Triple Cosines Lemma and π-sums of Arccosines

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Abstract. We obtain a relationship between cosines of two independent angles and cosine of the angle that depends on them in 3D space and then we use that relationship to obtain π-sums of Arccosines.

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1. Triple Cosines Lemma

In a Cartesian coordinate system for a three-dimensional space of an ordered triplet of axes: OX, OY, OZ that go through the origin O, let the angle AOX = α, the angle XOB = β. See Figure 1.

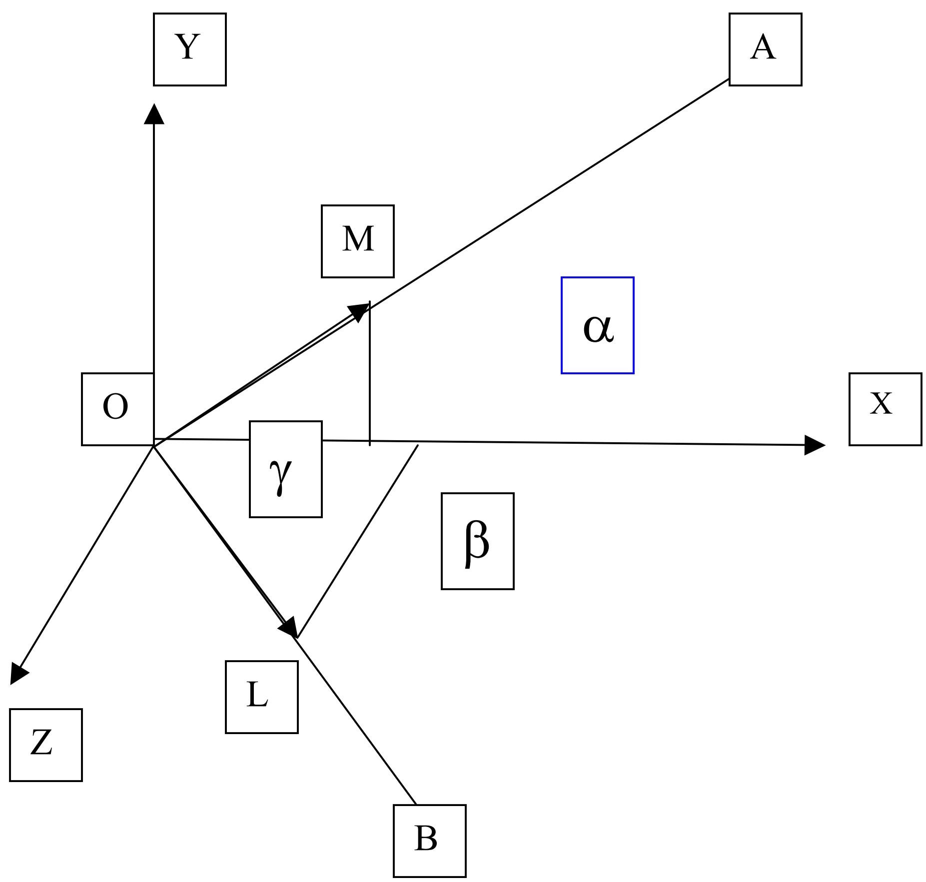

Figure 1. Triple Cosines Lemma

Let us find the angle AOB = γ.

Lemma 1. *$\cos \gamma = \cos \alpha \cos \beta$*

Proof. Let OM be a unit vector in the direction of OA, let OL be a unit vector in the direction of OB. OM = ($\cos \alpha$, $\sin \alpha$, 0), OL = ($\cos \beta$, 0, $\sin \beta$). Since the dot product of vectors OM and OL is: OM * OL = |OM| |OL| $\cos \gamma = \cos \gamma$, finally we have: $\cos \gamma = \cos \alpha \cos \beta$. □

2. Triple Arccosines Theorem

Theorem 1. *For any triangle cross section of a cube with a plane, where two sides of said cross section, meeting at first vertex of said cross section, said first vertex is located on first edge of the cube, said two sides and said first edge intersect at angles α_1 and β_3, two sides of said cross section, meeting at second vertex of said cross section, said second vertex is located on second edge of the cube, said two sides and said second edge intersect at angles α_2 and β_1, two sides of said cross section, meeting at third vertex of said cross section, said third vertex is located on third edge of the cube, said two sides and said third edge intersect at angles α_3 and β_2, we have:*

$$\alpha_1 + \beta_1 = \pi/2, \quad \alpha_2 + \beta_2 = \pi/2, \quad \alpha_3 + \beta_3 = \pi/2,$$

$$\arccos(\cos\alpha_1\cos\beta_3) + \arccos(\cos\alpha_2\cos\beta_1) + \arccos(\cos\alpha_3\cos\beta_2) = \pi$$

Proof. Let $ABCDA_1B_1C_1D_1$ be a cube (see Figure 2).

Let $M \in [AB]$, $N \in [BB_1]$, $L \in [BC]$.

Let $\alpha_1 = \angle NMB$, $\alpha_2 = \angle BNL$, $\alpha_3 = \angle BLM$,

$\beta_1 = \angle MNB$, $\beta_2 = \angle NLB$, $\beta_3 = \angle BML$,

Then, considering the triangles MNB, NBL and MBL we have:

$\alpha_1 + \beta_1 = \pi/2$, $\alpha_2 + \beta_2 = \pi/2$, $\alpha_3 + \beta_3 = \pi/2$.

By applying the Triple Cosines Lemma 1 to the triangle MNL, since

$\cos\angle LMN = \cos\alpha_1 \cos\beta_3$, $\cos\angle MNL = \cos\alpha_2 \cos\beta_1$,

$\cos\angle NLM = \cos\alpha_3 \cos\beta_2$, and $\angle LMN + \angle MNL + \angle NLM = \pi$, we prove the Theorem 1. □

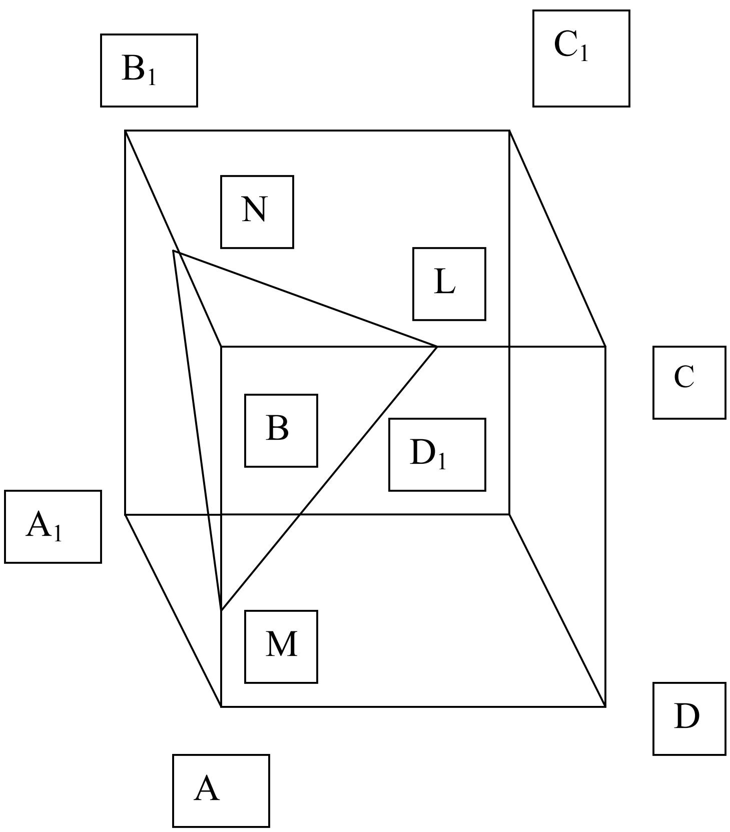

Figure 2. Triple Arccosines Theorem

Remark 1. Note that we could generalize Theorem 1 for the case of 4-sided, 5-sided and 6-sided cross section polygons: intersections of plane and cube, instead of the triangle cross section as well as for intersections of n-dimensional hypercubes(n >3).

3. π-sum of 6 Arccosines Theorem

Let us consider a tetrahedron SABC, having a triangle ABC as a base, a height SO, wherein the point O is located inside ABC. Since SO is perpendicular to the base ABC, then OAS, OBS and OCS are perpendicular to the base ABC as well(see Figure 3).

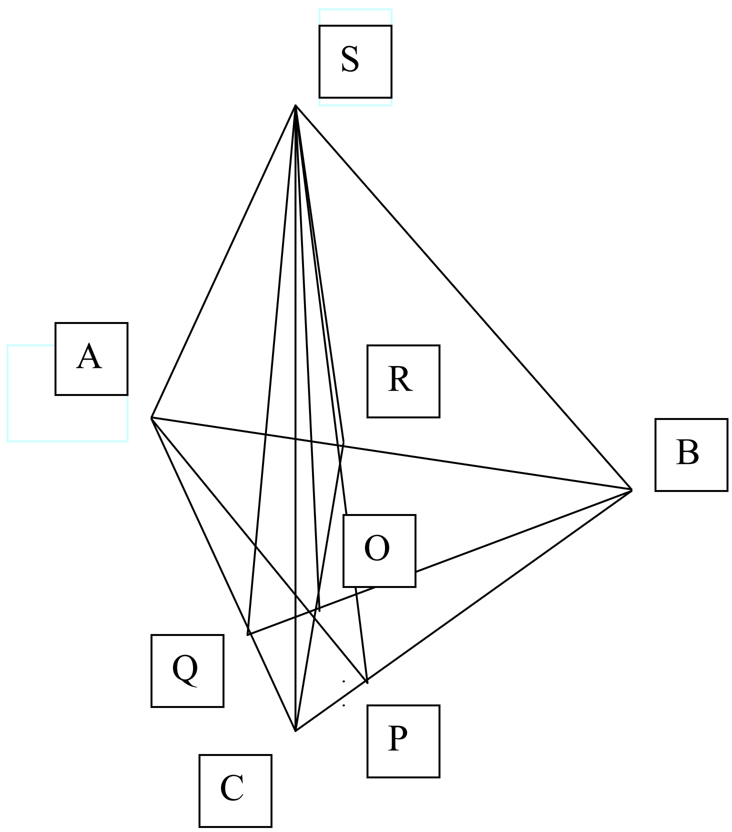

Figure 3. π-sum of 6 Arccosines Theorem

Let $\alpha_1 = \angle\, SAO$, $\beta_{11} = \angle\, OAC$, $\beta_{12} = \angle\, OAB$, $\gamma_1 = \angle\, SAC$, $\delta_1 = \angle SAB$,

$\alpha_2 = \angle\, SBO$, $\beta_{21} = \angle\, ABO$, $\beta_{22} = \angle\, OBC$, $\gamma_2 = \angle\, SBA$, $\delta_2 = \angle SBC$,

$\alpha_3 = \angle$ SCO, $\beta_{31} = \angle$ BCO, $\beta_{32} = \angle$ OCA, $\gamma_3 = \angle$ BCS, $\delta_3 = \angle$SCA,

Theorem 2. *For any tetrahedron, where a projection of the apex of the tetrahedron on the base is located inside the base and for the first vertex of the base: the angle between the edge, connecting the apex with said first vertex and a segment, connecting said first vertex with said projection is α_1, the angles between a the edge, connecting the apex with said first vertex and two base's edges, meeting at said first vertex are γ_1* and *δ_1, for the second vertex of the base: the angle between the edge, connecting the apex with said second vertex and a segment, connecting said second vertex with said projection is α_2, the angles between the edge, connecting the apex with said second vertex and two base's edges, meeting at said second vertex are γ_2* and *δ_2, for the third vertex of the base: the angle between the edge, connecting the apex with said third vertex and a segment, connecting said third vertex with said projection is α_3, the angles between the edge, connecting the apex with said third vertex and two base's edges, meeting at said third vertex are γ_3 and δ_3, we have:*

$$arccos(\cos\gamma_1 / \cos\alpha_1) + arccos(\cos\delta_1 / \cos\alpha_1) +$$

$$arccos(\cos\gamma_2 / \cos\alpha_2) + arccos(\cos\delta_2 / \cos\alpha_2) +$$

$$arccos(\cos\gamma_3 / \cos\alpha_3) + arccos(\cos\delta_3 / \cos\alpha_3) = \pi.$$

*Proof***.** By applying the Triple Cosines Lemma 1 to the angles at the vertex A of tetrahedron SABC, we have:

$$\cos\gamma_1 = \cos\alpha_1 \cos\beta_{11}, \cos\delta_1 = \cos\alpha_1 \cos\beta_{12}.$$

Thus, $\beta_{11} = \arccos(\cos\gamma_1 / \cos\alpha_1)$, $\beta_{12} = \arccos(\cos\delta_1 / \cos\alpha_1)$,
So, $\angle$ BAC $= \beta_{11} + \beta_{12} = \arccos(\cos\gamma_1 / \cos\alpha_1) + \arccos(\cos\delta_1 / \cos\alpha_1)$.
Similarly, $\angle$ ABC $= \arccos(\cos\gamma_2 / \cos\alpha_2) + \arccos(\cos\delta_2 / \cos\alpha_2)$ and
$\angle$ BCA $= \arccos(\cos\gamma_3 / \cos\alpha_3) + \arccos(\cos\delta_3 / \cos\alpha_3)$.

Since $\angle$ BAC + $\angle$ ABC + $\angle$ BCA = π, we prove the Theorem 2. □

Remark 2. Note that Theorem 2 can be generalized for the pyramid, having a convex n-sided polygon $A_1 \dots A_n$ as a base, a height SO, which is perpendicular to the base $A_1 \dots A_n$ of the pyramid $SA_1 \dots A_n$ and O locates inside the base $A_1 \dots A_n$. Similarly to the Theorem 2, considering the triangles SOA_1, ... , SOA_n, that are perpendicular to the base $A_1 \dots A_n$, we can prove that the sum of 2n Arccosines, (two Arccosines per each vertex of the base $A_1 \dots A_n$) is equal to the sum of interior angles of the polygon $A_1 \dots A_n$: $(n - 2)\,\pi$.

References

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