The Information Volume of Uncertain Information: (6) Information Multifractal Dimension

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Abstract

How to measure the uncertainty in the open world is a popular topic in recent study. Many entropy measures have been proposed to address this problem, but most have limitations. In this series of paper, a method for measuring the information volume of mass function is presented. The fractal property about the maximum information volume is shown in this paper, which indicates the inherent physical meanings of Deng entropy from the perspective of statistics. The results shows the multifractal property of this maximum information volume. Some experiment results are applied to support this perspective.

Keywords: Information volume, Deng entropy, Deng distribution, Fractal property, Mass function.

1. Introduction

Recently, numerous methods have been proposed to deal with the uncertain problem in the open world, such as Dempster-Shafer evidence theory [1, 2], and D number [3]. It can be used in several fields, like decision making [4], information fusion [5, 6] and pattern recognition. However, there are still some issues to be solved. Since firstly derived from thermodynamics, different kinds of entropy

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have been proposed, such as Shannon entropy [7], Tsallis entropy [8], and so on [9, 10]. Recently, a new entropy, called Deng entropy [11], is presented for measuring the uncertainty in evidence theory. Deng entropy is the generalization of Shannon entropy. Compared with traditional methods, Deng entropy is more reasonable, and it takes both discord and non-specificity into account. Some analyzed the properties of Deng entropy [12], some made improved work based on Deng entropy [13]. From the perspective of classical entropy theory, Deng entropy doest not verify the requirements of set consistency, range, subadditivity, additivity and monotonicity, which are defined by Klir & Wierman [14]. However, Deng entropy, considered as an extension of Shannon entropy, can not only deal with uncertain phenomenon in the probability field, but also be applied to absorb the complex imprecise (or unknown) phenomenon in the belief filed efficiently. When the BPA is degenerated as probability distribution, Deng entropy is degenerated as Shannon entropy. In this paper, we propose a method to obtain the maximum information volume of BPA distribution. Based on these maximum information volumes, we get the information multifractal dimension based on the information volume and the probability scale. The well linear relationship between these parameters shows the existence of the fractal property. Some numerical examples are given in this paper to show the fractal property of this dimension.

The organization of the rest of this paper is as follows. Section 2 introduces some concepts about Basic probability assignment and Deng entropy. This proposed information multifractal dimension is proposed in Section 3. Meanwhile, some numerical experiments are simulated to illustrate this proposed method in Section 4. The conclusion is conducted in Section 5.

2. Preliminaries

2.1. Basic probability assignment

Dempster-Shafer evidence theory[1, 2] can be used to deal with uncertainty. Besides, evidence theory satisfies the weaker conditions than the probability theory, which provides it with the ability to express uncertain information directly.

Let Θ, called the frame of discernment, denote an exhaustive nonempty set of hypotheses, where the elements are mutually exclusive. Let the set Θ have N elements, which can be expressed as follows,

$$\Theta = \{\theta_1, \theta_2, \theta_3, \cdots, \theta_N\} \tag{1}$$

The power set of Θ, denoted as 2^Θ, contains all possible subsets of Θ and has 2^N elements, and 2^Θ is represented as follows,

$$\begin{aligned} 2^\Theta &= \{A_1, A_2, A_3, \cdots, A_{2^N}\} \\ &= \{\ \emptyset, \{\theta_1\}, \{\theta_2\}, \cdots, \{\theta_N\}, \{\theta_1, \theta_2\}, \{\theta_1, \theta_3\}, \cdots, \{\theta_1, \theta_N\}, \cdots, \Theta\ \} \end{aligned} \tag{2}$$

where the element A_i is called the focal element of Θ, if A_i is nonempty.

A BPA is a mass function mapping m from 2^Θ to $[0,1]$, and it is defined as follows,

$$m : 2^\Theta \rightarrow [0,1] \tag{3}$$

which is constrained by the following conditions:

$$\sum_{A \in 2^\Theta} m(A) = 1 \tag{4}$$

$$m(\emptyset) = 0 \tag{5}$$

2.2. Deng entropy

In information theory, entropy can be used to measure the uncertainty of a system. Recently, a novel entropy, named as Deng entropy [11], is proposed to measure the uncertainty in evidence theory, and it is shown below,

$$H_{DE}(m) = -\sum_{A \in 2^\Theta} m(A) \log\left(\frac{m(A)}{2^{|A|} - 1}\right) \tag{6}$$

where $|A|$ is the cardinal of a certain focal element A. Through a simple transformation, Eq.(6) can be rewritten as follows:

$$H_{DE}(m) = \sum_{A \in 2^\Theta} \log(2^{|A|} - 1) - \sum_{A \in 2^\Theta} m(A) \log m(A) \tag{7}$$

where $\sum_{A\in 2^\Theta}\log(2^{|A|}-1)$ and $-\sum_{A\in 2^\Theta} m(A)\log m(A)$ are measurements of nonspecificity and discord, respectively. When every focal element is singleton, Deng entropy degenerates into Shannon entropy.

2.3. The maximum Deng entropy

Assume A is the focal element of a certain frame of discernment Θ and $m(A)$ is the BPA for A. According to [15], Deng entropy reaches its maximum value, and the analytic solution of the maximum Deng entropy is shown as follows,

$$H_{MDE}(m) = \log \sum_{A\in 2^\Theta} (2^{|A|}-1) \tag{8}$$

where the condition is if and only if $m(A) = \frac{(2^{|A|}-1)}{\sum\limits_{A\in 2^\Theta}(2^{|A|}-1)}$.

2.4. Fractal property

In fractal geometry, a fractal dimension is a ratio providing a statistical index of complexity comparing how detail in a pattern (strictly speaking, a fractal pattern) changes with the scale at which it is measured. A fractal dimension does not have to be an integer. And a fractal dimension can be presented as follows,

$$d = \frac{\ln N(s)}{\ln s} \tag{9}$$

where d is the fractal dimension, s is the magnification factor, and $N(s)$ is the number of self-similar pieces. They would follow a double logarithmic scale fitting curve.

3. Proposed Information Fractal Dimension

3.1. Information volume of mass function

Firstly, the maximum Deng entropy is introduced in Section 2.3. Deng entropy $E_d = -\sum\limits_i m(F_i)\log_2 \frac{m(F_i)}{2^{|F_i|}-1}$ reaches it maximum value if and only if $m(F_i) = \frac{2^{|F_i|}-1}{\sum\limits_i 2^{|F_i|}-1}$.

Besides, according to this series papers [16, 17, 18], the mass function which can yield the maximum Deng entropy is called the Deng distribution. In addition, the definition of information volume and the maximum information volume is introduced in [16].

In order to get the maximum information volume, the distribution should be modified. The Deng distribution with cardinal equal to 2 (universe discourse E) is shown here as an example, and it is $\left[\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right]$ for $\{X_0\}, \{Y_0\}$, and $\{X_0, Y_0\} = \{U_0\}$. So we separate its BPA based on this proportion: $\frac{1}{5} : \frac{1}{5} : \frac{3}{5}$. Namely, we focus on $m(U_0)$, and separate this BPA based on $\frac{1}{5} : \frac{1}{5} : \frac{3}{5}$. The result is that:

$$\begin{cases} m(X_0) = \frac{1}{5}, \\ m(Y_0) = \frac{1}{5}, \\ m(X_1) = \frac{1}{5} \times \frac{3}{5} = \frac{3}{25}, \\ m(Y_1) = \frac{1}{5} \times \frac{3}{5} = \frac{3}{25}, \\ m(U_1) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}. \end{cases} \tag{10}$$

where $m(X_1)$, $m(Y_1)$ and $m(U_1)$ can be seen as derivatives of $m(U_0)$. Based on Deng entropy, calculate the uncertainty of the new BPA distribution.

$$\begin{aligned} & H_1 = -\tfrac{1}{5}\log_2(\tfrac{1}{5}) - \tfrac{1}{5}\log_2(\tfrac{1}{5}) \\ & -\tfrac{3}{25}\log_2(\tfrac{3}{25}) - \tfrac{3}{25}\log_2(\tfrac{3}{25}) - \tfrac{9}{25}\log_2(\tfrac{9/25}{3}) \\ & = 2.76411 \end{aligned} \tag{11}$$

Based on this calculating steps, the uncertainty of this new BPA distribution is 2.76411, which is larger than 2.32193 (the maximum Deng entropy). This example shows that, the uncertainty increases when the BPA is separated.

Next, continuously separate the BPA of the element whose cardinal is larger than 1 until convergence. Concretely, repeat following steps until Deng entropy is convergent. Use index i to denote the times of this loop.

1) Focus on the element whose cardinal is larger than 1, and separate its BPA based on Deng distribution:

$$m(F_i) = \frac{2^{|F_i|} - 1}{\sum\limits_i 2^{|F_i|} - 1} \tag{12}$$

Since $U_0 = \{X_0, Y_0\}$, the proportion is that $\frac{1}{5} : \frac{1}{5} : \frac{3}{5}$. The ith times of separation divide $m(U_{i-1})$ and yield following new BPAs: $m(X_i)$, $m(Y_i)$, $m(U_i)$ which are derived from $m(U_{i-1})$. In addition, they satisfy these equations:

$$
\begin{aligned}
&m(X_i) + m(Y_i) + m(U_i) = m(U_{i-1}) \\
&m(X_i) : m(Y_i) : m(U_i) = \tfrac{1}{5} : \tfrac{1}{5} : \tfrac{3}{5}
\end{aligned}
\tag{13}
$$

2) Based on Deng entropy, calculate the uncertainty of the new BPA distribution. The result is denoted as H_i.

3) Calculate $\Delta_i = H_i - H_{i-1}$. When Δ_i satisfies following condition that $\Delta_i = H_i(m) - H_{i-1}(m) < \varepsilon$, jump out of this loop. ε is the allowable error, which equals to 0.001 in this paper.

For better understanding, the calculating procedure is illustrated in Fig. 1. Thus, the maximum information volume of the mass function is obtained, and it is shown below,

$$
MIV_{|E|} = -\sum_{F_i \in |E|} m(F_i) \log_2 \frac{m(F_i)}{2^{|F_i|} - 1} \tag{14}
$$

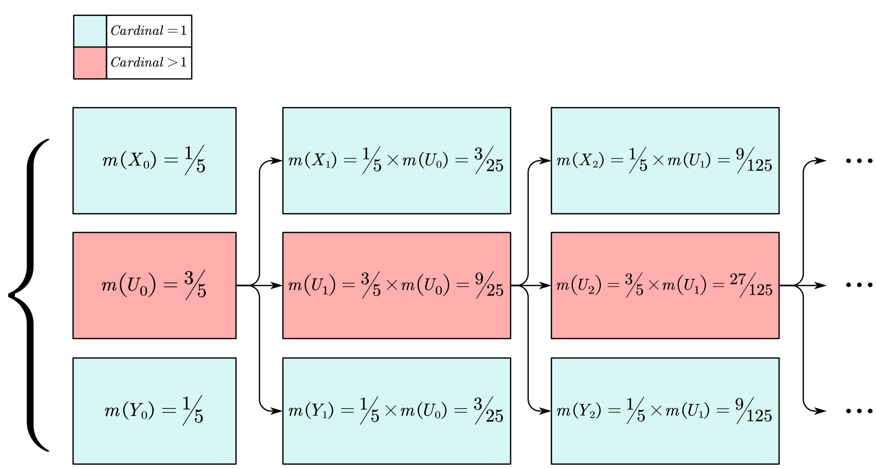

Fig. 1: The separate procedure of BPA.

3.2. Fractal information dimension

A power law function $P(r)$ with Deng information scale r is established using the following equation,

$$P(r) \sim r^{-d} \tag{15}$$

where r is the scale and $P(r)$ is information volume.

For the generated BPA distribution $m(F_i)$ from Deng distribution, a partition sum is considered as follows,

$$Z(q) = \sum_{m(F_i)} (m(F_i))^q \tag{16}$$

where q is a real number and $m(F_i)$ is the BPA distribution. The value of $Z(q)$ follows that $Z(q) > 0$ and $Z(0) = 1$. If $Z(q)$ has a power-law r dependence $Z(q) \sim r^{\tau(q)}$, the mass exponent function $\tau(q)$ is obtained as follows,

$$\tau(q) = \lim_{r \to 0} -\frac{\ln Z(q)}{\ln r} \tag{17}$$

where r equals to $\sum\limits_{F_i \in E} (2^{|F_i|} - 1)$, $\tau(q)$ is a linear function of q. $Z(q)$ is satisfied as follows,

$$Z(q) \sim r^{D_q(q-1)} \tag{18}$$

where the D_q is the generalized fractal dimension of the measure $m(F_i)$. We have the linear relation, which is shown as follows,

$$D_q = \frac{\tau(q)}{q-1}, q \neq 1 \tag{19}$$

for $q = 1$, where $D_1 = \lim\limits_{r \to 0} \frac{Z(1)}{\ln r}$ and $Z(1) = MIV_{|E|}$, $\sum\limits_{F_i \in E} (2^{|F_i|} - 1)$ is the ith BPA distribution when it reaches the maximum information volume.

4. Numerical examples and discussions

Firstly, the BPA distribution which cause the maximum Deng entropy is used as the initial BPA distribution, and it is $m(F_i) = \frac{2^{|F_i|}-1}{\sum\limits_i 2^{|F_i|}-1}$. Then some elements of the initial BPA distribution will be separated to numerous new elements and the maximum information volume $MIV_{|E|}$ under different cardinal $|E|$ can be obtained.

With these generated BPA distribution obtained by Eq. (14), the generalized fractal dimension can be obtained by Eq. (19). The double logarithmic scale fitting curve is shown in Fig. 2. Observed from Fig. 2, it can be found that the approximate linear relation is between these two parameters, which show the scale-free and fractal property.

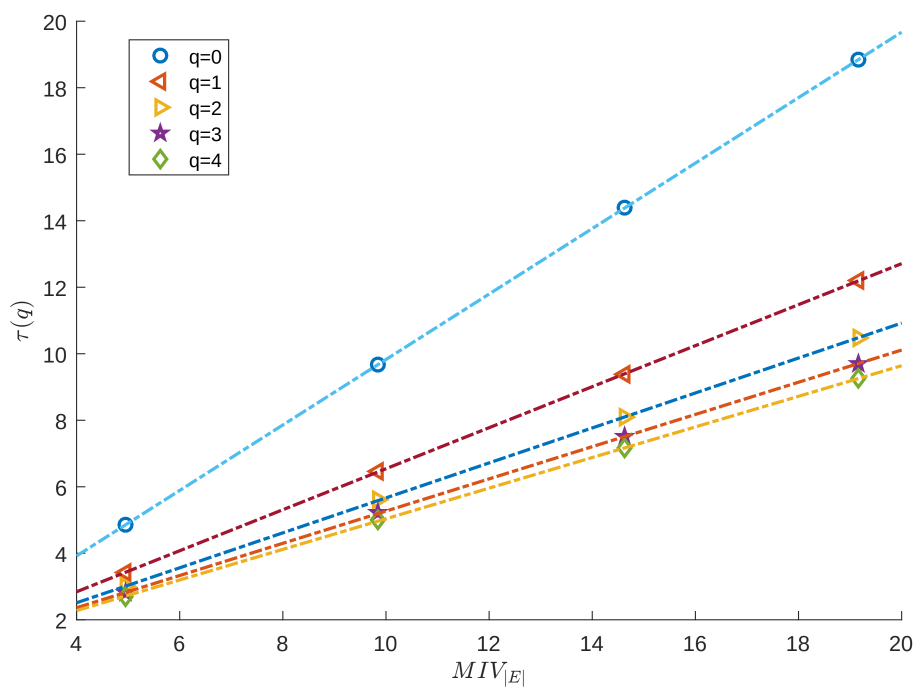

Fig. 2: The double logarithmic scale fitting curves

5. Conclusion

Deng entropy can not only deal with uncertain phenomenon in the probability field, but also measure uncertain degree with basic probability assignment in the belief filed. In this paper, we proposed a novel dimension to describe the uncertainty with different cardinal which is called information multifractal

Table 1: $\tau(q)$ and $\log_2 \sum\limits_{F_i \in E} (2^{|F_i|} - 1)$ in different cardinal and q.

$\|E\|$		2	3	4	5	D_q
	q=0	4.8579	9.6741	14.3890	18.8414	0.9843
	q=1	3.4259	6.4690	9.3704	12.2046	0.6166
$\tau(q)$	q=2	2.9999	5.6129	8.0826	0.4841	0.5257
	q=3	2.8073	5.2257	7.4993	9.7032	0.4843
	q=4	2.6958	5.0013	7.1605	9.2490	0.4602
$\log_2 \sum\limits_{F_i \in E} (2^{\|F_i\|} - 1)$		4.9542	9.8439	14.6294	19.1618	

dimension. Interestingly, the separate with these BPA shows the fractal property of this dimension. This indicated some work related to fractal property or scale-free can be analyzed using Deng entropy and the results of this paper may stimulate some further research.

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References

[1] A. P. Dempster, "Upper and lower probabilities induced by a multivalued mapping," in *Classic works of the Dempster-Shafer theory of belief functions*, pp. 57–72, Springer, 2008.

[2] G. Shafer, *A mathematical theory of evidence*, vol. 42. Princeton university press, 1976.

[3] J. Zhao and Y. Deng, "Performer selection in human reliability analysis: D numbers approach.," *International Journal of Computers, Communications & Control*, vol. 14, no. 3, 2019.

[4] X. Deng and Y. Deng, "D-ahp method with different credibility of information," *Soft Computing*, vol. 23, no. 2, pp. 683–691, 2019.

[5] F. Xiao, "Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy," *Information Fusion*, vol. 46, pp. 23–32, 2019.

[6] L. Yin, X. Deng, and Y. Deng, "The negation of a basic probability assignment," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 1, pp. 135–143, 2018.

[7] C. E. Shannon, "A mathematical theory of communication," *Bell system technical journal*, vol. 27, no. 3, pp. 379–423, 1948.

[8] C. Tsallis, "Possible generalization of boltzmann-gibbs statistics," *Journal of statistical physics*, vol. 52, no. 1-2, pp. 479–487, 1988.

[9] T. Wen and Y. Deng, "The vulnerability of communities in complex networks: An entropy approach," *Reliability Engineering & System Safety*, vol. 196, p. 106782, 2020.

[10] T. Wen and Y. Deng, "Identification of influencers in complex networks by local information dimensionality," *Information Sciences*, vol. 512, pp. 549–562, 2020.

[11] Y. Deng, "Deng entropy," *Chaos, Solitons & Fractals*, vol. 91, pp. 549–553, 2016.

[12] J. Abellán, "Analyzing properties of deng entropy in the theory of evidence," *Chaos, Solitons & Fractals*, vol. 95, pp. 195–199, 2017.

[13] X. Gao and Y. Deng, "The pseudo-pascal triangle of maximum deng entropy.," *International Journal of Computers, Communications & Control*, vol. 15, no. 1, 2020.

[14] G. J. Klir and M. J. Wierman, *Uncertainty-based information: elements of generalized information theory*, vol. 15. Physica, 2013.

[15] B. Kang and Y. Deng, "The maximum deng entropy," *IEEE Access*, vol. 7, pp. 120758–120765, 2019.

[16] Y. Deng, "The information volume of uncertain informaion: (1) mass function," *viXra preprint*, vol. viXra, p. 2006.0028, 2020.

[17] J. Deng and Y. Deng, "The information volume of uncertain information: (2) fuzzy membership function," *viXra preprint*, vol. viXra, p. 2006.0037, 2020.

[18] T. Wen and Y. Deng, "The information volume of uncertain information: (3) information fractal dimension," *viXra preprint*, vol. viXra, p. 2006.0035, 2020.