUCTION

The Hodge conjecture is one of the better known open problems in mathematics and was chosen as one of the Millennium Prize problems by the Clay Mathematics Institute [1]. The formulation of the conjecture in [1] is: On a projective non-sigular algebraic variety C, any Hodge class is a rational linear combination of classes $cl(\mathbb{Z})$ of algebraic cycles. This seems to say that every Hodge class on an algebraic projective complex manifold M is a linear combination with rational coefficients of the cohomology classes obtained by the Poincaré duality from homology classes of complex algebraic subvarieties of M.

The paper proposes a counterexample to the Hodge conjecure in the the algebraic projective complex manifold

$$M = \{(s_1 : s_2 : s_3 : s_4 : s_5 : s_6 : s_7 : s_8) | s_1^4 + s_2^4 + s_3^4 + s_4^4 = 0,$$
$$s_5^4 + s_6^4 + s_7^4 + s_8^4 = 0, s_4^4 + s_5^4 = 0\} \tag{1.1}$$

M is an algebraic subvariety of codimension 3 in the complex projective space $\mathbb{P}^7$ and a closed complex manifold of dimension 4. The manifold M is composed as a product of two Fermat quadratic surfaces tied with the third homogeneous polynomial equation in order to embed M into a projective space. As the Fermat quadratic surface is a $K3$ space, the manifold M inherits a (2,0)-form that is never zero from the first copy of $K3$ and a similar (0,2)-form from the second copy of $K3$. It is shown in the presented paper that the wedge product of these forms is a (2,2)-form that cannot be represented as a $\mathbb{Q}$-linear combination of cohomology classes of deriving from algebraic subvarieties of M. This form, having only one term $f(P)dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4$, is rational (it is a linear combination of this type of forms with rational coefficients) and thus a Hodge class.

The study of what cohomology classes can be obtained by the Poincaré duality from homology classes of submanifolds was started by René Thom, but in real manifolds. Thom's original paper is in French and rather difficult to read, but we can look at [2] that is available on-line and find there the following statement (page 1 in [2]): All homology classes with integral coefficients of compact orientable differentiable manifolds of dimension < 10 are realizable by submanifolds. As a 4-dimensional complex manifold can be understood as an 8-dimensional real manifold, this statement, combined with the Poincaré duality, implies that every singular cohomology class of the 8-dimensional manifold can be realized as a linear combination with

integer coefficients of classes of real submanifolds. However, these real submanifolds need not be complex submanifolds because a complex submanifold inherits the complex structure from the mother manifold.

If Z is a complex submanifold of M of codimension k then the inclusion map $i : Z \to M$ induces a cohomology class $[Z]$ that is in $H^{k,k}(M)$. In local coordinates the cohomology class $[Z]$ is then a form of the type

$$\psi = f(z, \bar{z})dz^I \wedge d\bar{z}^J \tag{1.2}$$

where $(z, \bar{z}) = (z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$ are the local coordinates. As $\psi \in H^{k,k}(M)$ there are the same number k of indices in the multi-indices I and J. This does not seem to say very much. However, if we look more carefully we see that the form (1.2) has the property: if z are the local coordinated of the submanifold, the differentials dz^j and $d\bar{z}^j$ always appear in pairs, as $dz^j \wedge d\bar{z}^j$. The reason is the following.

The tangent space of a 2-dimensional complex manifold in a chosen base point P is $\mathbb{C}^2$. There are two complex coordinates $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. (Let us use lower indices for a while as it is more natural.) A 2-dimensional real plane in the tangent space of the chosen basepoint can be spanned by any linear combination of four independent vectors $e_{x,1}, e_{y,1}, e_{x,2}, e_{y,2}$ but a complex line cannot separate x_j and y_j. They are not vectors, they are numbers in the field $\mathbb{C}$. Thus, a complex line through the basepoint can only be a linear combination $z = az_1 + bz_2 = c$, where a, b and c are complex numbers. Such a complex line defines a real plane as the x and iy coordinates of $z = x + iy$ correspond to the vectors e_x and e_y. In the local coordinates of the (real) mother manifold this (real) plane has the coordinates $e_x = ae_{x,1} + be_{x,2}$ and $e_y = ae_{y,1} + be_{y,2}$. It can be expressed by using a wedge product $xe_x \wedge ye_y$, which is a vector pointing to a direction orthogonal to the plane as the wedge product is a bit like the cross product. Instead of xe_x and ye_y we can use a linear combination of them: $z = x + iy$ and $\bar{z} = x - iy$. In this case the wedge product gets the form $z \wedge \bar{z}$. Here z is a local coordinate of the complex line. It is clearer if we denote the local coordinates of the tangent space of the complex submanifold by z'_j and the local coordinates of the tangent space of the mother manifold by z_m. Then z'_j is a linear combination (with complec coefficients) of the local coordinates z_m and a complex curve expressed as a (1,1)-form in is $f(P)dz' \wedge d\bar{z}'$ where $z' = z'_1$ is the only local coordinate of the 1-dimensional submanifold and P is the basepoint. If we take a complex surface as the submanifold, it has two local coordinates z'_1 and z'_2 and the form is of the type

$$f_1(P)f_2(P)dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \tag{1.3}$$

Consider a (2,2)-form of the type (changing the notation to upper indices)

$$\psi = f(z, \bar{z})dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \tag{1.4}$$

Does such a form exist in the cohomology of M and can such a form be obtained as a $\mathbb{Q}$-linear combination of forms of the type (1.3)?

Lemmas 3.1-3.3 are only needed to show that a form of the type (1.4) is not exact, i.e., zero in the cohomology. Lemmas 3.1 and 3.2 show that for a particular type of a form ψ we get $*\psi$ which is quite similar to ψ in form. Lemma 3.3 shows that the form is harmonic and thus not exact. Actually, we created the form so that both $d\psi = 0$ and $d(*\psi) = 0$ for trivial reasons. There is a special condition in this part of the proof: Lemmas 3.1-3.3 needs an assumption that the submanifolds are algebraic

varieties defined by polynomial equations with real coefficients. The important part is that the coefficients are real since the proof makes complex conjugation and requires that $f(P)^* = f(P^*)$, which is not always true. This assumption means that the manifold M must be selected in such a way that this assumption holds.

Lemma 3.4 shows that a form of the type (1.4) exists on $X = K3 \times K3$, which is a compact complex manifold of complex dimension 4. Lemma 3.3 shows that the form is not zero in $H^{2,2}(X)$. This is where we need the surface to be K3 and e.g. an Abelian surface would not work. Complex conjugation sends a harmonic form to a harmonic form, and thus we have a wedge of two harmonic forms, but the wedge of two harmonic forms is not necessarily harmonic - it can be exact. The nowhere vanishing (2,0)-form is needed in order to conclude that the wedge product is harmonic.

Lemma 3.5 proves that a form of the type (1.4) cannot arise as a $\mathbb{Q}$-linear combination from the cohomology classes of submanifolds. Here we again need a similar condition as in Lemma 3.1 that $f(P)^* = f(P^*)$. This condition follows if the mother manifold and all submanifolds are algebraic varieties defined by homogeneous polynomial equations with real coefficients. A restrictive assumption must be added to the lemma and fulfilling it requires a special form for M.

The manifold X in Lemma 3.4 is not quite what we want. It is a submanifold of $\mathbb{P}^3 \times \mathbb{P}^3$ but we want an algebraic subvariety of $\mathbb{P}^n$ for some n. Thus, we replace X by M. No changes are needed to Lemma 3.4.

The (2,2)-form from Lemma 3.4 is rational and represents a Hodge class. This fact seems obvious considering that it is like (1.4) and has only one term. However, in 2011 I received a comment from an expert of the field stating that the form is not rational. Therefore this issue must be addresses.

Finally I comment the apparent conflict of of the presented result with a published Ph.D. thesis [4]. The thesis proves that for some K3xK3 manifolds the Hodge conjecture holds. The conflict is only apparent since the proofs of Lemmas 3.1 and 3.5 require special conditions that $f(P)^* = f(P^*)$ and do not state anything for the general case of K3xK3.

2. NOTATIONS AND CONCEPTS

We will use the notations in Kodaira [7] page 147. Local coordinates of a complex manifold M of complex dimension n at the base point z_0 are denoted by $z^1, \ldots, z^n$, $\bar{f}(z^1, \ldots, z^n)$ is the complex conjugate of $f(z^1, \ldots, z^n)$, and the Hodge star operation is denoted by $*$.

Let φ and ψ be C^∞ (p,q)-forms in a complex manifold M.

$$\varphi = \frac{1}{p!q!} \sum \varphi_{\alpha_1,\ldots,\alpha_p,\bar{\beta}_1,\ldots,\bar{\beta}_q}(z) dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}$$
$$\psi = \frac{1}{p!q!} \sum \psi_{\alpha_1,\ldots,\alpha_p,\bar{\beta}_1,\ldots,\bar{\beta}_q}(z) dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}$$

$$(2.1)$$

The inner product is defined as

$$(\varphi, \psi)(z) = \frac{1}{p!q!} \sum \varphi_{\alpha_1,\ldots,\alpha_p,\bar{\beta}_1,\ldots,\bar{\beta}_q}(z) \bar{\psi}_{\alpha_1,\ldots,\alpha_p,\bar{\beta}_1,\ldots,\bar{\beta}_q}(z) \tag{2.2}$$

and

$$(\varphi, \psi) = \int_M (\varphi, \psi)(z) \frac{\omega^n}{n!} \tag{2.3}$$

where the volume element $\frac{\omega^n}{n!}$ is

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2} g(z) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \tag{2.4}$$

Here

$$g(z) = det(g_{\alpha\bar{\beta}}(z))_{\alpha,\beta=1,\ldots,n} \tag{2.5}$$

is given by a Hermitian metric

$$\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta \tag{2.6}$$

and

$$\omega = i \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \tag{2.7}$$

is the associated (1,1)-form, and $\omega^n = \omega \wedge \cdots \wedge \omega$ is the n-fold product. Then

$$
\begin{aligned}
(\varphi,\psi)\frac{\omega^n}{n!} &= (i)^n (-1)^{n(n-1)/2} g(z) \frac{1}{p!q!} \sum \varphi_{\alpha_1\ldots\alpha_p \bar{\beta}_1 \ldots \bar{\beta}_q}(z) \\
&\quad \times \bar{\psi}_{\alpha_1\ldots\alpha_p \bar{\beta}_1 \ldots \bar{\beta}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge \bar{z}^n \\
&= \frac{1}{p!q!} \sum \varphi_{\alpha_1\ldots\alpha_p \bar{\beta}_1 \ldots \bar{\beta}_q}(z) dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge d\bar{z}^{\beta_q} \wedge *\psi(z) \\
&= \varphi(z) \wedge *\psi(z)
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
*\psi(z) &= (i)^n (-1)^{\frac{1}{2}n(n-1)+(n-p)q} \sum_{A_p, B_q} sgn \begin{pmatrix} A_p & A_{n-p} \\ B_q & B_{n-q} \end{pmatrix} g(z) \bar{\psi}^{A_p B_q}(z) \\
&\quad \times dz^{B_{n-q}} \wedge d\bar{z}^{A_{n-p}}
\end{aligned}
\tag{2.9}
$$

The multi-indices are $A_p = \alpha_1 \ldots \alpha_p$, $A_{n-p} = \alpha_{p+1} \ldots \alpha_n$, $B_q = \beta_1 \ldots \beta_q$, $B_{n-q} = \beta_{q+1} \ldots \beta_n$. Notice, that Kodaira on page 117 defines the Hodge star differently as

$$(\varphi,\psi)\frac{\omega^n}{n!} = \varphi(z) \wedge *\bar{\psi}(z) \tag{2.10}$$

but the usual definition in the literature is

$$(\varphi,\psi)vol_x = \varphi(z) \wedge *\psi(z) \tag{2.11}$$

i.e.,

$$(\varphi,\psi)\frac{\omega^n}{n!} = \varphi(z) \wedge *\psi(z) \tag{2.12}$$

as we have defined.

Let us select local coordinates such that $g_{\alpha\bar{\beta}}(z_0) = \delta_{\alpha\beta}$ where z_0 is the base point. Then also $g^{\alpha\bar{\beta}}(z_0) = \delta_{\alpha\beta}$.

We will usually mark indices as upper indices but in a calculation in Lemma 3.5 upper indices get confused with powers and indices are marked as lower indices.

3. Lemmas and a Theorem

Lemma 3.1. *Let $p = q = 2$, $n = 4$, and*

$$\psi_{12\bar{3}\bar{4}}(z) = 4f(z^1, z^2, \bar{z}^3, \bar{z}^4)$$

$$\psi_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}(z) = 0 \quad if \quad (\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2) \neq (12\bar{3}\bar{4}) \tag{3.1}$$

and let $f(z^1, z^2, z^3, z^4)$ be a holomorphic function satisfying $f(z^1, z^2, z^3, z^4)^* = f(\bar{z}^1, \bar{z}^2, \bar{z}^3, \bar{z}^4)$. Then in local coordinates

$$\psi = f(z^1, z^2, \bar{z}^3, \bar{z}^4) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4$$

$$*\psi = g(z) f(\bar{z}^1, \bar{z}^2, z^3, z^4) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \tag{3.2}$$

Proof. The first claim is obvious since

$$\psi = \frac{1}{4}\psi_{12\bar{3}\bar{4}}(z) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \tag{3.3}$$

For the second claim we calculate

$$*\psi = (i)^4(-1)^{\frac{1}{2}4\cdot3+2\cdot2} sgn\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} g(z)\psi^{12\bar{3}\bar{4}}(z) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \tag{3.4}$$

Since

$$\psi^{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}(z) = \sum_{\lambda_1,\lambda_2,\mu_1,\mu_2} g^{\bar{\lambda}_1\alpha_1} g^{\bar{\lambda}_2\alpha_2} g^{\bar{\beta}_1\mu_1} g^{\bar{\beta}_2\mu_2} \bar{\psi}_{\lambda_1\lambda_2\bar{\mu}_1\bar{\mu}_2}(z)$$

$$= \sum_{\lambda_1,\lambda_2,\mu_1,\mu_2} \delta_{\lambda_1\alpha_1} \delta_{\lambda_2\alpha_2} \delta_{\beta_1\mu_1} \delta_{\beta_2\mu_2} \bar{\psi}_{\lambda_1\lambda_2\bar{\mu}_1\bar{\mu}_2}(z) \tag{3.5}$$

$$= \psi_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}(z) = \begin{cases} \bar{\psi}_{12\bar{3}\bar{4}}(z) & if \quad (\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2) = (12\bar{3}\bar{4}) \\ 0 & otherwise \end{cases}$$

Thus

$$*\psi = g(z)\bar{\psi}_{12\bar{3}\bar{4}}(z) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge \bar{z}^2$$

$$= g(z)\bar{f}(z^1, z^2, \bar{z}^3, \bar{z}^4) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge \bar{z}^2 \tag{3.6}$$

and the assumption on $f(z^1, z^2, z^3, z^4)$ gives

$$\bar{f}(z^1, z^2, \bar{z}^3, \bar{z}^4) = f(z^1, z^2, z^3, z^4)^* = f(\bar{z}^1, \bar{z}^2, z^3, z^4) \tag{3.7}$$

$\square$

Lemma 3.2. *Let $p = q = 2$, $n = 4$, and*

$$\psi_{12\bar{3}\bar{4}}(z) = 4f(z^1, z^2, \bar{z}^3, \bar{z}^4)$$

$$\psi_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}(z) = 0 \quad if \quad (\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2) \neq (12\bar{3}\bar{4}) \tag{3.8}$$

and let $f(z^1, z^2, z^3, z^4)$ be a holomorphic function satisfying $f(z^1, z^2, z^3, z^4)^* = f(\bar{z}^1, \bar{z}^2, \bar{z}^3, \bar{z}^4)$. We can select the metric such that in local coordinates

$$\psi = f(z^1, z^2, \bar{z}^3, \bar{z}^4) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4$$

$$*\psi = f(\bar{z}^1, \bar{z}^2, z^3, z^4) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \tag{3.9}$$

Proof. At the base point z_0 we can select the metric $g(z_0) = \det(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,\ldots,n}$ such that $g_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$. Then $g(z_0) = \det(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,\ldots,n} = 1$. We can make the same selection at all points z and thus $g(z) = 1$. $\square$

Lemma 3.3. *Let ψ be a (2,2)-form in a complex manifold M. Let ψ be expressed in local coordinates as*

$$\psi = f(z^1, z^2, \bar{z}^3, \bar{z}^4)dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \qquad (3.10)$$

where $f(z^1, z^2, \bar{z}^3, \bar{z}^4)$ is holomorphic satisfying $f(z^1, z^2, z^3, z^4)^* = f(\bar{z}^1, \bar{z}^2, \bar{z}^3, \bar{z}^4)$. Let us assume that the complex dimension of M is four, M is compact and without boundary (i.e., closed). Then $\triangle\psi = 0$.

Proof. For compact manifolds without boundary

$$\triangle\psi = 0 \quad \Leftrightarrow \quad d\psi = 0 \quad and \quad \delta\psi = 0 \qquad (3.11)$$

Let us calculate $d\psi$

$$d\psi = \sum_{j=1}^{4} \frac{\partial f}{\partial z^j}(z^1, z^2, \bar{z}^3, \bar{z}^4)dz^j \wedge dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 = 0 \qquad (3.12)$$

since $dz^j \wedge dz^1 = 0$ if $j = 1$, $dz^j \wedge dz^2 = 0$ if $j = 2$, and

$$\frac{\partial f}{\partial z^3}(z^1, z^2, \bar{z}^3, \bar{z}^4) = 0$$
$$\frac{\partial f}{\partial z^4}(z^1, z^2, \bar{z}^3, \bar{z}^4) = 0 \qquad (3.13)$$

since in the coordinate system $(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$ the coordinates z^j and $\bar{z}^j$ are considered independent. The second assertation follows from the definition of the codifferential: if $* : \Omega^k \to \Omega^{n-k}$ is the Hodge star operator then $\delta : \Omega^k \to \Omega^{k-1}$ is defined by

$$\delta\psi = (-1)^{n(k+1)+1}(*d*)\psi \qquad (3.14)$$

Inserting $n = 4$, $k = 2$, yields

$$\delta\psi = -(*d*)\psi = - * d(*\psi) \qquad (3.15)$$

We may assume that the metric is chosen such that $g(z) = 1$. By Lemma 3.2

$$*\psi = f(\bar{z}^1, \bar{z}^2, z^3, z^4)dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \qquad (3.16)$$

As in the previous case, we conclude that

$$d(*\psi) = \sum_{j=1}^{4} \frac{\partial f}{\partial z^j}(\bar{z}^1, \bar{z}^2, z^3, z^4)dz^j \wedge dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 = 0 \qquad (3.17)$$

$$\square$$

Lemma 3.4. *Let X be a product of two K3 surfaces. Then X allows a (2,2)-form of the type*

$$f(z^1, z^2)f(\bar{z}^3, \bar{z}^4)dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \qquad (3.18)$$

where $f(z^1, z^2)$ is holomorphic and nowhere vanishing.

Proof. The existence of a nowhere vanishing 2-form is often taken as the definition of a K3 surface, the additional condition guaranteeing that a 2-dimensional complex manifold X is a K3 surface is that the manifold X is connected. This nowhere vanishing 2-form λ is the generator of $H^{2,0}(X)$, i.e., every other element $\alpha \in H^{2,0}(X)$ can be expressed as $\alpha = c\lambda$ for some $c \in \mathbb{C}$. The complex conjugate

$\bar{\lambda}$ of λ is the generator of $H^{0,2}(X)$ as is shown by the Hodge duality pairing. Let the (2,0)-form λ be expressed in local coordinates as

$$\lambda = f(z^1, z^2)dz^1 \wedge dz^2 \qquad (3.19)$$

Then $f(z^1, z^2)$ is holomorphic and nowhere vanishing. If $M = K3 \times K3$ then there are two nowhere vanishing 2-forms λ_1 and λ_2. Let us remember that a complex K3-surface is compact and as a real manifold it is a 4-dimensional closed manifold. Thus, M is closed and we can use Lemma 3.3. We can make a (2,2)-form as a wedge product of

$$\lambda_1 = f(z^1, z^2)dz^1 \wedge dz^2 \qquad (3.20)$$

and

$$\bar{\lambda}_2 = \bar{f}(z^3, z^4)d\bar{z}^3 \wedge d\bar{z}^4 = f(\bar{z}^3, \bar{z}^4)d\bar{z}^3 \wedge d\bar{z}^4 \qquad (3.21)$$

as $f(z^3, z^4)$ is holomorphic. Thus, we define

$$\begin{aligned} \psi &= f(z^1, z^2)dz^1 \wedge dz^2 \wedge f(\bar{z}^3, \bar{z}^4)d\bar{z}^3 \wedge d\bar{z}^4 \\ &= f(z^1, z^2)f(\bar{z}^3, \bar{z}^4)dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \end{aligned} \qquad (3.22)$$

The form ψ satisfies $\triangle\psi = 0$ by Lemma 3.3. We still have to show that it is not zero in $H^{2,2}(X)$. Typical ways to show this are calculating periods or intersection products, but we will show it differently. For any complex manifold M the Hodge star gives an isomorphism from $H^k(M)$ to $H^{n-k}(M)$ and the Poincaré duality gives an isomprphism from $H_k(M)$ to $H^{n-k}(M)$. Especially, when $n = 4$, $H^2(X)$ and $H_2(X)$ are isomorphic and if $X = S_1 \times S_2$ $H^2(S_i)$ and $H_2(S_i)$, $i = 1, 2$ are isomorphic (for any coefficients, so coefficients are supressed in the notations). Let this isomorphism be $\Psi : H_2(X) \to H^2(X)$. Let C be a 2-chain in S_1 and let $(C, p_t) \in X$ be a family of pairs where p_t is a path in S_2. This family $C \times [0, 1]$ defines a homotopy from (C, t_0) to (C, t_1). By the homomorphism we have also a family $\lambda_1 \wedge \bar{\lambda}_2(P_t)$ parametrized by $t \in [0, 1]$. Then P_t defines a path in S_2. This is where we need the nowhere vanishing (2,0)-form. If there existed a point P_1 such that $\bar{\lambda}_2(P_1) = 0$ then $\psi = 0$ at any point $(Q, P_1) \in S_1 \times S_2$. As the preimage of zero is a point, this would yield a homotopy from C to a point, i.e., $(C, p_0) = 0$ in $\pi_2(X)$. Consequently $(C, p_0) = 0$ in $H_2(X)$ and $\Psi(C, p_0) = (\lambda_1, P_0) = 0$ in $H^2(X)$. As there is no such point P_1, (C, p_0) is not contractible in X and is therefore nonzero in $H_2(X)$. Its image under Ψ is therefore nonzero in $H^2(X)$. Thus, ψ is a harmonic form in $H^{2,2}(X)$. $\square$

Lemma 3.5. *A (2,2)-form of the type $f(z, \bar{z})dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4$ is not a linear combination with rational coefficients of cohomology classes deriving from complex submanifolds of complex codimension 2 in a complex submanifold of complex dimension 4 assuming two conditions: 1) the mother manifold is an algebraic variety defined by homogeneous polynomial equations with real coefficients, and 2) all submanifolds are algebraic varieties defined by homogeneous polynomial equations with real coefficient.*

Proof. Let N be a complex manifold of dimension n and let M be a complex submanifold of N of complex dimension k. Let $i : M \to N$ be the inclusion map and the complex structure of M be induced by the complex structure of N. Let $P \in M$ be a point in M and $(x'^1, y'^1, \ldots, x'^k, y'^k)$ be a local coordinate system in TM_P where M is considered as a real manifold of dimension $2k$. The coordinate

system can be completed to a coordinate system of $TN_{i(P)} = TN_P$ by adding $2(n-k)$ coordinate vectors. This yields a coordinate system

$$(x', y') = (x'^1, y'^1, \ldots, x'^k, y'^k, x'^{k+1}, y'^{k+1}, \ldots, x'^n, y'^n)$$

to TN_P where $P = (0,0) = (0, \ldots, 0)$. We can assume that the coordinates are orthonormal. The manifold N can be considered as a real manifold of dimension $2n$ and TN_P be given a local coordinate system

$$(x, y) = (x^1, y^1, \ldots, x^n, y^n) \quad , \quad P = i(P) = (0, 0)$$

There is an orthonormal coordinate transform $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $A \in SO(2n)$, such that

$$[x' \ y']^T = A[x \ y]^T \tag{3.23}$$

The coordinate systems at TN_P can be chosen such that

$$\begin{aligned}
z^j &= x^j + iy^j \quad , \quad \bar{z}^j = x^j - iy^j \quad j = 1, \ldots, n \\
z'^j &= x'^j + iy'^j \quad , \quad \bar{z}'^j = x'^j - iy'^j \quad j = 1, \ldots, n
\end{aligned} \tag{3.24}$$

and there is an orthonormal coordinate transform $B : \mathbb{C}^n \to \mathbb{C}^n$ such that for

$$\begin{aligned}
(z, \bar{z}) &= (z^1, \bar{z}^1, \ldots, z^n, \bar{z}^n) \\
(z, \bar{z}') &= (z'^1, \bar{z}'^1, \ldots, z'^n, \bar{z}'^n)
\end{aligned} \tag{3.25}$$

the transform takes $(z, \bar{z})$ to $(z', \bar{z}')$, $[z' \ \bar{z}']^T = B[z \ \bar{z}]^T$. As the complex structure of M is induced from the complex structure of N, the following statement holds

$$\begin{aligned}
if \quad &(z', \bar{z}') = (x'^1, x'^1, \ldots, x'^n, x'^n) \quad i.e. \quad y'^j = 0 \ for \ j = 1, \ldots, n \\
then \quad &(z, \bar{z}) = (x^1, x^1, \ldots, x^n, x^n) \quad i.e. \quad y^j = 0 \ for \ j = 1, \ldots, n
\end{aligned} \tag{3.26}$$

It follows that if

$$z'^j = \sum_m a_{m,j} z^m \tag{3.27}$$

then

$$\bar{z}'^j = \sum_m a_{m,j} \bar{z}^m \tag{3.28}$$

that is, $a_{m,j} \in \mathbb{R}$ for all j and m. The Poincaré dual $[M]$ of M considered as a $2k$-dimensional real manifold in the $2n$-dimensional real manifold N satisfies the condition that $[M]$ capped with the fundamental class of N is the homology class of M. Thus, $[M]$ is a form of the type

$$\varphi(x', y')dx'^{k+1} \wedge dy'^{k+1} \wedge \cdots \wedge dx'^n \wedge dy'^n \tag{3.29}$$

Since x'^j and y'^j are independent, the class $[M]$ is represented at p by a form of the type

$$\varphi(z', \bar{z}')dz'^{k+1} \wedge d\bar{z}'^{k+1} \wedge \cdots \wedge dz'^n \wedge d\bar{z}'^n \tag{3.30}$$

In coordinates $\{z^1, \bar{z}^1, \ldots, z^n, \bar{z}^n\}$ the form can be expressed by inserting dz'^j and $d\bar{z}'^j$ as linear combinations (with real coefficients) of dz^k and $d\bar{z}^k$, $k = 1, \ldots, n$.

Let us now take a two dimensional complex plane in a four dimensional complex manifold. We will use lower indices in this calculation as we obtain squares at some places and upper indices get easily confused with powers. The two local coordinates

z_1'', z_2'' of a complex plane in a four dimensional complex manifold can be expressed in the local coordinates z_1, z_2, z_3, z_4 of the mother manifold as

$$z_1'' = b_{11}z_1 + b_{12}z_2 + b_{13}z_3 + b_{14}z_4$$
$$z_1'' = b_{21}z_1 + b_{22}z_2 + b_{23}z_3 + b_{24}z_4 \qquad (3.31)$$

where b_{ij} are complex numbers. We want to see what terms come from

$$dz_1'' \wedge d\bar{z}''_1 \wedge dz_2' \wedge d\bar{z}'_2 \qquad (3.32)$$

when the local coordinates z_i'' of the submanifold are replaced by the local coordinates z_i of the mother manifold. We get 36 terms the form $dz_{m_1} \wedge dz_{m_2} \wedge d\bar{z}_{m_3} \wedge d\bar{z}_{m_4}$ where $m_1 < m_2$ and $m_3 < m_4$ because if f $m_1 = 1$, m_1 can be $2, 3, 4$, if $m_1 = 2$, m_2 can be $3, 4$, and if $m_1 = 3$ then $m_2 = 4$. This means that there are six possibilities for m_1, m_2. There are also six possibilities for m_3, m_4. Together there are 36 possibilities and thus 36 different terms. While the terms can be calculated we get simpler expressions by making a linear transform of the local coordinates z_1'', z_2''

$$z_1' = \alpha(-b_{22}z_1'' + b12z_2'')$$
$$z_2' = \alpha(b_{21}z_1'' - b11z_2'')$$
$$\alpha = (b_{21}b_{12} - b_{11}b_{22})^{-1} \qquad (3.33)$$

Then

$$z_1' = z_1 + a_{13}z_3 + a_{14}z_4$$
$$z_2' = z_2 + a_{23}z_3 + a_{24}z_4$$
$$a_{13} = \alpha(b_{12}b_{23} - b_{13}b_{22}) \quad , \quad a_{14} = \alpha(b_{12}b_{24} - b_{14}b_{22})$$
$$a_{23} = \alpha(b_{21}b_{13} - b_{11}b_{23}) \quad , \quad a_{24} = \alpha(b_{21}b_{14} - b_{11}b_{24}) \qquad (3.34)$$

These new coordinates are not orthogonal, but that does not matter here: we just want to see what terms come from

$$dz_1' \wedge d\bar{z}'_1 \wedge dz_2' \wedge d\bar{z}'_2 \qquad (3.35)$$

when the z_i coordinates are inserted. Again there are 36 terms, but the expressions are shorter than for the original coordinates z_i''. The first six terms are:

$$= -dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 - |a_{13}|^2 dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3 - |a_{14}|^2 dz_2 \wedge dz_4 \wedge d\bar{z}_2 \wedge d\bar{z}_4$$
$$-|a_{23}|^2 dz_1 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_3 - |a_{24}|^2 dz_1 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_4$$
$$-|a_{23}a_{14} - a_{24}a_{13}|^2 dz_3 \wedge dz_4 \wedge d\bar{z}_3 \wedge d\bar{z}_4$$

$$(3.36)$$

Terms 7-16 numbered in this order:

$$+a_{13}^* dz_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + a_{14}^* dz_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge d\bar{z}_4$$
$$+a_{13} dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 + a_{14} dz_2 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2$$
$$-a_{13}a_{14}^* dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_4 - a_{13}^* a_{14} dz_2 \wedge dz_4 \wedge d\bar{z}_2 \wedge d\bar{z}_3$$
$$+a_{23}(a_{23}^* a_{14}^* - a_{24}^* a_{13}^*)dz_1 \wedge dz_3 \wedge d\bar{z}_3 \wedge d\bar{z}_4 \qquad (3.37)$$
$$+a_{24}(a_{23}^* a_{14}^* - a_{24}^* a*_{13})dz_1 \wedge dz_4 \wedge d\bar{z}_3 \wedge d\bar{z}_4$$
$$+a_{23}^*(a_{23}a_{14} - a_{24}a_{13})dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_3$$
$$+a_{24}^*(a_{23}a_{14} - a_{24}a_{13})dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_4$$

Terms 17-22:

$$-a_{24}a_{23}^*dz_1 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_3 - a_{24}^*a_{23}dz_1 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_4$$
$$-a_{13}(a_{23}^*a_{14}^* - a_{24}^*a_{13}^*)dz_2 \wedge dz_3 \wedge d\bar{z}_3 \wedge d\bar{z}_4$$
$$-a_{14}(a_{23}^*a_{14}^* - a_{24}^*a_{13}^*)dz_2 \wedge dz_4 \wedge d\bar{z}_3 \wedge d\bar{z}_4 \qquad (3.38)$$
$$-a_{13}^*(a_{23}a_{14} - a_{24}a_{13})dz_3 \wedge dz_4 \wedge d\bar{z}_2 \wedge d\bar{z}_3$$
$$-a_{14}^*(a_{23}a_{14} - a_{24}a_{13})dz_3 \wedge dz_4 \wedge d\bar{z}_2 \wedge d\bar{z}_4$$

Terms 23-34:

$$+a_{23}^*a_{13}dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_3 + a_{23}^*a_{14}dz_2 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_3$$
$$+a_{24}^*a_{13}dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_4 + a_{24}^*a_{14}dz_2 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_4$$
$$+a_{23}a_{13}^*dz_1 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + a_{23}a_{14}^*dz_1 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_4$$
$$+a_{24}a_{13}^*dz_1 \wedge dz_4 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + a_{24}a_{14}^*dz_1 \wedge dz_4 \wedge d\bar{z}_2 \wedge d\bar{z}_4 \qquad (3.39)$$
$$-a_{23}^*dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3 - a_{24}^*dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3$$
$$-a_{23}dz_1 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 - a_{24}dz_1 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2$$

We are mainly interested in these two last terms, terms 35 and 36:

$$+(a_{23}a_{14} - a_{24}a_{13})dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2$$
$$+(a_{23}^*a_{14}^* - a_{24}^*a_{13}^*)dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4 \qquad (3.40)$$

We assume that the mother manifold is an algebraic variety defined by a finite number of homogeneus polynomial equations of a finite number of complex parameters s_i and real coefficients. An example of such a variety is the Fermat quadratic surface defined by the homogeneous polynomial equation

$$s_1^4 + s_2^4 + s_3^4 + s_4^4 = 0 \qquad (3.41)$$

In thie polynomial equation the coefficients are all integers (all are 1s).

If $P_o = (s_{10}, s_{20}, s_{30}, s_{40})$ is a chosen basepoint, the local coordinates of the manifold can be chosen as $z_i = s_i - s_{i0}$. Let us consider a (2,2)-form

$$f(P)dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4 \qquad (3.42)$$

that is linear combination with rational coefficients of classes of n complex submanifolds. Each submanifold gives 36 terms i of the type

$$\varphi_j(P)\varphi_k(P)A_{j,k,i}dz_{m_1} \wedge dz_{m_2} \wedge d\bar{z}_{m_3} \wedge d\bar{z}_{m_4} \qquad (3.43)$$

Here $A_{j,k,i}$ is the coefficient of $dz_{m_1} \wedge dz_{m_2} \wedge d\bar{z}_{m_3} \wedge d\bar{z}_{m_4}$ for two (1,1)-forms j and k and i is numbered as in (3.36-3.40) where we have explicitly written the coefficients $A_{1,2,i}$, $i = 1, \ldots, 36$. Thus, for $j = 1$ and $k = 2$ the coefficients of the last two terms are

$$A_{1,2,35} = a_{23}a_{14} - a_{24}a_{13}$$
$$A_{1,2,36} = a_{23}^*a_{14}^* - a_{24}^*a_{13}^* \qquad (3.44)$$

where a_{mn} are calculated for the submanifold corresponding to two (1,1)-forms that we numbered as $j =$ and $k = 2$. For different values of j and k we get different a_{mn} but it was inconvenient to add the indices j, k to (3.36)-(3.40).

If we use the coordinates z_i'' instead of the simpler coordinates z_i' and calculate $A_{1,2,35}$ and $A_{1,2,36}$, they are

$$A_{1,2,35} = -b_{13}b_{24}b_{11}^*b_{22}^* + b_{14}b_{23}b_{11}^*b_{22}^* + b_{13}b_{24}b_{12}^*b_{21}^* - b_{14}b_{23}b_{12}^*b_{21}^*$$
$$A_{1,2,36} = -b_{13}^*b_{24}^*b_{11}b_{22} + b_{14}^*b_{23}^*b_{11}b_{22} + b *_{13} b_{24}^*b_{12}b_{21} - b_{14}^*b_{23}^*b_{12}b_{21} \qquad (3.45)$$

Clearly, always holds

$$A_{j,k,36} = A_{j,k,35}{}^*$$
(3.46)

In order for a (2,2)-form of the type (3.42) to be a $\mathbb{Q}$-linear combination forms corresponding to submanifolds (of this type) in each point P must hold 35 equations of the type

$$\sum_{j,k} \varphi_j(P)\varphi_j(P)A_{j,k,i} = 0 \quad i = 1,\dots,35$$
(3.47)

and the last sum must be non-zero

$$\sum_{j,k} c_{jk}\varphi_j(P)\varphi_j(P)A_{j,k,36} \neq 0$$
(3.48)

where c_{jk} are rational coefficients. This is a set of linear equations for the unknowns c_{jk}. It is clear that in a chosen basepoint P_0 we can solve these linear equations and find complex numbers c_{jk} that satisfy these 36 equations, provided that there are at least 36 independent submanifolds. It is not clear if we can find rational numbers c_{jk} that satisfy all equations, but let us assume that we have found rational c_{jk} that satisfy these equations at the point P_0. The question is if these same c_{jk} can satisfy these 36 equations in every other point P.

It is easily shown that they cannot, assuming what we have assumed that the homogeneous polynomial equations defining the mother manifold and the submanifolds have real coefficients. With real coefficients we can find another point P_1 as a complex conjugate of P_0.

Thus, let us take the second point as $P_1 = (s_{10}^*, s_{20}^*, s_{30}^*, s_{30}^*)$. Then $P_1 = P_0^*$. The mother manifold and submanifolds are assumed to be algebraic varieties defined by a finite number of homogeneous polynomials with real coefficients. Therefore if P_0 is a solution to the homogeneous polynomial equations, then so it P_0^*. The functions $\varphi_j(P)$ have the same form at $P = P_0$ and $P = P_1$ but the values differ: $\varphi_j(P_1) = \varphi_j(P_0^*) = \varphi_j(P_0)^*$ because the coefficients of the homogeneous polynomial equations are real. This is the essence: in order to show that the 36 equations cannot be satisfied at every point P we need to find another point P_1 which is far from P_0 (in the close vicinity of P_0 the 36 equations are satisfied) and be able to calculate what the 36 equations are at the point $P - 1$. It is not likely that the 36 equations could be satisfied at each point P even if we would not make the restrictive assumption, but it is difficult to find a point P_1 without making this assumption.

There are two ways to transfer the local coordinates from P_0 to P_1. One way is to choose similar local coordinates z_i, $i = 1,\dots,4$, and z_i'', $j = 1,2$, in the basepoint P_1 as in P_0. In this case the form $dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4$ does not change in the transformation and the numbers $A_{jk,i}$, $i = 1,\dots,36$, do not change:

$$A_{j,k,36}(P_1) = A_{j,k,36}(P_0)$$
(3.49)

where $A_{j,k,i}(P)$ means $A_{j,k,i}$ calculated at the point P. The other method is to take conjugates. In this case both the form $dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4$ and the numbers $A_{jk,i}$, $i = 1,\dots,36$ change to complex conjugates.

Let us follow the first way. The equation 36 at the point $P = P_1$ gives the coefficient of the term $dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4$. The coefficient is

$$\sum_{j,k} c_{jk}\varphi_j(P_1)\varphi_j(P_1)A_{j,k,36}(P_1) = \sum_{j,k} c_{jk}\varphi_j(P_0)^*\varphi_j(P_0)^*A_{j,k,36}(P_0)$$

$$= \left(\sum_{j,k} c_{jk}\varphi_j(P_0)\varphi_j(P_0)\left(A_{j,k,36}(P_0)\right)^* \right)^*$$

$$= \left(\sum_{j,k} c_{jk}\varphi_j(P_0)\varphi_j(P_0)A_{j,k,35}(P_0) \right)^* = 0 \tag{3.50}$$

That is, at the point P_1 the form $dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4$ vanishes. The coefficient that does not vanish at P_1 is the coefficient of the term $dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2$.

The second way corresponds to finding the form at P_1 by conjugating the form at P_0. In this case

$$\left(\sum_{j,k} c_{jk}\varphi_j(P_0)\varphi_j(P_0)A_{j,k,36}(P_0)dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4 \right)^*$$

$$= \sum_{j,k} c_{jk}\varphi_j(P_1)\varphi_j(P_1)A_{j,k,36}(P_1)\left(dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4\right)^*$$

$$= \sum_{j,k} c_{jk}\varphi_j(P_1)\varphi_j(P_1)A_{j,k,36}(P_1)d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3 \wedge dz_4 \tag{3.51}$$

$$= \sum_{j,k} c_{jk}\varphi_j(P_1)\varphi_j(P_1)A_{j,k,36}(P_1)dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2$$

In this method the coefficient of the form stays as nonzero, but it is not the coefficent of $dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4$ at P_1. It is the coefficient of $dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2$. The coeffient of $dz_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4$ at P_1 is zero also in this method. Naturally, both ways give the same result.

We conclude that the form (3.42) cannot be created as a linear combination of terms (3.43). There should be at least two nonzero terms, those corresponding to $A_{j,k,36}$ and to $A_{j,k,35}$. $\qquad\square$

Theorem 3.6. *The algebraic variety*

$$M = \{(s_1 : s_2 : s_3 : s_4 : s_5 : s_6 : s_7 : s_8) | s_1^4 + s_2^4 + s_3^4 + s_4^4 = 0,$$
$$s_5^4 + s_6^4 + s_7^4 + s_8^4 = 0, s_4^4 + s_5^4 = 0\} \tag{3.52}$$

is an algebraic subvariety of the complex projective space $\mathbb{P}^7$ of codimension 3 and a complex manifold of dimension 4. There is a Hodge class in $H^{2,2}$ that cannot be represented as a Q-linear combination of the classes of algebraic subvarieties of M.

Proof. Clearly M is a submanifold of $\mathbb{P}^7$ and it is an algebraic variety. It contains two copies of the Fermat quadratic surface, which is a K_3 surface. The submanifolds of dimension 1 for the Fermat quadratic surface are defined by adding one homogeneous polynomial equation. The dimension of the space of (1,1)-forms in K_3 is 20, thus if we find 20 polynomial equations that give an essentially different

submanifold, we have represented all (1,1)-forms by an algebraic subvarity. By inspection, the homogeneous polynomial equations are

$$
\begin{aligned}
s_i^4 &= 0 \quad i = 1, 2, 3, 4 \\
s_i^4 &= 1 \quad i = 1, 2, 3, 4 \\
s_i^4 + s_j^4 &= 0 \quad (i, j) = (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \\
s_i^4 + s_j^4 &= 1 \quad (i, j) = (1,2), (1,3), (1,4), (2,3), (2,4), (3,4)
\end{aligned}
\tag{3.53}
$$

Clearly, fixing one of s_i to zero gives a submanifold and the submanifold is different for each $i = 1, 2, 3, 4$. In the complex projective space $\mathbb{P}^1$ there are only two numbers, zero and one. Thus, setting $s_i = 1$ gives another set of four different submanifolds and there are no more equations tying only one s_i. Setting the sum of two s_i variable terms together gives six different submanifolds if set the sum to zero and another six if we set the sum to one. There could be more equations of this type, but as these 20 equations give 20 different classes, we need not look further. There cannot be more classes: all other equations yield linear combinations of the classes. All of these 20 equations have real coefficients and we can use Lemmas 3.1-3.3 and 3.5.

Lemma 3.2 holds without any special considerations. In Lemma 3.3 we notice that a complex K_3 surface is compact and as a real 4-manifold it is closed. Thus, the product of two K_3 surfaces is closed. M is obtained from $K3 \times K3$ by adding an equation and is also closed. Lemma 3.3 can be used.

Lemma 3.4 applies also to M and gives the (2,2)-form.

The conditions for Lemma 3.5 are filled by M and the lemma shows the (2,2)-form of Lemma 3.4 cannot be represented as a $\mathbb{Q}$-linear combination of classes of submanifolds.

The (2,2)-form in Lemma 3.4 has only one term in the local coordinates of M everywhere. Thus, it is not a linear combination. It is a single term and therefore can be considered as a rational class. □

I wrote the first version of this paper in 2011 and sent it to arXiv for discussion purposes. I was not at all sure if my result was correct since the solution seemed too easy and there was a conflict with a statement in [4] on page 54. I got an answer from an expert of the field, Bert van Geemen. His comment was that the (2,2)-form in Lemma 3.4 does not represent a Hodge class since we cannot show that it is rational.

The explanation by Bert van Geeman is the following:
$''$ We remember that singular cohomology of a complex manifold of complex dimension n is defined as the singular cohomology of the underlying real manifold of dimension $2n$. This yields $H^*(X; \mathbb{Z})$. In order to get $H^*(X; k)$, where the field k is $\mathbb{Q}$ or $\mathbb{R}$, or $\mathbb{C}$, we form the tensor product of $H^*(X; \mathbb{Z})$ with k. When the base classes of $\mathbb{Z}$ are selected, the classes of the tensor products are linear combinations of the base classes with coefficients in k. We can multiply the created (2,2)-class with any number in k and get a harmonic class. If the class is a multiple of a base vector, we can always multiply it with a suitable number to get a class in $H^4(X; \mathbb{Q})$. However, if it is a linear combination with generic coefficients, multiplication by one number does not give a rational class. This seems to be the case with this (2,2)-form.

Theorem 2 in [4] on page 54, also published in [5], shows that the Hodge conjecture holds for certain K3xK3 spaces, showing that in the general case the constructed (2,2)-form cannot be a Hodge class. The space of Hodge classes $B(H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q}))$ is identified up to a Tate twist with $End_{Hdg}(H^2(S, \mathbb{Q}))$ on page 14 in [4]. Zarhin's theorem is used to characterize $End_{Hdg}(T)$ on page 16, and Mukai's theorem is used in the proof of Theorem 2, as in the proofs of other theorems in [4]. Mukai's theorem requires that the endomorphism $\varphi : T(S_1) \to T(S_1)$ is a Hodge isometry. A Hodge isometry maps $H^{2,0}(X)$ to $H^{2,0}(X)$, as is clear e.g. from [6] page 211. This method in [4] seems to derive from a paper of D. Morrison [7]. The endomorphism in the presented paper is complex conjugation, which sends $H^{2,0}(X)$ to $H^{0,2}(X)$ and is thus not a Hodge isometry. Mukai's theorem thus cannot be used, but the identification of the space of Hodge classes with endomorphisms preserving the Hodge structure still holds, and consequently the (2,2)-form created in Lemma 3.4 does not represent a Hodge class."

I accepted this answer from an expert at that time and wrote in revised version of the arXiv paper that the proof does not give a counterexample to the Hodge conjecture. However, when I have now checked the paper, I find that the presented (2,2)-form is not a linear combination of (2,2)-forms. It has a single term and therefore a multiplication with a number does give a rational form. Thus, it is a Hodge class. The complex manifold M can be understood as a real manifold of dimension eight. The form corresponds to a real submanifold of dimension four that is obtained by selecting local coordinates as $v_1 = e_{x,1} + e_{y,1}$, $v_2 = e_{x,1} - e_{y,1}$, $v_3 = e_{x,2} + e_{y,2}$ and $v_4 = e_{x,2} - e_{y,2}$. This submanifold gives a homology class that is the class of the submanifold. By duality there is a cohomology class corresponding to it. This cohomology class is inherited to singular cohomology of M and it is not a linear combination of anything.

The issue with the apparent conflict with [4] is avoided by noticing that the proofs of Lemma 3.1 and 3.5 require a special condition on the K3-surface. Therefore they do not state anything of K3xK3 spaces in the general case. Yet, if it follows from those results that the (2.2)-form in Lemma 3.4 cannot be rational, then the logic of that argument should be checked.

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E-mail address: jorma.o.jormakka@gmail.com