The theoretical average encoding length for micro-states in Boltzmann system based on Deng entropy

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Abstract

Because of the good performance of handling uncertainty, Dempster-Shafer evidence theory (evidence theory) has been widely used. Recently, a novel entropy, named as Deng entropy, is proposed in evidence theory, which is a generalization of Shannon entropy. Deng entropy and the maximum Deng entropy have been applied in many fields due to their efficiency and reliability of measuring uncertainty. However, the maximum Deng entropy lacks a proper explanation in physics, which limits its further application. Thus, in this paper, with respect to thermodynamics and Shannon's source coding theorem, the theoretical average encoding length for micro-states in Boltzmann system based on Deng entropy is proposed, which is a possible physical interpretation of the maximum Deng entropy.

Index Terms

Shannon's source coding theorem, average encoding length, Dempster-Shafer evidence theory, Deng entropy, Boltzmann system, thermodynamics.

I. INTRODUCTION

In the past decades, Plenty of theories have been developed for expressing and dealing with the uncertainty in the uncertain environment, for instance, probability theory [1], fuzzy set theory [2], Dempster-Shafer evidence theory [3], [4], rough sets [5], and D numbers [6].

Dempster-Shafer evidence theory (evidence theory) has been widely applied in many fields, like uncertainty measurements [7]–[9], data fusion [10]–[12], decision making [13], complex networks [14], [15], and so on [16]. However, there are still some issues to be solved in evidence theory. Among them, how to measure the uncertainty in evidence theory has attracted much attention. A lot of uncertainty measurements in evidence theory have been developed, such as Jousselme's AM [17], Harmanec's AU [18], Hohle's confusion [19], Yager's dissonance [20].

Entropy is one of the methods for measuring uncertainty, which can be extended to measure the uncertainty degree in evidence theory. Since firstly derived from thermodynamics, different kinds of entropy have been proposed, such as Shannon entropy [21], Tsallis entropy [22], nonadditive entropy [23], and so on [24], [25]. A comparative analysis of various entropy is discussed in [26]. Moreover, entropy has been widely applied in real practice, like decision making [27], [28], uncertainty measuring [29], stochastic signal processing [30]–[32], source encoding [33]–[35], data compression [36], and quantum communications [37].

Recently, a new entropy, called Deng entropy [38], is presented for measuring the uncertainty in evidence theory. Deng entropy is the generalization of Shannon entropy. Compared with

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traditional methods, Deng entropy is more reasonable, and it takes both discord and non-specificity into account. Because of these efficiency, Deng entropy has various applications, such as data fusion [39]–[41], decision making [42], [43], pattern classification [44], and so on [45], [46]. Moreover, based on the maximum entropy principle, the maximum form of Deng entropy, named as the maximum Deng entropy [47], is proposed, whose properties are analyzed in [48]. However, there are some issues of the maximum Deng entropy, especially lacking of explanation in physics, which is a limitation for its wider application.

To explain the maximum Deng entropy properly, some researches have been done. For example, Zhu, Chen and Kang interpret the maximum Deng entropy from a statistical point of view [49].

In this paper, based on Deng entropy, the theoretical average encoding length for micro-states in Boltzmann system is presented, which uncovers the physical explanation of the maximum Deng from the perspective of thermodynamics and Shannon's source coding theorem. Moreover, a simplified form of the maximum Deng entropy is defined, which is more convenient to be calculated. In addition, a Boltzmann system constrained by a special limitation is proposed. In this system, the number of microscopic states of a particular degenerate energy level is analyzed.

To summarize, the contributions of this paper are as follows:

1) The theoretical average encoding length for micro-states in Boltzmann system based on Deng entropy is proposed.

2) A simplified form of the maximum Deng entropy is defined.

3) In Boltzmann system, the number of microscopic states corresponding to a particular degenerate energy level is discussed.

4) The theoretical average encoding length and the efficiency of the simplified maximum Deng entropy are illustrated by some numerical examples.

The rest of this paper is organized as follows. In section II, some preliminaries are briefly reviewed. In section III, from the point of thermodynamics and Shannon's source coding theorem, we propose the theoretical average encoding length for micro-states in Boltzmann system based on Deng entropy. In section IV, numerical examples are expounded to illustrate the theoretical average encoding length and the simplified form of the maximum Deng entropy. In section V, we have a brief conclusion.

II. Preliminaries

In this section, some preliminaries are briefly introduced including Dempster-Shafer evidence theory, Deng entropy and the maximum Deng entropy.

A. Dempster-Shafer evidence theory

Dempster-Shafer evidence theory [3], [4] can be used to deal with uncertainty. Besides, evidence theory satisfies the weaker conditions than the probability theory, which provides it with the ability to express uncertain information directly. Some basic conceptions of evidence theory are given as follows:

Definition 2.1: *Frame of discernment and its power set*

Let Θ, called the frame of discernment, denote an exhaustive nonempty set of hypotheses, where the elements are mutually exclusive. Let the set Θ have N elements, which can be expressed as:

$$\Theta = \{\theta_1, \theta_2, \theta_3, \cdots, \theta_N\} \tag{1}$$

The power set of Θ*, denoted as* 2^Θ*, contains all possible subsets of* Θ *and has* 2^N *elements, and* 2^Θ *is represented by*

$$\begin{aligned} 2^\Theta &= \{A_1, A_2, A_3, \cdots, A_{2^N}\} \\ &= \{\ \emptyset, \{\theta_1\}, \{\theta_2\}, \cdots, \{\theta_N\}, \{\theta_1, \theta_2\}, \\ &\qquad \{\theta_1, \theta_3\}, \cdots, \{\theta_1, \theta_N\}, \cdots, \Theta\ \} \end{aligned} \tag{2}$$

where the element A_i *is called the focal element of* Θ*, if* A_i *is nonempty.*

Definition 2.2: *Basic probability assignment (BPA)*

A BPA is a mass function mapping m *from* 2^Θ *to* $[0,1]$*, and it is defined as follows:*

$$m : 2^\Theta \to [0,1] \tag{3}$$

which is constrained by the following conditions:

$$\sum_{A \in 2^\Theta} m(A) = 1 \tag{4}$$

$$m(\emptyset) = 0 \tag{5}$$

Definition 2.3: *Belief function and plausibility function*

Based on the BPA, the belief function $Bel(A)$ *is a mapping:* $2^\Theta \to [0,1]$*, which is defined as follows:*

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{6}$$

The plausibility function $Pl(A)$ *is a mapping:* $2^\Theta \to [0,1]$ *defined as follows:*

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B) \tag{7}$$

It should be noted that $Bel(A)$ and $Pl(A)$ are respectively the lower and upper degree of support for proposition A.

Definition 2.4: *Dempster's rule of combination*

Given two BPAs m_1 *and* m_2 *from two different evidence sources, the Dempster rule of combination, or the orthogonal sum of* m_1 *and* m_2*, is defined as:*

$$\begin{cases} m(A) = \frac{\sum_{B \cap C = A} m_1(B) \cdot m_2(C)}{1 - K(m_1, m_2)} & A \neq \emptyset \\ m(\emptyset) = 0 \end{cases} \tag{8}$$

where $K(m_1, m_2)$ *is the degree of conflict between* m_1 *and* m_2*, and it is defined as follows:*

$$K(m_1, m_2) = \sum_{B \cap C = \emptyset} m_1(B) \cdot m_2(C). \tag{9}$$

It worth noting that Dempster's rule of combination can only be used to combine such two BPAs, when $0 < K(m_1, m_2) < 1$.

B. Deng entropy

In information theory, entropy can be used to measure the uncertainty of a system. Recently, a novel entropy, named as Deng entropy [38], is proposed to measure the uncertainty in evidence theory.

Definition 2.5: *Deng entropy*

Deng entropy is defined as:

$$H_{DE}(m) = -\sum_{A\in 2^{\Theta}} m(A)\log(\frac{m(A)}{2^{|A|}-1}) \tag{10}$$

where $|A|$ is the cardinal of a certain focal element A.

Deng entropy is the generalization of Shannon entropy. When every focal element is singleton, Deng entropy degenerates into Shannon entropy.

Through a simple transformation, Eq.(10) can be rewritten as follows:

$$H_{DE}(m) = \sum_{A\in 2^{\Theta}} \log(2^{|A|}-1) - \sum_{A\in 2^{\Theta}} m(A)\log m(A) \tag{11}$$

where $\sum_{A\in 2^{\Theta}} \log(2^{|A|}-1)$ and $-\sum_{A\in 2^{\Theta}} m(A)\log m(A)$ are measurements of nonspecificity and discord, respectively. As a result, Deng entropy is a composite measurement of nonspecificity and discord, which means that it is a tool for measuring total uncertainty.

C. The maximum Deng entropy

Assume A is the focal element of a certain frame of discernment Θ and $m(A)$ is the BPA for A. According to [47], the analytic solution of the maximum Deng entropy and the conditions of BPA distribution is as follows:

Theorem 2.1: *The analytic solution of the maximum Deng Entropy and its BPA distribution*

If and only if $m(A) = \frac{(2^{|A|}-1)}{\sum_{A\in 2^{\Theta}}(2^{|A|}-1)}$, Deng entropy reaches its maximum value, and the analytic solution of the maximum Deng entropy is

$$H_{MDE}(m) = \log\sum_{A\in 2^{\Theta}}(2^{|A|}-1) \tag{12}$$

III. The theoretical average encoding length for micro-states in Boltzmann system based on Deng entropy

Since firstly derived from thermodynamics, different kinds of entropy have been proposed, such as Shannon entropy in information science [21], and Boltzmann-Gibbs entropy in thermodynamics [50]. Moreover, there are many connections between information science and thermodynamics [51]–[53].

In this section, inspired by thermodynamics and Shannon's source coding theorem, we propose the theoretical average encoding length for micro-states in Boltzmann system based on Deng entropy. The process is divided into three subsections. Firstly, we introduce a new expression of the maximum Deng entropy. Next, we assume a particular energy level in the Boltzmann system, and calculate the total number of microscopic states in this energy level. Finally, based on Deng entropy, the theoretical average encoding length is presented, which can encode micro-states corresponding to a particular energy level in the Boltzmann system.

A. A simplified expression of the maximum Deng entropy

According to [47], the analytic solution of the maximum Deng entropy is that

$$H_{MDE}(m) = \log \sum_{A \in 2^{\Theta}} (2^{|A|} - 1) \tag{13}$$

where Θ is the frame of discernment. However, this solution is a little bit complicated, which can be further simplified.

Theorem 3.1: *Given a frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \cdots, \theta_N\}$, the analytic solution of the maximum Deng entropy can be simplified as:*

$$H_{MDE}(m) = \log(3^N - 2^N) \tag{14}$$

where N is the cardinal of the frame of discernment.

For proofing **Theorem 3.1**, an important lemma should be introduced:

Lemma 3.1: *Let C_n^m denote combination which equals to $\frac{n!}{m!(n-m)!}$, where m and n are non-negative integers. Let a and b be non-negative integers. Then,*

$$(a+b)^N = \sum_{i=0}^{N} (C_N^i \times a^i \times b^{N-i}) \tag{15}$$

Proof 3.1: *Proof for theorem 1*

Let C_n^m denote combination, and the cardinal of frame of discernment be N. Then, the derivation process is as follows:

$$\begin{aligned} H_{MDE}(m) &= \log \sum_{A \in 2^{\Theta}} (2^{|A|} - 1) \\ &= \log \sum_{i=1}^{N} C_N^i (2^i - 1) \\ &= \log \sum_{i=1}^{N} (C_N^i 2^i - C_N^i) \qquad (16) \\ &= \log \sum_{i=0}^{N} ((C_N^i 2^i - C_N^0 2^0) - (C_N^i - C_N^0)) \\ &= \log \sum_{i=0}^{N} (C_N^i 2^i - C_N^i) \qquad (17) \end{aligned}$$

It should be noted that the index i of Eq.(16) and Eq.(17) is not identical. According to **Lemma 3.1**, Eq.(17) can be transformed and simplified as:

$$
\begin{aligned}
&\log(\sum_{i=0}^{N} C_N^i \times 2^i \times 1^{N-i} - \sum_{i=0}^{N} C_N^i \times 1^i \times 1^{N-i}) \\
=&\log((2+1)^N - (1+1)^N) \\
=&\log(3^N - 2^N)
\end{aligned} \tag{18}
$$

Therefore, the simplified maximum Deng entropy is obtained.

B. A particular energy level in the Boltzmann system and its total number of microscopic states

Boltzmann distribution [54] is a crucial conception in thermodynamics. It describes the probability of a certain state that a thermal equilibrium system will be in, as a function of the energy and the temperature of that system. This distribution shows the fact that a lower energy state always has a higher probability of being occupied.

Moreover, such system is called the Boltzmann system, whose implications are wide-ranging. It can range from a microscopic system like a atom to a macroscopic system like a society.

Because of the wide meaning of the Boltzmann system, it has been widely applied in many fields to solve various problems, such as information science [55]–[57], deep learning [58]–[60], and economics [61]–[63].

To better understand the conceptions of the Boltzmann system, some physical symbols and their physical meaning should be introduced, which are shown in TABLE I.

TABLE I: Physical symbols and their physical meaning

Symbol	Physical meaning
Ω	The total number of microscopic states.
N	The total number of particles in the Boltzmann system.
a_k	The number of particles at the kth energy level
ω_k	The degeneracy of the kth energy level

According to [54], the total number of microscopic states in a Boltzmann system is

$$
\Omega = \frac{N!}{\prod_k a_k!} \prod_k \omega_k^{a_k} \tag{19}
$$

where $\omega_k^{a_k}$ is the number of microscopic states corresponding to the kth degenerate energy level, whose degeneracy is ω_k. And $\frac{N!}{\prod_k a_k!}$ is the total number of microscopic states in the Boltzmann system, under the condition that the degeneracy of every energy level is 1. It should be noted that, in Boltzmann system, every particle is distinguishable, and every quantum state is different. As a result, the particles and quantum states can be marked by different index. The model of Boltzmann system is illustrated as Fig. 1.

If a Boltzmann system is affected by external conditions, such as magnetic field, pressure or heat, the quantum states of the particles at every energy level would be changed. When the system reaches thermal equilibrium, the number of microscopic states corresponding to every energy level would no longer be $\omega_k^{a_k}$.

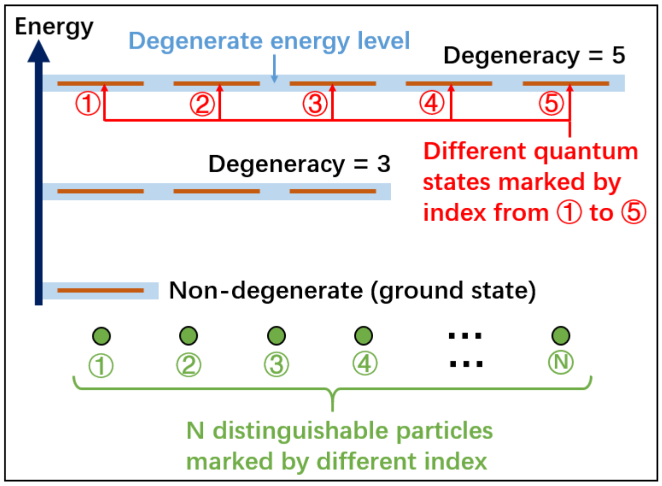

Fig. 1: The model of the Boltzmann system

Because the external conditions, such as heat, can be precisely controlled by researchers through apparatus, such as thermometer, researchers could precisely control the Boltzmann system and impose some limits on the microscopic states of it.

Suppose the limitation is that, at every energy level, when the system reaches thermal equilibrium, there should be at least one particle in one particular quantum state.

An example is given to better understand the meaning of the limitation, and the system under the condition of this limitation.

Example 3.1: Assume a particular degenerate energy level contains five quantum states marked from 1 to 5, and the 2nd quantum state is the particular quantum state. Consider that, in thermal equilibrium, this energy level contains N distinguishable particles. Under the limitation, there is at least one particle in the 2nd quantum state.

This example is shown in Fig. 2, where (a) is the non-equilibrium state of the energy level and (b) is the equilibrium state of the energy level.

Assume there is a Boltzmann system under the control of that limitation, one of whose degenerate energy level contains three different quantum states, which means the degeneracy of the energy level is 3. When the Boltzmann system is in thermal equilibrium, there are N distinguishable particles at this energy level.

Then, we calculate the number of microscopic states corresponding to this degenerate energy level by considering two scenarios.

Scenario A: If there was no particle in one particular quantum state (such as the 2nd state), all the particles of this energy level would in the rest of the two quantum states. So, the number of microscopic states in the energy level would be 2^N.

Scenario B: If the system was not limited, all the particles of this energy level could be in any one of the three quantum states. Thus, the number of microscopic states in the energy level would be 3^N.

These two scenarios are illustrated in Fig. 3, where (a) and (b) show the model of **Scenario A** and **Scenario B**, respectively.

Constrained by the limitation, the Boltzmann system can be seen as **Scenario B** minus **Scenario A**. As a result, the number of microscopic states corresponding to this degenerate energy level is $3^N - 2^N$.

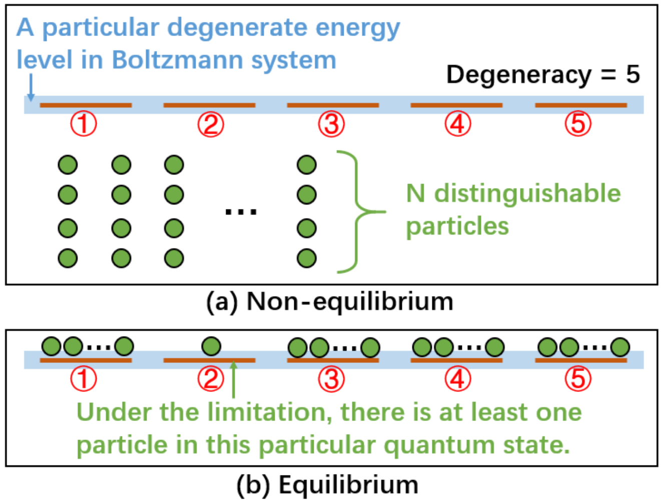

Fig. 2: The model of the degenerate energy level under the condition of the limitation

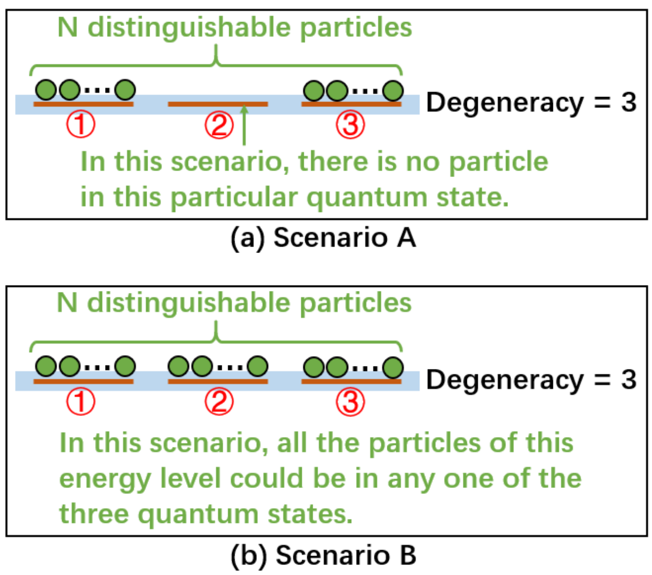

Fig. 3: The model of the two scenarios

C. The theoretical average encoding length for micro-states in Boltzmann system

In information theory, Shannon entropy [21] is widely used for quantifying the volume of information and measuring uncertainty in a system. Shannon entropy of a discrete random variable $X = \{x_1, x_2, \ldots, x_N\}$ with probability mass function $P(X)$ is defined as

$$H_S(P) = -\sum_{i=1}^{N} P(x_i) \log_b P(x_i) \tag{20}$$

where $p_i = P(x_i)$ is the probability of $x_i \in X$, b is the base of the logarithm used, and N is the total number of basic states. When b equals to 2, e or 10, the corresponding unit of Shannon entropy is bits, nats or bans.

When all the $p_i \in P(X)$ are equal to each other, Shannon entropy reaches its maximum, namely, the maximum Shannon entropy:

$$H_{MS}(P) = H_{MS}\left(\frac{1}{N}, \dots, \frac{1}{N}\right) = \log_b N \tag{21}$$

Moreover, Shannon's source coding theorem makes the limitation to possible information compression, and explain the physical meaning of the maximum Shannon entropy. To be more specific, the optimal encoding length for a input symbol is $\log_b \frac{1}{P}$, where P is the probability of that input symbol and b is the number of output symbols for encoding. If there are N input symbols and each of them appears equally, with 2 output symbols, the theoretical average encoding length for a input symbol is $\log_2 N$.

To better understand this, an example is given to show the explanation of the maximum Shannon entropy.

Example 3.2: Assume there are 32 different boxes and one ball is in one of the 32 boxes. If the owner of the boxes is only willing to answer "yes" or "no" to any question, for the purpose of knowing which box contains the ball, how many questions do we need to ask at most? This example is illustrated in Fig. 4.

Fig. 4: 32 different boxes with one ball

Essentially, this example is about encoding 32 input symbols by 2 output symbols. The two answers can be represented by 2 output symbols. Because there are 32 different boxes with one ball, the number of the basic states is 32, which means that there are 32 input symbols. Since every box can contain the ball, without prior information, each basic state has the equal probability to appear. Hence, this example can be solved by the maximum Shannon entropy:

$$H_{MS}(P) = \log_2 32 = 5 \tag{22}$$

As a result, we should at least use 5 bits on average to represent every different basic state, which means that we need to ask at most 5 questions to know which box has the ball.

Inspired by Shannon's source coding theorem, the maximum Deng entropy can also be seen as a way of encoding. Concretely, since the form of the maximum Deng entropy is that

$$H_{MDE}(m) = \log_b(3^N - 2^N) \tag{23}$$

where b is the base of logarithm, this form can be interpreted as using b output symbols to encode $3^N - 2^N$ different microscopic states corresponding to a particular degenerate energy level in the Boltzmann system (input symbols).

As a result, the inherent physical interpretation of the maximum Deng entropy is the theoretical average encoding length for one of the $3^N - 2^N$ microscopic states corresponding to a particular degenerate energy level in Boltzmann system.

IV. Numerical examples and Discussions

In this section, some examples are expounded to illustrate the simplified form of the maximum Deng entropy and the theoretical average encoding length for micro-states in Boltzmann system. The discussion is followed after every example. In the following examples, the base of the logarithmic function is 2.

Example 4.1: Consider that the cardinal of frame of discernment N changes from 1 to 10. Then, the value of $3^N - 2^N$, the simplified maximum Deng entropy and the maximum Deng entropy changing with N are shown in TABLE II.

TABLE II: The value of $3^N - 2^N$, the simplified maximum Deng entropy and the maximum Deng entropy under the condition of different N.

N	$3^N - 2^N$	$\log_2(3^N - 2^N)$	$\log_2 \sum_{A \in 2^\Theta}(2^{\lvert A\rvert} - 1)$
1	1	0	0
2	5	2.32193	2.32193
3	19	4.24793	4.24793
4	65	6.02237	6.02237
5	211	7.72110	7.72110
6	665	9.37721	9.37721
7	2059	11.00773	11.00773
8	6305	12.62228	12.62228
9	19171	14.22664	14.22664
10	58025	15.82439	15.82439

This example further proves that the simplified maximum Deng entropy is identical to the maximum Deng entropy. Besides, it illustrates that the simplified form is more convenient to be calculated compared with the maximum Deng entropy. In addition, it shows that, with the changing of N, the number of the microscopic states corresponding to a particular degenerate energy level rises faster and faster.

Example 4.2: Assume that there are 2 distinguishable particles at a particular degenerate energy level, when the Boltzmann system is in thermal equilibrium. Then, under the limitation, the number of the microscopic states corresponding to that energy level can be calculated as $3^2 - 2^2 = 5$. Assume that the 2nd quantum state is the particular quantum state. These five different microscopic states can be marked by serial number from No.1 to No.5, which are shown in Fig. 5.

Using the simplified maximum Deng entropy, the result is that $\log_2 5 = 2.32193$, which means that we should at least use 2.32193 bits on average to represent every different microscopic state. Taking Huffman code for example, the binary code corresponding to the serial number of microscopic states are shown in TABLE III.

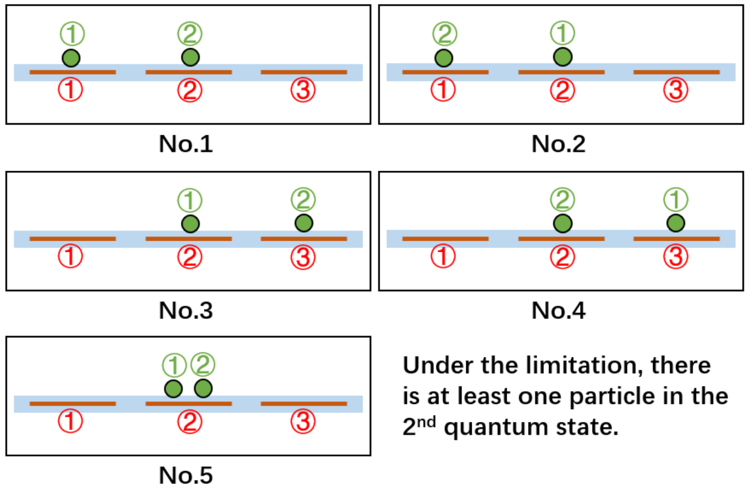

Fig. 5: Five microscopic states marked by serial number from No.1 to No.5

TABLE III: The Huffman binary code corresponding to the serial number of microscopic states.

Serial number	Binary code
No. 1	0 0
No. 2	0 1
No. 3	1 0
No. 4	1 1 0
No. 5	1 1 1

In this example, based on Huffman code, the actual average encoding length for a microscopic state is

$$(2 + 2 + 2 + 3 + 3) \div 5 = 2.4 \tag{24}$$

This actual average encoding length is larger than the theoretical average encoding length 2.32193 calculated by the maximum Deng entropy, which further proves that, the physical interpretation of the maximum Deng entropy is the theoretical average encoding length for one of the $3^N - 2^N$ microscopic states corresponding to a particular energy level in Boltzmann system. We can not use less than $\log_b(3^N - 2^N)$ bits on average to represent every different microscopic state.

To better understand **Example 4.2**, the process of this example is summarized in Fig. 6.

V. Conclusion

In this paper, inspired by thermodynamics and Shannon's source coding theorem, the theoretical average encoding length for micro-states in Boltzmann system based on Deng entropy is presented, which uncovers that the possible inherent physical explanation of the maximum Deng entropy is the theoretical average code length for encoding of microscopic states corresponding to a particular degenerate energy level in Boltzmann system. Moreover, a simplified form of the

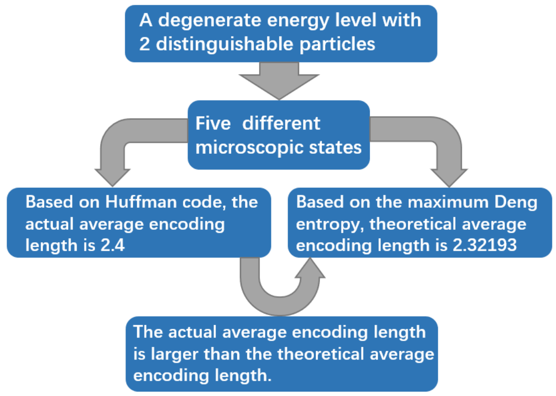

Fig. 6: The process of example 4.2

maximum Deng entropy is defined, which is more convenient to be calculated. In addition, a Boltzmann system constrained by a special limitation is proposed. In this system, the number of microscopic states of a particular degenerate energy level is analyzed. Some numerical examples are presented to illustrate the simplified maximum Deng entropy, and the relationship between theoretical value and actual value of encoding length. This paper establishes the relationship of Deng entropy, quantum physics and thermodynamics, which provides Deng entropy with a promising way to measure uncertainty in quantum field.

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