INITIAL-BOUNDARY VALUE PROBLEMS FOR A SYSTEM OF PARABOLIC CONSERVATION LAWS ARISING FROM A KELLER-SEGEL TYPE CHEMOTAXIS MODEL

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ABSTRACT. In this paper, we investigate the time-asymptotically nonlinear stability to the initial-boundary value problem for a coupled system in (p, q) of parabolic conservation laws derived from a Keller-Segel type repulsive model for chemotaxis with singular sensitivity and nonlinear production rate of $g(p) = p^\gamma$, where $\gamma > 1$. The proofs are based on basic energy method without any smallness assumption.

1. INTRODUCTION

In this paper, we investigate the global ,strong, large-amplitude solution and long time behavior of initial-boundary value problems for a system of parabolic conservation laws

$$\begin{cases} p_t - (pq)_x = p_{xx}, \ x \in (0,1), t > 0, \\ q_t - (g(p) + \varepsilon q^2)_x = \varepsilon q_{xx}, \quad \varepsilon > 0. \end{cases} \tag{1.1}$$

By taking the Cole-Hopf transformation $p = n$, $q = [\ln(c)]_x$ and assuming $D = -\chi = 1$ without loss of generality since specific values of χ and D are not important in our analysis, we can derive this system from the following chemotactic model proposed in [1] with logarithmic sensitivity and nonlinear production rate

$$\begin{cases} n_t = Dn_{xx} - [\chi n(\ln(c))_x]_x, \ x \in (0,1), \ t > 0, \\ c_t = \varepsilon c_{xx} + g(n)c - \mu c, \ x \in (0,1), \ t > 0, \varepsilon > 0, \end{cases} \tag{1.2}$$

where n and c represent the cell density and the chemical signal concentration, respectively. The parameter D denotes cell diffusion rate ($D > 0$), ε describes chemical diffusion rate and χ stands for chemotactic sensitivity coefficient. If $\chi > 0$(the positive chemotaxis), the chemotaxis means to be attractive, while if $\chi < 0$(the negative chemotaxis), the chemotaxis is repulsive. The constant $\mu > 0$ stands for the natural degradation rate of the chemical signal. The function $\ln c$ denotes logarithmic chemotactic sensitivity function, which describes the signal detection mechanism of the cellular population. Such a kind of sensitivity function can be found in works [11–13]. The nonlinear function $g(n)$ denotes the chemical production rate, which satisfies $g'(n) \geq 0$ when $n \geq 0$.

When $g(p) = p$, the mathematical analysis about global well-posedness, long-time behavior, diffusion limit, boundary layer, stability of traveling wave, etc. of (1.1) with subject to various initial and/or boundary conditions in one and multiple space dimensions has been made in significant progresses in the past few years, please refer [2–7, 10, 14, 16–23, 29, 30, 32–35] and the references therein. On the other hand, when the chemical production rate is a nonlinear function, there are a few results. In [36], the global well-posedness for the Cauchy problem of (1.1) in one dimension space for general initial data under the assumption that $|g''(p)|$ is uniformly bounded was proved. Later, in [37], Zhu, Liu, Martinez and Zhao adopted a new Lyapunov functional and removed this assumption to get the global well-posedness, the long time behavior and the diffusion limit for the Cauchy problem of (1.1) with $g(p) = p^\gamma$ for all

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$\gamma > 1$. However, the well posedness and long-time behavior to the system (1.1) in bounded domain remains open in the literature, considering that time-dependent Dirichlet boundary conditions are dynamic in vivo environment for tumor angiogenesis, which means this type of boundary value problem is more meaningful than the Cauchy problem from the biological point view. The main purpose of this manuscript is to study the well posedness and long-time behavior for the system (1.1) with the following nonhomogeneous boundary conditions: Initial-Dirichlet boundary value

$$\begin{cases}(p,q)(x,0)=(p_0,q_0)(x),\ p_0(x)\geq 0,\ x\in I;\\ p|_{x=0,x=1}=\alpha(t)>0,\ q|_{x=0,x=1}=\beta(t),\ if\ \varepsilon>0;\end{cases}\tag{1.3}$$

Now it is the place to state our main results of this paper. The first result addresses the global well-posedness and long-time behavior of large-amplitude global solutions to (1.1) and (1.3).

Theorem 1.1. *Assume that the initial data satisfy $p_0 > 0$ and $(p_0, q_0) \in H^2(I)$ and are compatible with the boundary conditions. Assume that*

- *there exist constants $\underline{\alpha}, \bar{\alpha}, \bar{\beta}$, such that $0 < \underline{\alpha} = \inf \alpha(t) \leq \sup \alpha(t) = \bar{\alpha} < \infty$ and $\sup |\beta(t)| = \bar{\beta} < \infty$, for all $t \geq 0$,*
- *$(\alpha_t, \beta_t) \in L^1(0,\infty) \cap H^1(0,\infty)$.*

Then for any $\varepsilon > 0$ there exists a unique global-in-time solution (p, q) to (1.1), such that $(p - \alpha(t), q - \beta(t)) \in L^\infty(0,\infty; H^2(0,1)) \cap L^2(0,\infty; H^3(0,1))$ and satisfies

$$\lim_{t\to\infty}(\|p(\cdot,t)-\alpha(t)\|_{H^2}^2+\|q(\cdot,t)-\beta(t)\|_{H^2}^2)=0.$$

Remark 1.1. *Compared with the Cauchy problem in [37], the distinction is that the energy estimate for the derivatives of the solution is inversely proportional to ε. We can't utilize the same method to estimate the first order spatial derivatives due to the influence of the boundary conditions. Compared with the mixed Neumann-Dirichlet boundary value problem in [18], under the Dirichlet type boundary conditions, the L^1 norm of p is not a conserved quantity. Hence the method established in [19] can't be utilized for the Dirichlet boundary conditions. Fortunately, such an issue was previously resolved in [18] for constant Dirichlet boundary problem by developing a new approach through higher order nonlinear cancellation. But such a technique not completely works for the time-dependent Dirichlet boundary conditions problem. The estimation method in the literature [37] is no longer completely applicable. To overcome these trouble, we explore the convexity of the expansion $E(p, \alpha)$ and construct a nonlinear function to obtain the L^1 estimate of p in terms of the entropy expansion.*

Remark 1.2. *If we remove the condition that α has a positive lower bound, we can use the general L^2 method to get a bound that depends on time variable t, that is, we can still get the global existence of the solution to system (1.1), but we cannot get the large time behavior of the solution. For brevity, we omit the details of the proof in this paper.*

The rest of paper is organized as follows: we first give some preliminaries in Section 2. In section 3, we give the proof of Theorem1.1.

Notations: Throughout this paper, we denote $\|\cdot\|_{L^2}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{H^s}$ by the usual norms of Lebesgue measurable spaces L^2, L^∞ and Hilbert's space H^s, respectively. The values of positive constants C may vary line by line according to the context. For two quantities A and B, we write $A \sim B$ if $C^{-1}A \leq B \leq CA$. The notation $A \lesssim B$ means that $A \leq CB$ for a universal constant $C > 0$ independent of time t.

2. PRELIMINARIES

In this section, we shall introduce some Algebraic inequalities which will be frequently used in the subsequent analysis (cf. [8, 9, 37]).

Lemma 2.1. *Let $a \geq -1$ and $\gamma \geq 2$. Then it holds that*
$$(a+1)^\gamma - 1 - \gamma a \geq \frac{\gamma}{2} a^2.$$

Lemma 2.2. *Let $a \geq -1$ and $\gamma \geq 2$. Then it holds that*
$$(a+1)^\gamma - 1 - \gamma a \geq |a|^\gamma.$$

Lemma 2.3. *Let $a \geq -1$ and $\gamma > 1$. Then it holds that*
$$(a+1)^\gamma \geq 1 + \gamma a.$$

Lemma 2.4. *Let $a \geq 0$ and $0 \leq \gamma \leq 1$. Then it holds that*
$$|a^\gamma - 1| \leq |a - 1|.$$

Lemma 2.5. *Let $a \geq -1$ and $1 < \gamma < 2$. Then it holds that*
$$(a+1)^\gamma - 1 - \gamma a \leq a^2.$$

3. Global dynamics when $\gamma > 1$

In this section, we are devoted to studying the dynamic of solutions to the problem (1.1) with Dirichlet Boundary condition. First, using the standard arguments (e.g. see [24–26, 31]), one can show the local existence of solutions to (1.1)-(1.3). We omit the technical details of the routine arguments in order to simplify the presentation. Next we derive some *a priori* uniform-in-t estimates of solutions, which not only extend the local solutions to global ones, but also play important role in investigating the long time behavior of solutions.

Lemma 3.1 (Entropy estimate)**.** *Let the assumptions in Theorem1.1 hold. Then there exists a constant $C > 0$ which is independent on t and ε, such that*
$$E(p(\cdot,t),\alpha(t)) + \|q(\cdot,t) - \beta(t)\|^2 + \gamma\int_0^t\int_0^1 p^{\gamma-2}p_x^2 dxd\tau + \varepsilon\int_0^t \|q_x\|^2 d\tau \leq C, \tag{3.1}$$
where $E(p(\cdot,t),\alpha(t)) = \frac{1}{\gamma-1}\int_0^1 (p^\gamma - \alpha^\gamma - \gamma\alpha^{\gamma-1}(p-\alpha)) \geq 0$ denotes the entropy expansion.

Proof. We divide the proof into three steps.
Step 1. By a direct calculation, we can obtain that
$$\begin{aligned}
&\frac{1}{\gamma-1}(p^\gamma - \alpha^\gamma - \gamma\alpha^{\gamma-1}(p-\alpha))_t \\
&= \frac{\gamma}{\gamma-1}p^{\gamma-1}p_t - \gamma\alpha^{\gamma-1}\alpha_t - \gamma\alpha^{\gamma-2}\alpha_t(p-\alpha) - \gamma\alpha^{\alpha-1}(p-\alpha)_t \\
&= \frac{\gamma}{\gamma-1}(p^{\gamma-1} - \alpha^{\gamma-1})p_t - \gamma\alpha^{\gamma-2}(p-\alpha)\alpha_t.
\end{aligned} \tag{3.2}$$
By employing $(1.1)_1$ and observing α depends only on t, one has
$$\begin{aligned}
\frac{\gamma}{\gamma-1}(p^{\gamma-1} - \alpha^{\gamma-1})p_t &= \frac{\gamma}{\gamma-1}(p^{\gamma-1} - \alpha^{\gamma-1})(p_{xx} + (pq)_x) \\
&= \frac{\gamma}{\gamma-1}[(p^{\gamma-1} - \alpha^{\gamma-1})pq]_x + \frac{\gamma}{\gamma-1}[(p^{\gamma-1} - \alpha^{\gamma-1})p_x]_x \\
&\quad - \gamma p^{\gamma-1}qp_x - \gamma p^{\gamma-2}(p_x)^2.
\end{aligned} \tag{3.3}$$
Substituting (3.3)into (3.2), we deduce that
$$\begin{aligned}
&\frac{1}{\gamma-1}(p^\gamma - \alpha^\gamma - \gamma\alpha^{\gamma-1}(p-\alpha))_t \\
&= \frac{\gamma}{\gamma-1}[(p^{\gamma-1} - \alpha^{\gamma-1})(pq + p_x)]_x - \gamma p^{\gamma-1}qp_x - \gamma p^{\gamma-2}(p_x)^2 - \gamma\alpha^{\gamma-2}(p-\alpha)\alpha_t.
\end{aligned} \tag{3.4}$$

Upon integrating in I, we deduce

$$\begin{aligned}&\frac{1}{\gamma-1}\frac{d}{dt}\int_I(p^\gamma-\alpha^\gamma-\gamma\alpha^{\gamma-1}(p-\alpha))dx+\gamma\int_I p^{\gamma-2}(p_x)^2dx\\&=-\gamma\int_I p^{\gamma-1}qp_xdx-\gamma\alpha^{\gamma-2}\int_I(p-\alpha)\alpha_tdx.\end{aligned}\tag{3.5}$$

On the other hand, we derive from $(1.1)_2$ that

$$\begin{aligned}(q-\beta)_t-(p^\gamma)_x&=\varepsilon(q-\beta)_{xx}+2\varepsilon q(q-\beta)_x-\beta_t\\&=\varepsilon(q-\beta)_{xx}+2\varepsilon(q-\beta)(q-\beta)_x+2\varepsilon\beta(q-\beta)_x-\beta_t.\end{aligned}\tag{3.6}$$

Multiplying (3.6) by $q-\beta$ and integrating the result equation over I yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|q-\beta\|^2+\varepsilon\|q_x\|^2&=\int_I(q-\beta)(p^\gamma)_xdx-\int_I(q-\beta)\beta_tdx\\&=\gamma\int_I p^{\gamma-1}qp_x-\int_I(q-\beta)\beta_tdx.\end{aligned}\tag{3.7}$$

We add (3.7) into (3.5) to arrive at

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{\gamma-1}\left(\int_I p^\gamma-\alpha^\gamma-\gamma\alpha^{\gamma-1}(p-\alpha)dx\right)+\frac{1}{2}\|q-\beta\|^2\right)\\&+\gamma\int_I p^{\gamma-2}(p_x)^2dx+\varepsilon\|q_x\|^2\\&=-\gamma\int_I\alpha^{\gamma-2}(p-\alpha)\alpha_tdx-\int_I\beta_t(q-\beta)dx\\&\le\gamma\alpha^{\gamma-2}|\alpha_t|\int_I|p-\alpha|dx+|\beta_t|\int_I|q-\beta|dx.\end{aligned}\tag{3.8}$$

Step 2. In this step, we obtain a bound for the L^1 norm of p in terms of the $E(p,\alpha)$. We make use of the convexity of the entropy expansion $E(p,\alpha)$ and compare it with a nonlinear function. For this purpose, we set

$$F_\alpha(p)=\frac{1}{\gamma-1}(p^\gamma-\alpha^\gamma-\gamma\alpha^{\gamma-1}(p-\alpha))-p+(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}.$$

It's easy to check that

$$F_\alpha(0)=\alpha^\gamma+(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}>0,$$

$$F'_\alpha(p)=\frac{1}{\gamma-1}(\gamma p^{\gamma-1}-\gamma\alpha^{\gamma-1})-1,$$

$$F''_\alpha(p)=\gamma p^{\gamma-2}\ge 0,$$

$$F'_\alpha((\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}})=0,$$

$$F_\alpha((\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}})=\frac{1}{\gamma-1}[(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{\gamma}{\gamma-1}}-\alpha^\gamma-\gamma\alpha^{\gamma-1}((\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}-\alpha)]\ge 0,$$

which imply that $F_\alpha(p)\ge 0$ for any $p\ge 0$. This lead to

$$0\le p\le\frac{1}{\gamma-1}(p^\gamma-\alpha^\gamma-\gamma\alpha^{\gamma-1}(p-\alpha))+(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}},$$

and therefore,

$$0\le\int_I p(x,t)dx\le E(p,\alpha)+(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}.\tag{3.9}$$

Step 3. Substituting (3.9) into (3.7), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(E(p(\cdot,t),\alpha(t))+\frac{1}{2}\|q-\beta\|^2\right)+\int_I p^{\gamma-2}(p_x)^2dx+\varepsilon\|q_x\|^2\\
&=-\gamma\int_I \alpha^{\gamma-2}\alpha_t(p-\alpha)dx-\int_I\beta_t(q-\beta)dx\\
&\leq \gamma\alpha^{\gamma-2}|\alpha_t|\int_I|p-\alpha|dx+|\beta_t|\int_I|q-\beta|dx\\
&\leq \gamma\alpha^{\gamma-2}|\alpha_t|E(p,\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|q-\beta\|^2. \quad (3.10)
\end{aligned}$$

By using the first assumption of Theorem1.1 and applying the Gronwall's inequality to (3.10), we have

$$\begin{aligned}
&E(p(\cdot,t),\alpha)+\frac{1}{2}\|q-\beta\|^2\\
&\leq \exp\left\{\int_0^t(\gamma\bar{\alpha}^{\gamma-2}|\alpha_\tau|+|\beta_t|)d\tau\right\}\times\left[\int_0^t(\gamma\bar{\alpha}^{\gamma-2}|\alpha_\tau|(\bar{\alpha}^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\bar{\alpha}^{\gamma-1}|\alpha_t|+\frac{|\beta_\tau|}{2})d\tau\right.\\
&\qquad\left.+E(p_0,\alpha_0)+\frac{1}{2}\|q_0-\beta_0\|^2\right]. \quad (3.11)
\end{aligned}$$

By using the second assumption of Theorem1.1, we deduce from (3.11) that

$$E(p(\cdot,t),\alpha(t))+\frac{1}{2}\|q(\cdot,t)-\beta(t)\|^2\leq C,\forall t>0,\forall\varepsilon>0. \quad (3.12)$$

By substituting (3.12) into (3.11), then integrating the resulting inequality with rspect to time, we have

$$\int_0^t\int_I p^{\gamma-2}p_x^2dxd\tau+\varepsilon\int_0^t\|q_x\|^2d\tau\leq C,\forall\ t>0,\forall\varepsilon\geq 0, \quad (3.13)$$

This together with (3.12) completes the entropy estimate and hence the proof of Lemma 3.1. □

Now, we shall show the key estimate in this paper.

3.1. L^2 **estimate.** To carry out further energy estimates, we reformulate (1.1) by letting $\tilde{p}=p-\alpha$, $\tilde{q}=q-\beta$, then $(\tilde{p},\tilde{q})$ satisfies

$$\begin{cases}\tilde{p}_t-(\tilde{p}\tilde{q})_x-\alpha\tilde{q}_x-\beta\tilde{p}_x=\tilde{p}_{xx}-\alpha_t,\\ \tilde{q}_t-(\tilde{p}+\alpha)_x^\gamma=\varepsilon\tilde{q}_{xx}+2\varepsilon\tilde{q}\tilde{q}_x+2\varepsilon\beta\tilde{q}_x-\beta_t,\\ (\tilde{p},\tilde{q})(x,0)=(p_0-\alpha,q_0-\beta)(x),\\ \tilde{p}|_{x=0,x=1}=0,\ \tilde{q}|_{x=0,x=1}=0.\end{cases} \quad (3.14)$$

In terms of the perturbation, (3.1) can be rewritten as

$$E(\tilde{p}(\cdot,t),\alpha(t))+\frac{1}{2}\|\tilde{q}(\cdot,t)\|^2+\gamma\int_0^t\int_I(\tilde{p}+\alpha(t))^{\gamma-2}(\tilde{p}_x)^2dx+\varepsilon\|\tilde{q}_x\|^2d\tau\leq C, \quad (3.15)$$

where

$$E(\tilde{p}(\cdot,t),\alpha(t))=\frac{1}{\gamma-1}\int_I[(\tilde{p}+\alpha)^\gamma-\alpha^\gamma-\gamma\alpha^{\gamma-1}\tilde{p}](x,t)dx$$

denotes the first order "entropy expansion." First, to obtain the basic L^2 estimate of $\tilde{p}$, we note that for the case $\gamma\geq 2$, letting $a=\frac{\tilde{p}}{\alpha}$ in Lemma 2.1, we obtain

$$(\tilde{p}+\alpha)^\gamma-\alpha^\gamma-\gamma\alpha^{\gamma-1}\tilde{p}\geq\frac{\gamma}{2}\alpha^{\gamma-2}\tilde{p}^2\geq\frac{\gamma}{2}\underline{\alpha}^{\gamma-2}\tilde{p}^2, \quad (3.16)$$

which implies

$$E(\tilde{p},\alpha)\geq\frac{\gamma}{2(\gamma-1)}\underline{\alpha}^{\gamma-2}\|\tilde{p}\|^2. \quad (3.17)$$

By letting $a = \frac{\tilde{p}}{\alpha}$ in Lemma 2.2, one has

$$E(\tilde{p}, \alpha) \geq \frac{1}{\gamma - 1} \int_I |\tilde{p}|^\gamma dx = \frac{1}{\gamma - 1} \|\tilde{p}\|_{L^\gamma}^\gamma. \tag{3.18}$$

In light of (3.15)-(3.18), we can deduce that

$$\|\tilde{p}\|_{L^\gamma}^\gamma \leq (\gamma - 1)C, \ \|\tilde{p}\|_{L^2}^2 \leq \frac{2(\gamma - 1)C}{\gamma \underline{\alpha}^{\gamma - 2}}, \tag{3.19}$$

which will be frequently used in the subsequent energy estimates. Next, motivated by [37], we shall investigate a nondegenerate dissipative mechanism of $\tilde{p}$ by performing some delicate energy estimations. Then we have the following Lemma.

Lemma 3.2. *Under the conditions of Theorem 1.1, for any $\gamma > 1$, $\varepsilon > 0$, and $t > 0$, it holds that*

$$\int_0^t \|\tilde{p}_x(\tau)\|_{L^2}^2 d\tau \leq C,$$

where C is a positive constant which is independent of t and ε.

To prove this lemma, we have to divide the proof into four kinds of subcases: $2 \leq \gamma \leq 3$, $3 < \gamma \leq 4$, $\gamma > 4$ and $1 < \gamma \leq 2$, for present technical reasons.

3.1.1. *L^2 estimate when $2 \leq \gamma \leq 3$.*

Proof. Step 1. First of all, let $E(\tilde{p}, \bar{p}) = \frac{1}{\gamma-1} \int_I [(\tilde{p} + \bar{p})^\gamma - \bar{p}^\gamma - \gamma \bar{p}^{\gamma-1}\tilde{p}](x,t)dx$, then (3.10) can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\left(E(\tilde{p}, \alpha) + \frac{1}{2}\|\tilde{q}\|_{L^2}^2\right) + \gamma \int_I [(\tilde{p} + \alpha)^{\gamma-2} - \alpha^{\gamma-2}](\tilde{p}_x)^2 dx \\
&\quad + \gamma\alpha^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 + \varepsilon\|\tilde{q}_x\|_{L^2}^2 \\
&\leq \gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde{p}, \alpha) + \gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1} + \frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}} + \gamma\alpha^{\gamma-1}|\alpha_t| + \frac{|\beta_t|}{2} + \frac{|\beta_t|}{2}\|\tilde{q}\|^2.
\end{aligned} \tag{3.20}$$

Since $2 \leq \gamma \leq 3$, choosing $a = (\tilde{p} + \alpha)/\alpha$, by employing Lemma 2.3 and Young's inequality, we can deduce that

$$|(\tilde{p} + \alpha)^{\gamma-2} - \alpha^{\gamma-2}| \leq \alpha^{\gamma-3}|\tilde{p}| \leq \frac{\alpha^{\gamma-4}\tilde{p}^2}{2} + \frac{\alpha^{\gamma-2}}{2},$$

which implies

$$(\tilde{p} + \alpha)^{\gamma-2} - \alpha^{\gamma-2} \geq -\frac{\alpha^{\gamma-4}\tilde{p}^2}{2} - \frac{\alpha^{\gamma-2}}{2} \geq -\frac{\underline{\alpha}^{\gamma-4}\tilde{p}^2}{2} - \frac{\alpha^{\gamma-2}}{2}. \tag{3.21}$$

Plugging (3.21) into (3.20) yields

$$\begin{aligned}
&\frac{d}{dt}\left(E(\tilde{p}, \alpha) + \frac{1}{2}\|\tilde{q}\|_{L^2}^2\right) + \frac{\gamma}{2}\underline{\alpha}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2 - \frac{\gamma}{2}\underline{\alpha}^{\gamma-4}\int_I (\tilde{p}\tilde{p}_x)^2 dx + \varepsilon\|\tilde{q}_x\|_{L^2}^2 \\
&\leq \gamma\bar{\alpha}^{\gamma-2}|\alpha_t|E(\tilde{p}, \alpha) + \gamma\bar{\alpha}^{\gamma-2}|\alpha_t|(\bar{\alpha}^{\gamma-1} + \frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}} + \gamma\bar{\alpha}^{\gamma-1}|\alpha_t| + \frac{|\beta_t|}{2} + \frac{|\beta_t|}{2}\|\tilde{q}\|^2.
\end{aligned} \tag{3.22}$$

Step 2. To control the term $-\frac{\alpha^{\gamma-4}}{2}\int_I(\tilde{p}\tilde{p}_x)^2dx$, we multiplying the first equation of (3.14) by $4\tilde{p}^3$, then integrating by parts and using the Hölder's inequality to derive

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2\\
&=-12\int_I\tilde{p}^2\tilde{p}_x(\tilde{p}+\alpha)qdx-4\alpha_t\int_I\tilde{p}^3dx\\
&\le 12\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left(\int_I(\tilde{p}+\alpha)^{4-\gamma}\tilde{p}^4q^2dx\right)^{\frac12}+4|\alpha_t|\int_I|\tilde{p}|^3dx\\
&=12\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left(\int_I(\tilde{p}^3+\alpha\tilde{p}^2)^{4-\gamma}|\tilde{p}|^{2\gamma-4}q^2dx\right)^{\frac12}+4|\alpha_t|\int_I|\tilde{p}|^3dx\\
&\le 12\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\|\tilde{p}^3+\alpha\tilde{p}^2\|_{L^\infty}^{\frac{4-\gamma}{2}}\|\tilde{p}\|_{L^\infty}^{\gamma-2}\|q\|_{L^2}+4|\alpha_t|\int_I|\tilde{p}|^3dx\\
&\le C\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left(\|\tilde{p}^3\|_{L^\infty}^{\frac{4-\gamma}{2}}+\|\tilde{p}^2\|_{L^\infty}^{\frac{4-\gamma}{2}}\right)\|\tilde{p}\|_{L^\infty}^{\gamma-2}+C|\alpha_t|.
\end{aligned}\tag{3.23}$$

We have to estimate the L^∞ norms on the right-hand side of (3.23). Using Hölder's inequality and (3.19), we obtain

$$\begin{aligned}
|\tilde{p}^3(x,t)|&=\left|3\int_0^x\tilde{p}^2\tilde{p}_xdx\right|\le 3\int_I|\tilde{p}|^2|\tilde{p}_x|dx\\
&\le 3\left(\int_I|\tilde{p}|^\gamma dx\right)^{\frac12}\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac12}\\
&\le C\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac12},
\end{aligned}$$

which implies

$$\|\tilde{p}^3\|_{L^\infty}^{\frac{4-\gamma}{2}}\le C\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{4}}.\tag{3.24}$$

Substituting (3.24) into (3.23), and by virtue of Sobolev's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2\\
&\le C\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left[\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{4}}+\|\tilde{p}_x\|_{L^2}^{\frac{4-\gamma}{2}}\right]\|\tilde{p}_x\|_{L^2}^{\frac{\gamma-2}{2}}+C|\alpha_t|\\
&\le C(\delta)\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)+C|\alpha_t|\\
&\quad+\delta\left[\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-2}+\|\tilde{p}_x\|_{L^2}^2\right],
\end{aligned}\tag{3.25}$$

where $\delta>0$ is a constant to be determined. Noting that when $2\le\gamma\le3$, by employing Young's inequality, we derive that

$$\left(\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-2}\le\int_I|\tilde{p}|^{4-\gamma}(\tilde{p}_x)^2dx+\|\tilde{p}_x\|_{L^2}^2.\tag{3.26}$$

We employ Young's inequality and (3.26) to derive

$$\left(\int_I |\tilde p|^{4-\gamma}(\tilde p_x)^2dx\right)^{\frac{4-\gamma}{2}}\|\tilde p_x\|_{L^2}^{\gamma-2}\le \|\tilde p\tilde p_x\|_{L^2}^2+2\|\tilde p_x\|_{L^2}^2. \tag{3.27}$$

Substituting (3.27) into (3.25), we can infer that

$$\frac{d}{dt}\|\tilde p\|_{L^4}^4+12\|\tilde p\tilde p_x\|_{L^2}^2\le C\int_I(\tilde p+\alpha)^{\gamma-2}(\tilde p_x)^2dx+\delta(\|\tilde p\tilde p_x\|^2+3\|\tilde p_x\|_{L^2}^2)+C|\alpha_t|. \tag{3.28}$$

Let

$$M_1=\frac{4+\gamma\underline{\alpha}^{\gamma-4}}{24}.$$

Multiplying (3.28) by M_1, then inserting the result to (3.22), we deduce that

$$\begin{aligned}&\frac{d}{dt}\left(E(\tilde p,\alpha)+\frac12\|\tilde q\|_{L^2}^2+M_1\|\tilde p\|_{L^4}^4\right)+\frac{\gamma}{2}\underline{\alpha}^{\gamma-2}\|\tilde p_x\|_{L^2}^2+2\|\tilde p\tilde p_x\|_{L^2}^2+\varepsilon\|\tilde q_x\|_{L^2}^2\\&\le\gamma\bar\alpha^{\gamma-2}|\alpha_t|E(\tilde p,\alpha)+\gamma\bar\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\bar\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde q\|^2\\&\quad+M_1C(\delta)\int_I(\tilde p+\alpha)^{\gamma-2}(\tilde p_x)^2dx+\delta M_1(\|\tilde p\tilde p_x\|_{L^2}^2+3\|\tilde p_x\|_{L^2}^2)+CM_1|\alpha_t|.\end{aligned} \tag{3.29}$$

Letting

$$\delta=\frac{1}{M_1}\min\{1,\frac{\gamma}{12}\underline{\alpha}^{\gamma-2}\},$$

then we have

$$\begin{aligned}&\frac{d}{dt}\left(E(\tilde p,\alpha)+\frac12\|\tilde q\|_{L^2}^2+M_1\|\tilde p\|_{L^4}^4\right)+\frac{\gamma}{4}\underline{\alpha}^{\gamma-2}\|\tilde p_x\|_{L^2}^2+\|\tilde p\tilde p_x\|_{L^2}^2+\varepsilon\|\tilde q_x\|_{L^2}^2\\&\le\gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde p,\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\bar\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde q\|^2\\&\quad+C\int_I(\tilde p+\alpha)^{\gamma-2}(\tilde p_x)^2dx+CM_1|\alpha_t|.\end{aligned} \tag{3.30}$$

We integrate (3.30) over $[0,t]$ and employ (3.2) to deduce

$$\begin{aligned}&E(\tilde p,\alpha)+\frac12\|\tilde q\|_{L^2}^2+M_1\|\tilde p\|_{L^4}^4+\int_0^t\left(\frac{\gamma}{4}\underline{\alpha}^{\gamma-2}\|\tilde p_x\|_{L^2}^2+\|\tilde p\tilde p_x\|_{L^2}^2+\varepsilon\|\tilde q_x\|_{L^2}^2\right)d\tau\\&\le\exp\left\{\int_0^t\gamma\bar\alpha^{\gamma-2}|\alpha_\tau|+|\beta_\tau|d\tau\right\}\times\left[\int_0^t(\gamma\bar\alpha^{\gamma-2}|\alpha_\tau|(\bar\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+(\gamma\bar\alpha^{\gamma-1}+CM_1)|\alpha_t|\right.\\&\left.+\frac12|\beta_\tau|+C\int_I(\tilde p+\alpha)^{\gamma-2}(\tilde p_x)^2)d\tau+E(p_0,\alpha(0))+\frac12\|q_0-\beta_0\|^2+M_1\int_I\tilde p_0^4dx\right].\end{aligned} \tag{3.31}$$

□

3.1.2. L^2 *Estimate when* $3\le\gamma\le4$.

Proof. Step 1. Since $3\le\gamma\le4$, from Lemma 2.3, by choosing $a=\frac{\tilde p}{\alpha}$, we have

$$(\tilde p+\alpha)^{\gamma-2}\ge\alpha^{\gamma-2}+(\gamma-2)\alpha^{\gamma-3}\tilde p. \tag{3.32}$$

Inserting (3.32) into (3.22), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(E(\tilde p,\alpha)+\frac12\|\tilde q\|_{L^2}^2\right)+\gamma\alpha^{\gamma-2}\|\tilde p_x\|_{L^2}^2+\gamma(\gamma-2)\alpha^{\gamma-3}\int_I\tilde p(\tilde p_x)^2dx+\varepsilon\|\tilde q_x\|_{L^2}^2\\&\le\gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde p,\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde q\|^2.\end{aligned} \tag{3.33}$$

Noting that for any constant $\eta > 0$, it holds that

$$\tilde{p} \geq -|\tilde{p}| \geq -\frac{\tilde{p}^2}{2(\eta)^2} - \frac{(\eta)^2}{2}. \tag{3.34}$$

Plugging (3.34) into (3.33), one has

$$\begin{aligned}&\frac{d}{dt}\left(E(\tilde{p},\alpha)+\frac{1}{2}\|\tilde{q}\|_{L^2}^2\right)+\gamma\alpha^{\gamma-3}\left(\alpha-\frac{(\gamma-2)(\eta)^2}{2}\right)\|\tilde{p}_x\|_{L^2}^2-\frac{\gamma(\gamma-2)}{2(\eta)^2}\alpha^{\gamma-3}\int_I(\tilde{p}\tilde{p}_x)^2dx+\varepsilon\|\tilde{q}_x\|_{L^2}^2\\&\leq\gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde{p},\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde{q}\|^2.\end{aligned} \tag{3.35}$$

By choosing

$$\eta=\left(\frac{\alpha}{\gamma-2}\right)^{\frac{1}{2}},$$

we have

$$\begin{aligned}&\frac{d}{dt}\left(E(\tilde{p},\alpha)+\frac{1}{2}\|q\|_{L^2}^2\right)+\frac{\gamma}{2}\underline{\alpha}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2-\frac{\gamma(\gamma-2)^2}{2}\underline{\alpha}^{\gamma-4}\int_I(\tilde{p}\tilde{p}_x)^2dx+\varepsilon\|\tilde{q}_x\|_{L^2}^2\\&\leq\gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde{p},\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde{q}\|^2.\end{aligned} \tag{3.36}$$

Step 2. Multiplying the first equation of (3.10) by $4\tilde{p}^3$, integrating by parts, and employing the Hölder's inequality, we deduce that

$$\begin{aligned}&\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2\\&=-12\int_I\tilde{p}^2\tilde{p}_x(\tilde{p}+\alpha)qdx-4\alpha_t\int_I\tilde{p}^3dx\\&\leq12\left(\int_I(\tilde{p}+\bar{p})^{\gamma-2}(\tilde{p}_x)^2\right)^{\frac{1}{2}}\left(\int_I(\tilde{p}+\alpha)^{4-\gamma}\tilde{p}^4q^2dx\right)^{\frac{1}{2}}-4\alpha_t\int_I\tilde{p}^3dx\\&=12\left((\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2\right)^{\frac{1}{2}}\left(\int_I(\tilde{p}^5+\alpha\tilde{p}^4)^{4-\gamma}|\tilde{p}|^{4\gamma-12}q^2dx\right)^{\frac{1}{2}}-4\alpha_t\int_I\tilde{p}^3dx\\&\leq12\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac{1}{2}}\|\tilde{p}^5+\alpha\tilde{p}^4\|_{L^\infty}^{\frac{4-\gamma}{2}}\|\tilde{p}\|_{L^\infty}^{2\gamma-6}\|q\|_{L^2}+C|\alpha_t|.\end{aligned} \tag{3.37}$$

It suffices to bound the L^∞ norm of $\tilde{p}^5+\alpha\tilde{p}^4$ on the right-hand side of (3.37), for this purpose, we observe that

$$\begin{aligned}\tilde{p}^5(x,t)+\alpha\tilde{p}^4(x,t)&=5\int_0^x\tilde{p}^4\tilde{p}_xdx+4\alpha\int_0^x\tilde{p}^3\tilde{p}_xdx\\&\leq5\int_0^x|\tilde{p}|^3(\tilde{p}+\alpha)|\tilde{p}_x|dx+\alpha\int_0^x|\tilde{p}|^3|\tilde{p}_x|dx\\&\leq5\left(\int_I\tilde{p}^6(\tilde{p}+\alpha)dx\right)^{\frac{1}{2}}\left(\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx\right)^{\frac{1}{2}}+\alpha\int_I|\tilde{p}|^3|\tilde{p}_x|dx.\end{aligned} \tag{3.38}$$

Noting that

$$5\left(\int_I\tilde{p}^6(\tilde{p}+\alpha)dx\right)^{\frac{1}{2}}\leq5\|\tilde{p}^5+\alpha\tilde{p}^4\|_{L^\infty}^{\frac{1}{2}}\|\tilde{p}\|_{L^2},$$

we have

$$\|\tilde{p}^5+\alpha\tilde{p}^4\|_{L^\infty} \le C\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx+2\alpha\int_I|\tilde{p}|^3|\tilde{p}_x|dx$$
$$. \le C\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx+\bar{\alpha}\|\tilde{p}_x\|_{L^2}^2. \tag{3.39}$$

Substituting (3.39) into (3.37), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2\\
&\le 12\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left(C\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx+\|\tilde{p}_x\|_{L^2}^2\right)^{\frac{4-\gamma}{2}}\|\tilde{p}\|_{L^\infty}^{2\gamma-6}\|q\|_{L^2}+C|\alpha_t|\\
&\le C\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left(\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx+\|\tilde{p}_x\|_{L^2}^2\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-3}+C|\alpha_t|\\
&\le C\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right)^{\frac12}\left[\left(\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx\right)^{\frac{4-\gamma}{2}}\|\tilde{p}_x\|_{L^2}^{\gamma-3}+\|\tilde{p}_x\|_{L^2}\right]+C|\alpha_t|.
\end{aligned}\tag{3.40}$$

We employ the Young's inequality to the right-hand side of (3.40) to obtain

$$\begin{aligned}
\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2 &\le C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx+C|\alpha_t|\\
&\quad+\delta\left[\left(\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2\right)^{4-\gamma}\|\tilde{p}_x\|_{L^2}^{2(\gamma-3)}+\|\tilde{p}_x\|_{L^2}^2\right].
\end{aligned}\tag{3.41}$$

When $3\le\gamma\le4$, by virtue of Young's inequality, we can get

$$\begin{aligned}
\left(\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx\right)^{4-\gamma}\|\tilde{p}_x\|_{L^2}^{2(\gamma-3)} &\le (4-\gamma)\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx+(\gamma-3)\|\tilde{p}_x\|_{L^2}^2\\
&\le \int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx+\|\tilde{p}_x\|_{L^2}^2.
\end{aligned}\tag{3.42}$$

We substitute (3.42) into (3.41), and use the elementary inequality $2|\tilde{p}|\le\tilde{p}^2+1$ to derive

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{p}\|_{L^4}^4+12\|\tilde{p}\tilde{p}_x\|_{L^2}^2\\
&\le C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx+\delta\left(\int_I(\tilde{p}+\alpha)(\tilde{p}_x)^2dx+2\|\tilde{p}_x\|_{L^2}^2\right)+C|\alpha_t|\\
&\le C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx+\delta(\|\tilde{p}\tilde{p}_x\|_{L^2}^2+(\bar{\alpha}+3)\|\tilde{p}_x\|_{L^2}^2)+C|\alpha_t|.
\end{aligned}\tag{3.43}$$

Step 3. Let

$$M_2=\frac{4+\gamma(\gamma-2)^2\underline{\alpha}^{\gamma-4}}{24}.$$

Multiplying (3.43) by M_2, then adding the result to (3.36), we have

$$\begin{aligned}
&\frac{d}{dt}\left(E(\tilde{p},\alpha)+\frac12\|\tilde{q}\|_{L^2}^2+M_2\|\tilde{p}\|_{L^4}^4\right)+\frac{\gamma}{2}\alpha^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+2\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\varepsilon\|\tilde{q}_x\|_{L^2}^2\\
&\le M_2C(\delta)\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx+\delta M_2(\|\tilde{p}\tilde{p}_x\|_{L^2}^2+(\bar{\alpha}+3)\|\tilde{p}_x\|_{L^2}^2)+CM_2|\alpha_t|.
\end{aligned}\tag{3.44}$$

Choosing

$$\delta=\frac{1}{M_2}\min\{1,\frac{\gamma\underline{\alpha}^{\gamma-2}}{4(\bar{\alpha}+3)}\},$$

we obtain from (3.44)

$$\frac{d}{dt}\left(E(\tilde{p},\alpha)+\frac{1}{2}\|\tilde{q}\|_{L^2}^2+M_2\|\tilde{p}\|_{L^4}^4\right)+\frac{\gamma}{4}\underline{\alpha}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\varepsilon\|\tilde{q}_x\|_{L^2}^2$$
$$\leq \gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde{p},\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+(\gamma\bar{\alpha}^{\gamma-1}+CM_2)|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde{q}\|^2$$
$$+C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx. \tag{3.45}$$

Integrating (3.45) over $[0,t]$ and using (3.2), we have

$$E(\tilde{p},\alpha)+\frac{1}{2}\|\tilde{q}\|_{L^2}^2+M_2\|\tilde{p}\|_{L^4}^4+\int_0^t(\frac{\gamma}{4}\underline{\alpha}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\|\tilde{p}\tilde{p}_x\|_{L^2}^2+\varepsilon\|\tilde{q}_x\|_{L^2}^2)d\tau$$
$$\leq \exp\left\{\int_0^t\gamma\bar{\alpha}^{\gamma-2}|\alpha_\tau|+|\beta_\tau|d\tau\right\}\times[\int_0^t(\gamma\bar{\alpha}^{\gamma-2}|\alpha_\tau|(\bar{\alpha}^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+(\gamma\bar{\alpha}^{\gamma-1}+CM_2)|\alpha_t|$$
$$+\frac{1}{2}|\beta_\tau|+C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2)d\tau+E(p_0,\alpha(0))+\frac{1}{2}\|q_0-\beta_0\|^2+M_1\int_I\tilde{p}_0^4dx]. \tag{3.46}$$

□

3.1.3. L^2 *estimate when* $\gamma>4$.

Proof. Step 1. Since $\gamma>4$, as an application of Lemma 2.3, by letting $a=\frac{\tilde{p}}{\alpha}$, we have

$$(\tilde{p}+\alpha)^{\gamma-2}\geq\alpha^{\gamma-2}+(\gamma-2)\alpha^{\gamma-3}\tilde{p}. \tag{3.47}$$

Substituting (3.47) into (3.6), we have

$$\frac{d}{dt}\left(E(\tilde{p},\alpha)+\frac{1}{2}\|\tilde{q}\|_{L^2}^2\right)+\gamma\alpha^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\gamma(\gamma-2)\alpha^{\gamma-3}\int_I\tilde{p}(\tilde{p}_x)^2dx+\varepsilon\|\tilde{q}_x\|_{L^2}^2$$
$$\leq\gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde{p},\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde{q}\|^2. \tag{3.48}$$

Since $\gamma>4$, by applying the Young's inequality, we can show that

$$|\tilde{p}|=\eta_1\cdot\frac{|\tilde{p}|}{\eta_1}\leq\left(\frac{\gamma-2}{\gamma-1}\right)\cdot\eta_1^{\frac{\gamma-1}{\gamma-2}}+\left(\frac{1}{\gamma-1}\right)\cdot\frac{|\tilde{p}|^{\gamma-1}}{\eta_1^{\gamma-1}}\leq\eta_1^{\frac{\gamma-1}{\gamma-2}}+\frac{|\tilde{p}|^{\gamma-1}}{\eta_1^{\gamma-1}},$$

which implies

$$\tilde{p}\geq-|\tilde{p}|\geq-\eta_1^{\frac{\gamma-1}{\gamma-2}}-\frac{|\tilde{p}|^{\gamma-1}}{\eta_1^{\gamma-1}}. \tag{3.49}$$

Substituting (3.49)into (3.48), we have

$$\frac{d}{dt}\left(E(\tilde{p},\alpha)+\frac{1}{2}\|\tilde{q}\|_{L^2}^2\right)+\gamma\alpha^{\gamma-3}[\alpha-(\gamma-2)\eta_1^{\frac{\gamma-1}{\gamma-2}}]\|\tilde{p}_x\|_{L^2}^2-\frac{\gamma(\gamma-2)}{\eta_1^{\gamma-1}}\alpha^{\gamma-3}\int_I|\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx$$
$$+\varepsilon\|\tilde{q}_x\|_{L^2}^2$$
$$\leq\gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde{p},\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde{q}\|^2. \tag{3.50}$$

Next, by choosing

$$\eta_1=\left(\frac{\alpha}{2(\gamma-2)}\right)^{\frac{\gamma-2}{\gamma-1}},$$

we have

$$\frac{d}{dt}\left(E(\tilde p,\alpha)+\frac12\|\tilde q\|_{L^2}^2\right)+\frac{\gamma}{2}\underline{\alpha}^{\gamma-2}\|\tilde p_x\|_{L^2}^2-\frac{2^{\gamma-2}\gamma(\gamma-2)^{\gamma-1}}{\underline{\alpha}}\int_I|\tilde p|^{\gamma-1}(\tilde p_x)^2dx+\varepsilon\|\tilde q_x\|_{L^2}^2$$
$$\le\gamma\alpha^{\gamma-2}|\alpha_t|E(\tilde p,\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde q\|^2. \tag{3.51}$$

Step 2. By multiplying the first equation of (3.10) by $(\gamma+1)|\tilde p|^{\gamma-1}\tilde p$ and integrating by parts with respect to x over I, we obtain

$$\frac{d}{dt}\left(\int_I|\tilde p|^{\gamma+1}dx\right)+\gamma(\gamma+1)\int_I|\tilde p|^{\gamma-1}(\tilde p_x)^2dx$$
$$=-\gamma(\gamma+1)\int_I(\tilde p+\alpha)\tilde q|\tilde p|^{\gamma-1}\tilde p_xdx-(\gamma+1)\alpha_t\int_I|\tilde p|^{\gamma-1}\tilde pdx. \tag{3.52}$$

Since $\gamma>4$ and $\tilde p+\alpha>0$, by using the Holder's inequality, we estimate the right-hand side of (3.52) as

$$\left|\int_I(\tilde p+\alpha)\tilde q|\tilde p|^{\gamma-1}\tilde p_xdx\right|\le\left(\int_I(\tilde p+\alpha)^2|\tilde p_x|^{\frac{4}{\gamma-2}}|\tilde p_x|^{2\frac{\gamma-4}{\gamma-2}}dx\right)^{\frac12}\|\tilde q\|_{L^2}\|\tilde p^{\gamma-1}\|_{L^\infty}$$
$$\le C\left(\int_I(\tilde p+\alpha)^{\gamma-2}|\tilde p_x|^2dx\right)^{\frac{1}{\gamma-2}}\|\tilde p\|_{L^\infty}^{\gamma-1}\|\tilde p_x\|_{L^2}^{\frac{\gamma-4}{\gamma-2}}. \tag{3.53}$$

To control the L^∞ norm of $\tilde p$ on the right-hand side of (3.53), we note that

$$\begin{aligned}\tilde p(x,t)^{\gamma-1}&=\int_0^x\left((\tilde p^2)^{\frac{\gamma-1}{2}}\right)_xdx\\&=(\gamma-1)\int_0^x|\tilde p|^{\gamma-3}\tilde p\tilde p_xdx\\&\le(\gamma-1)\int_I|\tilde p|^{\gamma-2}|\tilde p_x|dx\\&=(\gamma-1)\int_I|\tilde p|^{\frac{\gamma}{2}}|\tilde p|^{\frac{\gamma}{2}-2}|\tilde p_x|dx\\&\le(\gamma-1)\left(\int_I|\tilde p|^\gamma dx\right)^{\frac12}\left(\int_I|\tilde p|^{\gamma-4}|\tilde p_x|^2dx\right)^{\frac12},\end{aligned}$$

which implies

$$\begin{aligned}\|\tilde p^{\gamma-1}\|_{L^\infty}&\le(\gamma-1)\left(\int_I|\tilde p|^\gamma dx\right)^{\frac12}\left(\int_I|\tilde p|^{\gamma-4}|\tilde p_x|^2dx\right)^{\frac12}\\&\le C\left(\int_I|\tilde p|^{\gamma-4}|\tilde p_x|^2dx\right)^{\frac12}\\&\le C\left(\int_I|\tilde p|^{\gamma-1}(\tilde p_x)^2dx+\|\tilde p_x\|_{L^2}^2\right)^{\frac12}.\end{aligned} \tag{3.54}$$

Substituting (3.54) into (3.53), we have

$$\begin{aligned}\left|\int_I (\tilde p+\alpha)\tilde q|\tilde p|^{\gamma-1}\tilde p_x dx\right| &\le C\left(\int_I(\tilde p+\alpha)^{\gamma-2}|\tilde p_x|^2dx\right)^{\frac{1}{\gamma-2}}\left(\int_I|\tilde p|^{\gamma-4}|\tilde p_x|^2dx\right)^{\frac12}\|\tilde p_x\|_{L^2}^{\frac{\gamma-4}{\gamma-2}}\\ &\le C(\delta)\left(\int_I(\tilde p+\alpha)^{\gamma-2}|\tilde p_x|^2dx\right)+\delta\left(\int_I|\tilde p|^{\gamma-4}|\tilde p_x|^2dx\right)^{\frac{\gamma-2}{2(\gamma-3)}}\|\tilde p_x\|_{L^2}^{\frac{\gamma-4}{\gamma-3}}\\ &\le C(\delta)\int_I(\tilde p+\alpha)^{\gamma-2}|\tilde p_x|^2dx+\delta\left(\int_I|\tilde p|^{\gamma-4}|\tilde p_x|^2dx+\|\tilde p_x\|_{L^2}^2\right)\\ &\le C(\delta)\int_I(\tilde p+\alpha)^{\gamma-2}|\tilde p_x|^2dx+\delta\left(\int_I|\tilde p|^{\gamma-1}|\tilde p_x|^2dx+2\|\tilde p_x\|_{L^2}^2\right),\qquad(3.55)\end{aligned}$$

where we have used the Young's inequality $|\tilde p|^{\gamma-4}\le|\tilde p|^{\gamma-1}+1$ due to $\gamma>4$. For the second term of right hand side of (3.52), using (3.18), we have

$$-(\gamma+1)\alpha_t\int_I|\tilde p|^{\gamma-1}\tilde p dx\le(\gamma+1)\|\tilde p\|_{L^\gamma}^{\gamma}|\alpha_t|\le C|\alpha_t|.\qquad(3.56)$$

Substituting (3.55) and (3.56) into (3.52), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\int_I|\tilde p|^{\gamma+1}dx\right)+\gamma(\gamma+1)\int_I|\tilde p|^{\gamma-1}(\tilde p_x)^2dx\\ &\le C(\delta)\int_I(\tilde p+\alpha)^{\gamma-2}|\tilde p_x|^2dx+\delta\left(|\tilde p|^{\gamma-1}|\tilde p_x|^2dx+2\|\tilde p_x\|_{L^2}^2\right)+C|\alpha_t|.\qquad(3.57)\end{aligned}$$

Step 3. It suffices to bound the last term on the right-hand side. To this end, let

$$M_3=\frac{2\underline{\alpha}+2^{\gamma-2}\gamma(\gamma-2)^{\gamma-1}}{\gamma(\gamma+1)\underline{\alpha}}.$$

Multiplying (3.57) by M_3, then adding the result to (3.51), we get

$$\begin{aligned}&\frac{d}{dt}\left(E(\tilde p,\alpha)+\frac12\|\tilde q\|_{L^2}^2+M_3\int_I|\tilde p|^{\gamma+1}dx\right)+\frac{\gamma}{2}\underline{\alpha}^{\gamma-2}\|\tilde p_x\|_{L^2}^2+2\int_I|\tilde p|^{\gamma-1}(\tilde p_x)^2dx+\varepsilon\|q_x\|_{L^2}^2\\ &\le\gamma\alpha^{\gamma-2}|\alpha_t|E(p,\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde q\|^2\\ &\quad+C(\delta)\gamma(\gamma+1)M_3\int_I(\tilde p+\alpha)^{\gamma-2}|\tilde p_x|^2dx+CM_3|\alpha_t|\\ &\quad+\delta\gamma(\gamma+1)M_3\left(\int_I|\tilde p|^{\gamma-1}(\tilde p_x)^2dx+2\|\tilde p_x\|_{L^2}^2\right).\qquad(3.58)\end{aligned}$$

Choosing

$$\delta=\frac{1}{\gamma(\gamma+1)M_3}\min\{\frac{\gamma}{8}\underline{\alpha}^{\gamma-2},1\},$$

we obtain from (3.58)

$$\begin{aligned}&\frac{d}{dt}\left(E(\tilde p,\alpha)+\frac12\|\tilde q\|_{L^2}^2+M_3\int_I|\tilde p|^{\gamma+1}dx\right)+\frac{\gamma}{4}\underline{\alpha}^{\gamma-2}\|\tilde p_x\|_{L^2}^2+\int_I|\tilde p|^{\gamma-1}(\tilde p_x)^2dx+\varepsilon\|\tilde q_x\|_{L^2}^2\\ &\le\gamma\alpha^{\gamma-2}|\alpha_t|E(p,\alpha)+\gamma\alpha^{\gamma-2}|\alpha_t|(\alpha^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+\gamma\alpha^{\gamma-1}|\alpha_t|+\frac{|\beta_t|}{2}+\frac{|\beta_t|}{2}\|\tilde q\|^2\\ &\quad+C\int_I(\tilde p+\alpha)^{\gamma-2}|\tilde p_x|^2dx+CM_3|\alpha_t|.\qquad(3.59)\end{aligned}$$

Integrating (3.59) over $[0,t]$ and using (3.2), we end up with

$$\begin{aligned}
&E(\tilde{p},\alpha)+\frac{1}{2}\|\tilde{q}\|_{L^2}^2+M_3\int_I|\tilde{p}|^{\gamma+1}dx+\int_0^t\left(\frac{\gamma}{4}\underline{\alpha}^{\gamma-2}\|\tilde{p}_x\|_{L^2}^2+\int_I|\tilde{p}|^{\gamma-1}(\tilde{p}_x)^2dx+\varepsilon\|\tilde{q}_x\|_{L^2}^2\right)d\tau\\
&\leq \exp\left\{\int_0^t\gamma\bar{\alpha}^{\gamma-2}|\alpha_\tau|+|\beta_\tau|d\tau\right\}\times\left[\int_0^t(\gamma\bar{\alpha}^{\gamma-2}|\alpha_\tau|(\bar{\alpha}^{\gamma-1}+\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}+(\gamma\bar{\alpha}^{\gamma-1}+CM_3)|\alpha_t|\right.\\
&\quad\left.+\frac{1}{2}|\beta_\tau|+C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2)d\tau+E(p_0,\alpha(0))+\frac{1}{2}\|q_0-\beta_0\|^2+M_3\int_I\tilde{p}_0^{\gamma+1}dx\right].
\end{aligned}\tag{3.60}$$

From (3.31), (3.46) and (3.60), we conclude that for any $\gamma\geq 2$, it holds that

$$\int_0^t\|\tilde{p}_x\|^2d\tau\leq C.\tag{3.61}$$

□

3.1.4. L^2 *estimate when* $1<\gamma<2$.

Proof. **Step 1.** Multiplying the first equation (3.10) by $(\tilde{p}+\alpha)^\gamma$ and the second equation by $\alpha\tilde{q}$, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{\gamma+1}\int_I\hat{E}(\tilde{p},\alpha)dx+\frac{\alpha}{2}\|\tilde{q}\|^2\right)+\gamma\int_I(\tilde{p}+\alpha)^{\gamma-1}(\tilde{p}_x)^2dx+\varepsilon\alpha\|\tilde{q}_x\|^2\\
&=-\gamma\int_I(\tilde{p}+\alpha)^{\gamma-1}\tilde{p}\tilde{q}\tilde{p}_xdx-\gamma\alpha^{\gamma-1}\alpha_t\int_I\tilde{p}dx-\alpha\beta_t\int_I\tilde{q}dx+\frac{\alpha_t}{2}\|\tilde{q}\|^2,
\end{aligned}\tag{3.62}$$

where $\hat{E}(\tilde{p},\alpha)=(\tilde{p}+\alpha)^{\gamma+1}-\alpha^{\gamma+1}-(\gamma+1)\alpha^\gamma\tilde{p}$. Since $1<\gamma<2$, by choosing $a=\frac{\tilde{p}}{\alpha}$ in Lemma 2.1, we obtain

$$\hat{E}(\tilde{p},\alpha)=(\tilde{p}+\alpha)^{\gamma+1}-\alpha^{\gamma+1}-(\gamma+1)\alpha^\gamma\tilde{p}\geq\frac{\gamma+1}{2}\alpha^{\gamma-1}|\tilde{p}|^2,\tag{3.63}$$

which implies

$$\frac{1}{\gamma+1}\int_I\hat{E}(\tilde{p},\alpha)dx\geq\frac{\alpha^{\gamma-1}}{2}\|\tilde{p}\|^2\geq\frac{\underline{\alpha}^{\gamma-1}}{2}\|\tilde{p}\|^2.\tag{3.64}$$

Step 2. Multiplying (3.10) by $-\frac{\gamma}{2}\alpha^{\gamma-2}\tilde{p}^2$ and integrating by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\left(-\frac{\gamma}{6}\alpha^{\gamma-2}\int_I\tilde{p}^3dx\right)-\gamma\alpha^{\gamma-2}\int_I\tilde{p}(\tilde{p}_x)^2dx\\
&=\gamma\alpha^{\gamma-2}\int_I\tilde{p}^2\tilde{q}\tilde{p}_xdx+\gamma\alpha^{\gamma-1}\int_I\tilde{p}\tilde{q}\tilde{p}_xdx+\frac{\gamma}{2}\alpha^{\gamma-2}\alpha_t\int_I\tilde{p}^2dx-\frac{\gamma(\gamma-2)}{6}\alpha^{\gamma-3}\alpha_t\int_I\tilde{p}^3dx.
\end{aligned}\tag{3.65}$$

Multiplying $(3.10)_1$ by $\frac{\gamma}{3}\alpha^{\gamma-3}\tilde{p}^3$ and integrating by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\gamma}{12}\alpha^{\gamma-3}\int_I\tilde{p}^4dx\right)+\gamma\alpha^{\gamma-3}\int_I(\tilde{p}\tilde{p}_x)^2dx\\
&=-\gamma\alpha^{\gamma-3}\int_I\tilde{p}^3\tilde{q}\tilde{p}_xdx-\gamma\alpha^{\gamma-2}\int_I\tilde{p}^2\tilde{q}\tilde{p}_xdx-\frac{\gamma}{3}\alpha^{\gamma-3}\alpha_t\int_I\tilde{p}^3dx+\frac{\gamma(\gamma-3)}{12}\alpha^{\gamma-4}\alpha_t\int_I\tilde{p}^4dx.
\end{aligned}\tag{3.66}$$

Combining (3.62),(3.65) and (3.66), we have

$$\begin{aligned}
&\frac{d}{dt}G(t)+H(t)\\
&=-\gamma\int_I[(\tilde p+\alpha)^{\gamma-1}-\alpha^{\gamma-1}]\tilde p\tilde q\tilde p_x dx-\gamma\alpha^{\gamma-3}\int_I\tilde p^3\tilde q\tilde p_x dx-\gamma\alpha^{\gamma-1}\alpha_t\int_I\tilde p dx-\alpha\beta_t\int_I\tilde q dx\\
&\quad+\frac{\alpha_t}{2}\|\tilde q\|^2+\frac{\gamma}{2}\alpha^{\gamma-2}\alpha_t\int_I\tilde p^2dx-\frac{\gamma^2}{6}\alpha^{\gamma-3}\alpha_t\int_I\tilde p^3dx+\frac{\gamma(\gamma-3)}{12}\alpha^{\gamma-4}\alpha_t\int_I\tilde p^4dx\\
&\le-\gamma\int_I[(\tilde p+\alpha)^{\gamma-1}-\alpha^{\gamma-1}]\tilde p q\tilde p_x dx-\gamma\alpha^{\gamma-3}\int_I\tilde p^3 q\tilde p_x dx+(\gamma|\alpha_t|+|\beta_t|+\frac{\gamma+7}{\underline{\alpha}}|\alpha_t|)G(t)\\
&\quad+\frac{\gamma\bar\alpha^{\gamma-1}}{2}|\alpha_t|+\frac{\bar\alpha|\beta_t|}{2},
\end{aligned}\tag{3.67}$$

where

$$G(t)=\frac{1}{\gamma+1}\int_I\hat E(\tilde p,\alpha)dx+\frac{\alpha}{2}\|q\|^2-\frac{\gamma}{6}\alpha^{\gamma-2}\int_I\tilde p^3dx+\frac{\gamma}{12}\alpha^{\gamma-3}\int_I\tilde p^4dx,$$

$$\begin{aligned}
H(t)&=\gamma\int_I(\tilde p+\alpha)^{\gamma-1}(\tilde p_x)^2dx+\varepsilon\alpha\|\tilde q_x\|^2-\gamma\alpha^{\gamma-2}\int_I\tilde p(\tilde p_x)^2dx+\gamma\alpha^{\gamma-3}\int_I(\tilde p\tilde p_x)^2dx\\
&=\gamma\int_I[(\tilde p+\alpha)^{\gamma-1}-\alpha^{\gamma-1}](\tilde p_x)^2dx+\frac{\gamma}{2}\alpha^{\gamma-1}\|\tilde p_x\|^2+\frac{\gamma}{2}\alpha^{\gamma-3}\|\alpha\tilde p_x-\tilde p\tilde p_x\|^2\\
&\quad+\frac{\gamma}{2}\alpha^{\gamma-3}\|\tilde p\tilde p_x\|^2+\varepsilon\alpha\|\tilde q_x\|^2
\end{aligned}\tag{3.68}$$

We have the following observations regarding $G(t)$ and $H(t)$. First, by virtue of (3.64), we have

$$\begin{aligned}
G(t)\ge&\frac{\alpha^{\gamma-1}}{2}\int_I\tilde p^2dx-\frac{\gamma}{6}\alpha^{\gamma-2}\int_I\tilde p^3dx+\frac{\gamma}{12}\alpha^{\gamma-3}\int_I\tilde p^4dx+\frac{\alpha}{2}\|\tilde q\|\\
=&\frac{(3-\gamma)}{6}\alpha^{\gamma-1}\|\tilde p\|^2+\frac{\gamma}{24}\alpha^{\gamma-3}\|2\alpha\tilde p-\tilde p^2\|^2+\frac{\gamma}{24}\alpha^{\gamma-3}\|\tilde p\|_{L^4}^4+\frac{\alpha}{2}\|\tilde q\|^2\\
\ge&\frac{1}{6}\alpha^{\gamma}\|\tilde p\|^2+\frac{\gamma}{24}\alpha^{\gamma-3}\|2\alpha\tilde p-\tilde p^2\|^2+\frac{\gamma}{24}\alpha^{\gamma-3}\|\tilde p\|_{L^4}^4+\frac{\alpha}{2}\|\tilde q\|^2.
\end{aligned}\tag{3.69}$$

For $H(t)$, from Lemma 2.4, it holds that

$$(\tilde p+\alpha)^{\gamma-1}-\alpha^{\gamma-1}\ge-\alpha^{\gamma-2}|\tilde p|\ge-\alpha^{\gamma-3}(\tilde p)^2-\frac14\alpha^{\gamma-1}.\tag{3.70}$$

Hence,

$$H(t)\ge\frac{\gamma}{4}\alpha^{\gamma-1}\|\tilde p_x\|^2+\frac{\gamma}{2}\alpha^{\gamma-3}\|\alpha\tilde p_x-\tilde p\tilde p_x\|^2-\frac{\gamma}{2}\alpha^{\gamma-3}\|\tilde p\tilde p_x\|^2-\frac{\gamma}{2}\alpha^{\gamma-3}\|\tilde p\tilde p_x\|^2+\varepsilon\alpha\|\tilde q_x\|^2.\tag{3.71}$$

By adding (3.66) to (3.67), we have

$$\begin{aligned}
\frac{d}{dt}X(t)+Y(t)\le&-\gamma\int_I[(\tilde p+\alpha)^{\gamma-1}-\alpha^{\gamma-1}]\tilde p\tilde q\tilde p_x dx-2\gamma\alpha^{\gamma-3}\int_I\tilde p^3\tilde q\tilde p_x dx\\
&-\gamma\alpha^{\gamma-2}\int_I\tilde p^2\tilde q\tilde p_x dx+(\gamma|\alpha_t|+|\beta_t|+\frac{\gamma+7}{\underline{\alpha}}|\alpha_t|)X(t)\\
&+\frac{\gamma\bar\alpha^{\gamma-1}}{2}|\alpha_t|+\frac{\bar\alpha}{2}|\beta_t|.
\end{aligned}\tag{3.72}$$

where, according to (3.71),

$$X(t)=G(t)+\frac{\gamma}{12}\alpha^{\gamma-3}\int_I\tilde p^4dx,$$

$$
\begin{aligned}
Y(t) &= H(t) + \gamma\alpha^{\gamma-3}\|\tilde{p}\tilde{p}_x\|^2 \\
&\geq \frac{\gamma}{4}\alpha^{\gamma-1}\|\tilde{p}_x\|^2 + \frac{\gamma}{2}\alpha^{\gamma-3}\|\alpha\tilde{p}_x - \tilde{p}\tilde{p}_x\|^2 + \frac{\gamma}{2}\alpha^{\gamma-3}\|\tilde{p}\tilde{p}_x\|^2 + \varepsilon\alpha\|\tilde{q}_x\|^2. \qquad (3.73)
\end{aligned}
$$

For the first term on the right-hand side of (3.72), due to Lemma 2.4, it holds that

$$
|(\tilde{p}+\alpha)^{\gamma-1} - \alpha^{\gamma-1}| \leq \alpha^{\gamma-2}|\tilde{p}|, \qquad (3.74)
$$

by which we can update (3.72) as

$$
\begin{aligned}
&\frac{d}{dt}X(t) + Y(t) \leq (\gamma|\alpha_t| + |\beta_t| + \frac{\gamma+7}{\underline{\alpha}}|\alpha_t|)X(t) \\
&\quad + \frac{\gamma\bar{\alpha}^{\gamma-1}}{2}|\alpha_t| + \frac{\bar{\alpha}}{2}|\beta_t| + 2\gamma\alpha^{\gamma-2}\int_I |\tilde{p}|^2|\tilde{q}||\tilde{p}_x|dx + 2\gamma\alpha^{\gamma-3}\int_I |\tilde{p}|^3|q||\tilde{p}_x|dx \\
&. \leq (\gamma|\alpha_t| + |\beta_t| + \frac{\gamma+7}{\underline{\alpha}}|\alpha_t|)X(t) \\
&\quad + \frac{\gamma\bar{\alpha}^{\gamma-1}}{2}|\alpha_t| + \frac{\bar{\alpha}}{2}|\beta_t| + J_1 + J_2. \qquad (3.75)
\end{aligned}
$$

Next, the terms J_1 and J_2 can be estimated by the same method [37, Lemma5.1]. For convenience and simplicity, we only state the results here but omit the specific proofs.

$$
J_1 \leq C(\delta)\|\tilde{p}\|^2\left(\int_I (\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right) + \frac{\delta}{4\gamma\underline{\alpha}^{\gamma-2}}(\|\tilde{p}\tilde{p}_x\|^2 + \|\tilde{p}_x\|^2) \qquad (3.76)
$$

and

$$
J_2 \leq C(\delta)\|\tilde{p}\|_{L^4}^4\left(\int_I (\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\right) + \frac{\delta}{4\gamma\underline{\alpha}^{\gamma-3}}(\|\tilde{p}\tilde{p}_x\|^2 + \|\tilde{p}_x\|^2). \qquad (3.77)
$$

Substituting (3.77) and (3.76) into (3.75), we obtain

$$
\frac{d}{dt}X(t) + \frac{1}{2}Y(t) \leq C(|\alpha_t| + |\beta_t| + \int_I (\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx)X(t) + \frac{\gamma\bar{\alpha}^{\gamma-1}}{2}|\alpha_t| + \frac{\bar{\alpha}}{2}|\beta_t|. \qquad (3.78)
$$

Applying Gronwall's inequality to (3.78) and using Lemma 3.1 and (3.73), we have

$$
\|\tilde{p}\|^2 + \|\tilde{p}\|_{L^4}^4 + \|\tilde{q}\|^2 + \int_0^t (\|\tilde{p}_x\|^2 + \|\tilde{p}\tilde{p}_x\|^2 + \varepsilon\|\tilde{q}_x\|^2)d\tau \leq C,
$$

where C is independent on t and ε. This completes the proof of Lemma 3.2. □

Next, we shall give the estimation of the first order derivatives of the solution.

3.2. H^1 and H^2 estimates.

Lemma 3.3. *Let the assumptions in Theorem 1.1 hold. Then it follows that*

$$
\|\tilde{p}_x(\cdot,t)\|^2 + \|\tilde{q}_x(\cdot,t)\|^2 + \int_0^t (\|\tilde{p}_{xx}\|^2 + \varepsilon\|\tilde{q}_{xx}\|^2)d\tau \leq C,
$$

where the constant C is independent of t, but is inversely proportional to ε.

Proof. Taking the L^2 inner products of the first equation of (3.10) with $-\tilde{p}_{xx}$, and the second with $-\tilde{q}_{xx}$, respectively, then adding the results, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\tilde{p}_x\|^2 + \|\tilde{q}_x\|^2) + \|\tilde{p}_{xx}\|^2 + \varepsilon\|\tilde{q}_{xx}\|^2 \\
&= -\int_I (\tilde{q}\tilde{p}_x + \tilde{p}\tilde{q}_x + \alpha\tilde{q}_x + \beta\tilde{p}_x)\tilde{p}_{xx}dx + \alpha_t\int_I \tilde{p}_{xx}dx \\
&\quad - 2\varepsilon\int_I \tilde{q}\tilde{q}_x\tilde{q}_{xx}dx - 2\varepsilon\beta\int_I \tilde{q}_x\tilde{q}_{xx}dx - \gamma\int_I(\tilde{p}+\alpha)^{\gamma-1}\tilde{p}_x\tilde{q}_{xx}dx + \beta_t\int_I \tilde{q}_{xx}dx \\
&= \sum_{i=1}^{6} J_i. \qquad (3.79)
\end{aligned}
$$

For J_1-J_4, by using the Cauchy-Schwarz inequality and Sobolev inequality, we have

$$
\begin{aligned}
J_1 &\le \frac{1}{4}\|\tilde{q}_{xx}\|^2 + 4(\|\tilde{q}\|_{L^\infty}^2\|\tilde{p}_x\|^2 + \|\tilde{p}\|_{L^\infty}^2\|\tilde{q}_x\|^2 + \alpha^2\|\tilde{q}_x\|^2 + \beta^2\|\tilde{p}_x\|^2) \\
&\le \frac{1}{4}\|\tilde{p}_{xx}\|^2 + 8\|\tilde{q}_x\|^2\|\tilde{p}_x\|^2 + 4\alpha^2\|\tilde{q}_x\|^2 + 4\beta^2\|\tilde{p}_x\|^2; \qquad (3.80)
\end{aligned}
$$

$$J_2 \le \frac{1}{4}\|\tilde{p}_{xx}\|^2 + |\alpha_t|^2;$$

$$J_3 \le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + 8\varepsilon\|\tilde{q}\|_{L^\infty}^2\|\tilde{q}_x\|^2 \le \frac{\varepsilon}{8}\|\tilde{p}_{xx}\|^2 + 8\varepsilon\|\tilde{q}_x\|^2\|\tilde{q}_x\|^2;$$

$$J_4 \le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + 8\varepsilon\bar{\beta}^2\|\tilde{q}_x\|^2;$$

For J_5, when $1 \le \gamma \le 4$, we have

$$
\begin{aligned}
J_5 &\le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + \frac{2\gamma^2}{\varepsilon}\int_I(\tilde{p}+\alpha)^{2(\gamma-1)}(\tilde{p}_x)^2dx \\
&\le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + \frac{2\gamma^2}{\varepsilon}\|(\tilde{p}+\alpha)^\gamma\|_{L^\infty}\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx \\
&\le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + C(\|\tilde{p}_x\|^{\frac{\gamma}{2}}\|p\|^{\frac{\gamma}{2}} + \|p\|^\gamma)\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx \\
&\le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + \|\tilde{p}_x\|^2\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx + C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx. \qquad (3.81)
\end{aligned}
$$

When $\gamma > 4$, by using (3.54) and Cauchy-Schwarz inequaity, we have

$$
\begin{aligned}
J_5 &\le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + \frac{2\gamma^2}{\varepsilon}\|(\tilde{p}+\alpha)^{2(\gamma-1)}\|_{L^\infty}\|\tilde{p}_x\|^2 \\
&\le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + C\int_I\tilde{p}^\gamma dx\int_I\tilde{p}^{\gamma-4}(\tilde{p}_x)^2dx\|p_x\|_{L^2}^2 + \alpha^{2(\gamma-1)}\|\tilde{p}_x\|^2 \\
&\le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + C\left(\int_I\tilde{p}^{\gamma-1}(\tilde{p}_x)^2dx + \|\tilde{p}_x\|_{L^2}^2\right)\|\tilde{p}_x\|_{L^2}^2 + C\|\tilde{p}_x\|^2. \qquad (3.82)
\end{aligned}
$$

For J_6, using the Cauchy inequality, we have

$$J_6 \le \frac{\varepsilon}{8}\|\tilde{q}_{xx}\|^2 + \frac{2}{\varepsilon}|\beta_t|^2.$$

Plugging these estimates into (3.79), when $1 \le \gamma \le 4$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\tilde{p}_x\|^2+\|\tilde{q}_x\|^2)+\frac{1}{2}\|\tilde{p}_{xx}\|^2+\frac{\varepsilon}{2}\|\tilde{q}_{xx}\|^2\\
&\le 8\|\tilde{q}_x\|^2\|\tilde{p}_x\|^2+4\bar{\alpha}^2\|\tilde{q}_x\|^2+4\bar{\beta}^2\|\tilde{p}_x\|^2+|\alpha_t|^2\\
&\quad+8\varepsilon\|\tilde{q}_x\|^2\|\tilde{q}_x\|^2+8\varepsilon\bar{\beta}^2\|\tilde{q}_x\|^2+C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\|\tilde{p}_x\|^2+\frac{2}{\varepsilon}|\beta_t|^2+C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx\\
&\le C\left(\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx+\|\tilde{p}_x\|^2+\varepsilon\|\tilde{q}_x\|^2\right)(\|\tilde{p}_x\|^2+\|\tilde{q}_x\|^2)\\
&\quad+(\frac{4\bar{\alpha}^2}{\varepsilon}+8\bar{\beta}^2)\varepsilon\|\tilde{q}_x\|^2+(4\bar{\beta}^2+C)\|\tilde{p}_x\|^2+C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx+|\alpha_t|^2+\frac{2}{\varepsilon}|\beta_t|^2,
\end{aligned}\tag{3.83}$$

When $\gamma > 4$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\tilde{p}_x\|^2+\|\tilde{q}_x\|^2)+\frac{1}{2}\|\tilde{p}_{xx}\|^2+\frac{\varepsilon}{2}\|\tilde{q}_{xx}\|^2\\
&\le 8\|\tilde{q}_x\|^2\|\tilde{p}_x\|^2+4\bar{\alpha}^2\|\tilde{q}_x\|^2+4\bar{\beta}^2\|\tilde{p}_x\|^2+|\alpha_t|^2\\
&\quad+8\varepsilon\|\tilde{q}_x\|^2\|\tilde{q}_x\|^2+8\varepsilon\bar{\beta}^2\|\tilde{q}_x\|^2+C\left(\int_I\tilde{p}^{\gamma-1}(\tilde{p}_x)^2dx+\|\tilde{p}_x\|^2\right)\|\tilde{p}_x\|^2+\frac{2}{\varepsilon}|\beta_t|^2+C\|\tilde{p}_x\|^2\\
&\le C\left(\int_I\tilde{p}^{\gamma-1}(\tilde{p}_x)^2dx+\|\tilde{p}_x\|^2+\varepsilon\|\tilde{q}_x\|^2\right)(\|\tilde{p}_x\|^2+\|\tilde{q}_x\|^2)\\
&\quad+\left(\frac{4\bar{\alpha}^2}{\varepsilon}+8\bar{\beta}^2\right)\varepsilon\|\tilde{q}_x\|^2+(4\bar{\beta}^2+C)\|\tilde{p}_x\|^2+C\int_I(\tilde{p}+\alpha)^{\gamma-2}(\tilde{p}_x)^2dx+|\alpha_t|^2+\frac{2}{\varepsilon}|\beta_t|^2,
\end{aligned}\tag{3.84}$$

where we have used the first assumption of Theorem 1.1. Applying the Gronwall's inequality to (3.83) and (3.84) and using (3.14),(3.60), we obtain

$$\|\tilde{p}_x(\cdot,t)\|^2+\|\tilde{q}_x(\cdot,t)\|^2+\int_0^t(\|\tilde{p}_{xx}\|^2+\varepsilon\|\tilde{q}_{xx}\|^2)d\tau\le C,\forall\ t>0\tag{3.85}$$

which implies

$$\int_0^t\|\tilde{p}_\tau(\tau)\|^2d\tau+\|\tilde{q}_\tau(\cdot,\tau)\|^2d\tau\le C.\tag{3.86}$$

Taking ∂_t to the two equations in (3.14), then taking the L^2 inner products of the resulting equations with the first order temporal derivatives of the solution, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\tilde{p}_t\|^2+\|\tilde{q}_t\|^2)+\|\tilde{p}_{xt}\|^2+\varepsilon\|\tilde{q}_{xt}\|^2\\
&=-\int_I(\tilde{p}\tilde{q})_t\tilde{p}_{xt}-\alpha\int_I\tilde{p}_{xt}\tilde{q}_tdx+\alpha_t\int_I\tilde{q}_x\tilde{p}_t+\beta_t\int_I\tilde{p}_x\tilde{p}_tdx-\alpha_{tt}\int_I\tilde{p}_tdx\\
&\quad+\gamma\int_I(\tilde{p}+\alpha)^{\gamma-1}\tilde{p}_{xt}\tilde{q}_tdx+\gamma(\gamma-1)\int_I(\tilde{p}+\alpha)^{\gamma-2}\tilde{p}_t\tilde{p}_x\tilde{q}_tdx+\gamma(\gamma-1)\int_I(\tilde{p}+\alpha)^{\gamma-2}\alpha_t\tilde{p}_x\tilde{q}_tdx\\
&\quad-\varepsilon\int_I(\tilde{q}^2)_t\tilde{q}_{xt}+2\varepsilon\beta_t\int_I\tilde{q}_x\tilde{q}_t-\beta_{tt}\int_I\tilde{q}_tdx\\
&=\sum_{i=1}^{11}J_i.
\end{aligned}\tag{3.87}$$

For the right hand side of (3.87), we apply the Cauchy-Schwarz inequality and Sobolev inequality to deduce

$$J_1\le\frac{1}{6}\|\tilde{p}_{xt}\|^2+C\|\tilde{p}_x\|^2\|\tilde{q}_t\|^2+\|\tilde{q}_x\|^2\|\tilde{p}_t\|^2,$$

$$J_2 \le \frac{1}{6}\|\tilde{p}_{xt}\|^2 + C\alpha^2\|\tilde{q}_t\|^2,$$

$$J_3 \le \frac{1}{2}\|\tilde{q}_x\|^2 + \frac{\alpha_t^2}{2}\|\tilde{p}_t\|^2,$$

$$J_4 \le \frac{1}{2}\|\tilde{p}_x\|^2 + \frac{\beta_t^2}{2}\|\tilde{p}_t\|^2,$$

$$J_5 \le \frac{1}{2}\alpha_{tt}^2 + \frac{1}{2}\|\tilde{p}_t\|^2,$$

$$J_6 \le \frac{1}{6}\|\tilde{p}_{xt}\|^2 + C\|\tilde{q}_t\|^2,$$

$$J_7 \le \frac{1}{2}\|\tilde{p}_t\|^2 + \frac{1}{2}(\|\tilde{p}_x\|^2 + \|\tilde{p}_{xx}\|^2)\|\tilde{q}_t\|^2,$$

$$J_8 \le C\|\tilde{p}_x\|^2 + \frac{{\alpha_t}^2}{2}\|\tilde{q}_t\|^2,$$

$$J_9 \le \frac{\varepsilon}{2}\|\tilde{q}_{xt}\|^2 + 2\varepsilon\|\tilde{q}_x\|^2\|\tilde{q}_t\|^2,$$

$$J_{10} \le \varepsilon\|\tilde{q}_x\|^2 + \varepsilon|\beta_t|^2\|\tilde{q}_t\|^2,$$

$$J_{11} \le \frac{\beta_{tt}^2}{2} + \frac{1}{2}\|\tilde{q}_t\|^2.$$

Plugging these estimates into (3.87), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\tilde{p}_t\|^2 + \|\tilde{q}_t\|^2) + \frac{1}{2}\|\tilde{p}_{xt}\|^2 + \frac{\varepsilon}{2}\|\tilde{q}_{xt}\|^2 \\
&\le C(\|\tilde{p}_x\|^2 + \|\tilde{q}_x\|^2 + |\alpha_t|^2 + |\beta_t|^2 + \|\tilde{p}_{xx}\|^2)(\|\tilde{p}_t\|^2 + \|\tilde{q}_t\|^2) \\
&\quad + C(\|\tilde{p}_t\|^2 + \|\tilde{q}_t\|^2 + \|\tilde{p}_x\|^2 + \|\tilde{q}_x\|^2 + |\beta_{tt}|^2 + |\alpha_{tt}|^2).
\end{aligned} \tag{3.88}$$

Applying the Gronwall's inequality to (3.88), and using (3.85), (3.86) and the assumption of Theorem 1.1, we have

$$\|\tilde{p}_t(\cdot,t)\|^2 + \|\tilde{q}_t(\cdot,t)\|^2 + \int_0^t \|\tilde{p}_{xt})\|^2 + \varepsilon\|\tilde{q}_{xt}\|^2 d\tau \le C, \tag{3.89}$$

which implies

$$\|\tilde{p}_{xx}(\cdot,t)\| + \|\tilde{q}_{xx}(\cdot,t)\|^2 + \int_0^t \|\tilde{p}_{xxx}(\cdot,\tau)\|^2 + \|\tilde{q}_{xxx}(\cdot,\tau)\|^2 d\tau \le C, \tag{3.90}$$

where the constant C is independent of t, but depends reciprocally on ε. □

Next, we prove the asymptotic stability of the solution.

3.3. **Asymptotic stability.** In what follows, we use the energy estimates obtained in the previous subsections to prove $\|(\tilde{p},\tilde{q})(\cdot,t)\|_{H^2}^2 \in W^{1,1}(0,\infty)$. From (3.1) and (3.61), we can show that

$$\|\tilde{p}_x(\cdot,t)\|^2 + \varepsilon\|\tilde{q}_x(\cdot,t)\|^2 \in L^1(0,\infty). \tag{3.91}$$

Next, multiplying the equation (1.1) by $\tilde{p}$, $\tilde{q}$ respectively, we obtain

$$\begin{aligned}
\frac{d}{dt}(\|\tilde{p}\|^2 + \|\tilde{q}\|^2) &= -2\|\tilde{p}_x\|^2 - 2\varepsilon\|\tilde{q}_x\|^2 - 2\int_I \tilde{p}\tilde{q}\tilde{p}_x dx \\
&\quad - 2\alpha_t\int_I \tilde{p}dx + 2\alpha\int_I \tilde{q}_x\tilde{p}dx + 2\gamma\int_I (\tilde{p}+\alpha)^{\gamma-1}\tilde{p}_x\tilde{q}dx - 2\beta_t\int_I \tilde{q}dx,
\end{aligned} \tag{3.92}$$

from which we can deduce

$$\left|\frac{d}{dt}(\|\tilde{p}\|^2+\|\tilde{q}\|^2)\right| \le 2\|\tilde{p}_x\|^2+2\varepsilon\|\tilde{q}_x\|^2+\|\tilde{p}\|_{L^\infty}(\|\tilde{q}\|^2+\|\tilde{p}_x\|^2)$$
$$+|\alpha_t|^2+\|\tilde{p}\|^2+\bar{\alpha}(\|\tilde{q}_x\|^2+\|\tilde{p}\|^2)+\gamma\|(\tilde{p}+\alpha)\|_{L^\infty}^{\gamma-1}(\|\tilde{p}_x\|^2+\|\tilde{q}\|^2)$$
$$+|\beta_t|^2+\|\tilde{q}\|^2. \tag{3.93}$$

Using the Sobolev embedding inequality and Poincaré's inequality, we obtain

$$\left|\frac{d}{dt}(\|\tilde{p}\|^2+\|\tilde{q}\|^2)\right| \le C(\|\tilde{p}_x\|^2+\|\tilde{q}_x\|^2+|\alpha_t|^2+|\beta_t|^2) \tag{3.94}$$

From, we find that each term of the right hand side of (3.94) is uniformly integrable with respect to time, which implies

$$\frac{d}{dt}(\|\tilde{p}(\cdot,t)\|^2+\|\tilde{q}(\cdot,t)\|^2)\in L^1(0,\infty), \tag{3.95}$$

which combing with (3.91), we have

$$\|\tilde{p}(\cdot,t)\|^2+\|\tilde{q}(\cdot,t)\|^2\in W^{1,1}(0,\infty).$$

Hence,

$$\lim_{t\to\infty}(\|\tilde{p}(\cdot,t)\|^2+\|\tilde{q}(\cdot,t)\|^2)=0.$$

Similarly, using the same argument we can show that

$$\lim_{t\to\infty}(\|\tilde{p}_x(\cdot,t)\|^2+\|\tilde{q}_x(\cdot,t)\|^2)=0.$$

This completes the proof of Theorem 1.1.

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