Triple Cosines Lemma and π-sums of Arccosines

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Abstract. We obtain a relationship between cosines of two independent angles and cosine of the angle that depends on them in 3D space and then we use that relationship to obtain π-sums of Arccosines.

1. Triple Cosines Lemma

In a Cartesian coordinate system for a three-dimensional space of an ordered triplet of axes: OX, OY, OZ that go through the origin O, let the angle AOX = α, the angle XOB = β. See Figure 1.

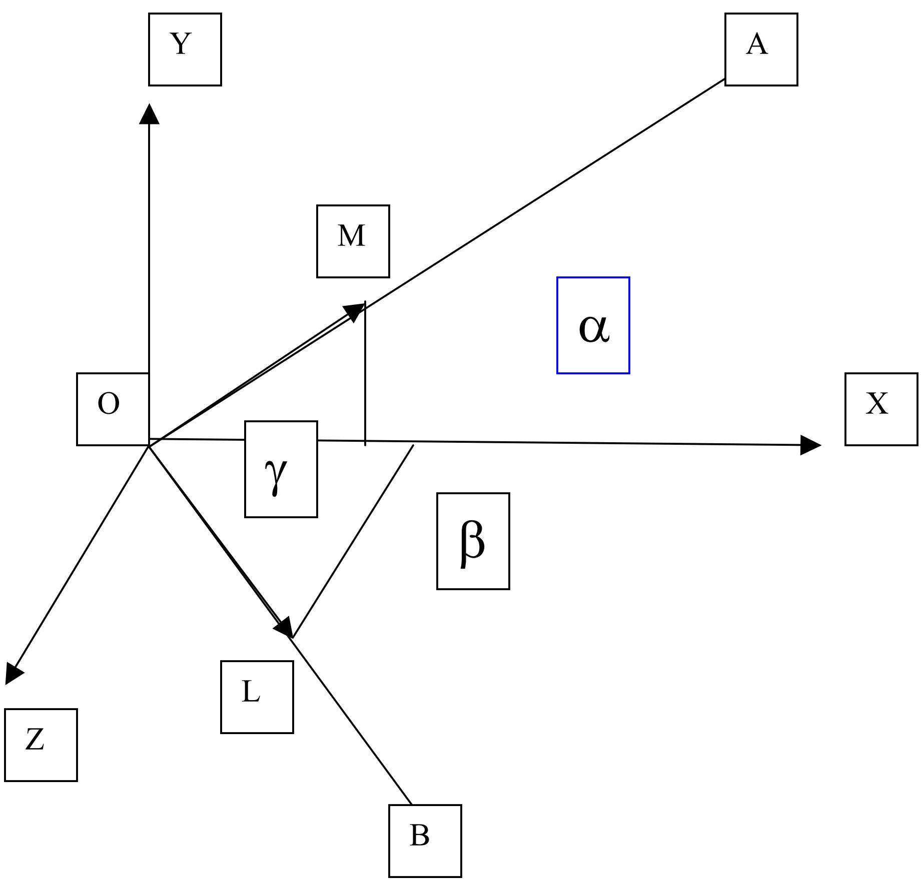

Figure 1. Triple Cosines Lemma

Let us find the angle AOB = γ.

Lemma 1. *$\cos\gamma = \cos\alpha \cos\beta$*

Proof. Let OM be a unit vector in the direction of OA, let OL be a unit vector in the direction of OB. OM = (cos α, sin α, 0), OL = (cos β, 0, sin β). Since the dot product of vectors OM and OL is: OM * OL = |OM| |OL| cos γ = cos γ, finally we have: $\cos\gamma = \cos\alpha \cos\beta$. □

2. Triple Arccosines Theorem

Theorem 1. *If $\alpha_1 + \beta_1 = \pi/2$, $\alpha_2 + \beta_2 = \pi/2$, $\alpha_3 + \beta_3 = \pi/2$, then:*

$$\arccos(\cos\alpha_1\cos\beta_3) + \arccos(\cos\alpha_2\cos\beta_1) + \arccos(\cos\alpha_3\cos\beta_2) = \pi$$

Proof. Let $ABCDA_1B_1C_1D_1$ be a cube (see Figure 2).

Let $M \subseteq [AB]$, $N \subseteq [BB_1]$, $L \subseteq [BC]$.

Let $\alpha_1 = \angle NMB$, $\alpha_2 = \angle BNL$, $\alpha_3 = \angle BLM$,

$\beta_1 = \angle MNB$, $\beta_2 = \angle NLB$, $\beta_3 = \angle BML$,

Then, considering the triangles MNB, NBL and MBL we have:

$\alpha_1 + \beta_1 = \pi/2$, $\alpha_2 + \beta_2 = \pi/2$, $\alpha_3 + \beta_3 = \pi/2$.

By applying the Triple Cosines Lemma 1 to the triangle MNL we finally have:

$$\arccos(\cos\alpha_1\cos\beta_3) + \arccos(\cos\alpha_2\cos\beta_1) + \arccos(\cos\alpha_3\cos\beta_2) = \pi. \qquad \square$$

Remark. Note that we could generalize Theorem 1 for the case of 4-sided and 6-sided polygons: intersections of plane and cube, instead of the triangle cross section as well as for intersections of n-dimensioned hypercubes(n >3).

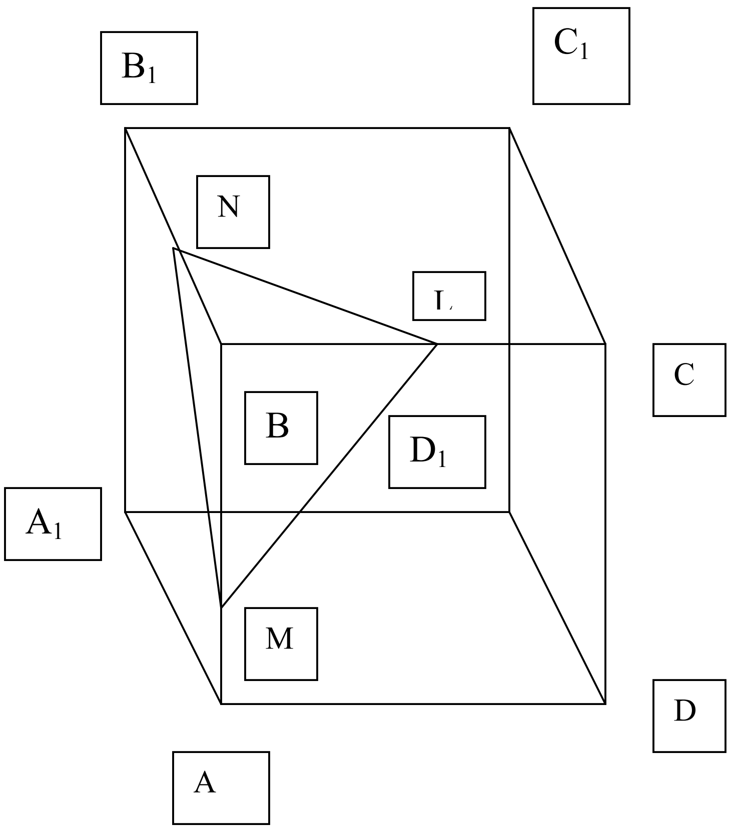

Figure 2. Triple Arccosines Theorem

3. π-sum of 6 Arccosines Theorem

Let us consider a tetrahedron SABC, having the base ABC, the height SO, where O is the orthocenter of ABC: the three altitudes - AP, BQ and CR of ABC intersect at the orthocenter O. Thus, SO, ASP, BSQ, CSR are perpendicular to the base ABC (see Figure 3).

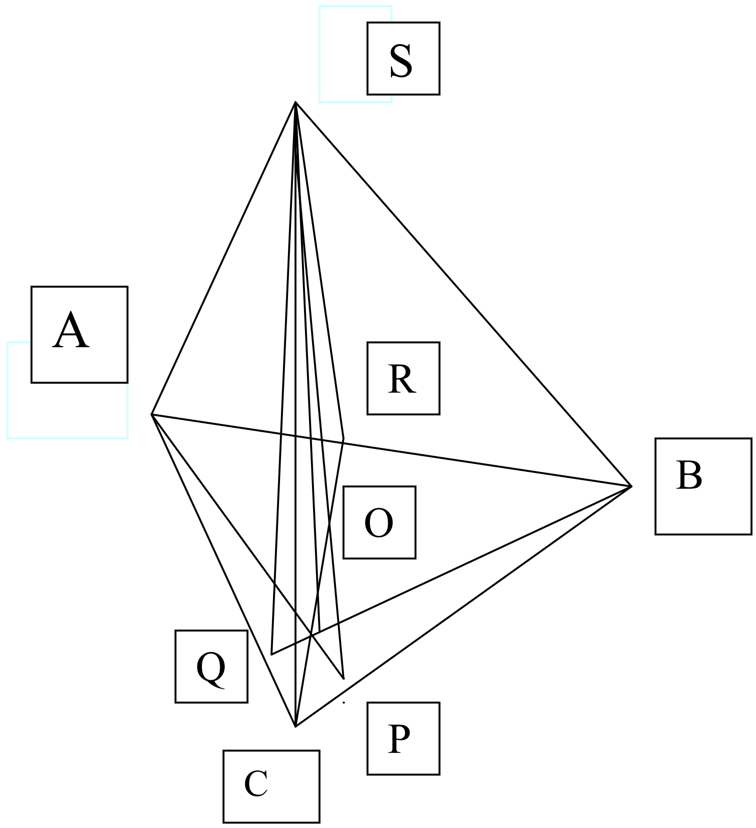

Figure 3. π-sum of 6 Arccosines Theorem

Let $\alpha_1 = \angle SAP$, $\beta_{11} = \angle PAC$, $\beta_{12} = \angle PAB$, $\gamma_1 = \angle SAC$, $\delta_1 = \angle SAB$,

$\alpha_2 = \angle SBQ$, $\beta_{21} = \angle ABQ$, $\beta_{22} = \angle QBC$, $\gamma_2 = \angle SBA$, $\delta_2 = \angle SBC$,

$\alpha_3 = \angle SCR$, $\beta_{31} = \angle BCR$, $\beta_{32} = \angle RCA$, $\gamma_3 = \angle BCS$, $\delta_3 = \angle SCA$,

Theorem 2. *If* $\alpha_1, \delta_1 + \gamma_2 < \pi, \quad \alpha_2, \delta_2 + \gamma_3 < \pi, \alpha_3, \delta_3 + \gamma_1 < \pi,$ *then:*

$$\arccos(\cos\gamma_1/\cos\alpha_1) + \arccos(\cos\delta_1/\cos\alpha_1) +$$

$$\arccos(\cos\gamma_2/\cos\alpha_2) + \arccos(\cos\delta_2/\cos\alpha_2) +$$

$$\arccos(\cos\gamma_3/\cos\alpha_3) + \arccos(\cos\delta_3/\cos\alpha_3) = \pi.$$

*Proof***.** By applying the Triple Cosines Lemma 1 to the angles at the vertex A of tetrahedron SABC, we have:

$$\cos \gamma_1 = \cos \alpha_1 \cos \beta_{11},\ \cos \delta_1 = \cos \alpha_1 \cos \beta_{12}\,.$$

Thus, $\beta_{11} = \arccos(\cos \gamma_1/\cos\alpha_1)$, $\beta_{12} = \arccos(\cos \delta_1/\cos\alpha_1)$,
So, $\angle BAC = \beta_{11} + \beta_{12} = \arccos(\cos \gamma_1/\cos\alpha_1) + \arccos(\cos \delta_1/\cos\alpha_1)$.
Similarly, $\angle ABC = \arccos(\cos \gamma_2/\cos\alpha_2) + \arccos(\cos\delta_2/\cos\alpha_2)$ and
$\angle BCA = \arccos(\cos \gamma_3/\cos\alpha_3) + \arccos(\cos\delta_3/\cos\alpha_3)$.
Since $\angle BAC + \angle ABC + \angle BCA = \pi$, we prove the Theorem 2. □

References

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