# A geometrical proof of Ptolemy's theorem

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**Abstract**

A geometrical proof of Ptolemy's theorem is presented. It shows the equality of the sum of the areas of the rectangles formed from the lengths of opposite sides of a cyclic quadrilateral to be equal to the area of the rectangle formed from the lengths of the diagonals. Introducing symmetry by choosing one of the component triangles of the quadrilateral to be an equilateral triangle, we prove the theorem for different cases. We then show that the specific case of maximum area configuration corresponds to that of a kite. By changing the kite configuration to that of a rectangle, we derive Pythagoras theorem as a special case of Ptolemy's theorem.

**Keywords**

Ptolemy's theorem, cyclic quadrilateral, geometrical proof, equilateral triangle, kite configuration, rectangles, Pythagoras theorem

**Introduction**

In a recent article[1] we presented a geometrical proof of an application of Ptolemy's theorem. There, the emphasis was laid on showing the sum of two distances DA, DB from a point D, lying on the circumcircle of the equilateral triangle ABC, to the apexes A, B is equal to the longest distance DC to the third vertex of the triangle. The sum of the two distances varies between a minimum, equal to the length of the side of the triangle, to a maximum, equal to the length of the diameter of the circle. Shirali's algebraic proof of Ptolemy's theorem[2] forms the basis of this geometrical proof. For an inspiring proof of the theorem I refer the reader to Zwezda's youtube videoclip[3]. Her proof is based on inversion.

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In this article we present an elegant proof of Ptolemy's theorem, laying emphasis on showing the sum of the areas of the rectangles formed from the opposite sides of a cyclic quadrilateral ABCD being equal to the area of the rectangle formed from its diagonals.

**Statement of Ptolemy's theorem**

Ptolemy's theorem states that the sum of the products of the opposite pairs of sides of a cyclic quadrilateral is equal to the product of the diagonals[1].

1. **Geometrical proof of Ptolemy's theorem**

Let us consider a convex quadrilateral ABCD along with its circumcircle (see Fig. 1). Applied to this cyclic quadrilateral, the statement of Ptolemy's theorem becomes:

$$AB.DC + AD.BC = AC.BD \qquad (1)$$

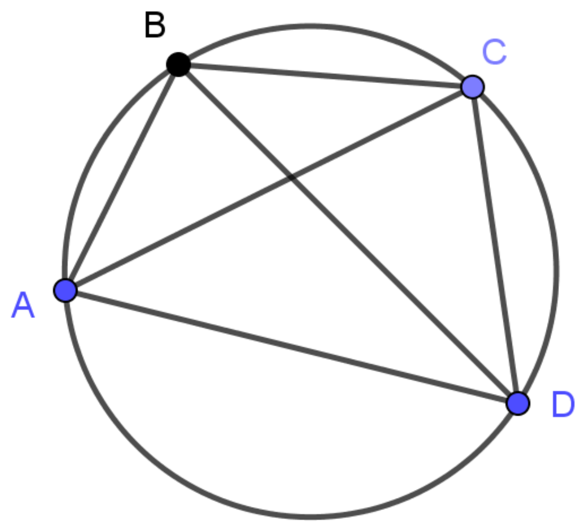


Fig. 1. ABCD is a cyclic quadrilateral. Ptolemy's theorem says, AB.DC + AD.BC = AC.BD.

We now construct the rectangle ABEF on the side AB of width AF = DC (see Fig. 2). Similarly, we construct the rectangle BCGH on the side BC of width BH = AD. Finally, we construct the rectangle ACIJ on the diagonal AC of width AJ = BD.

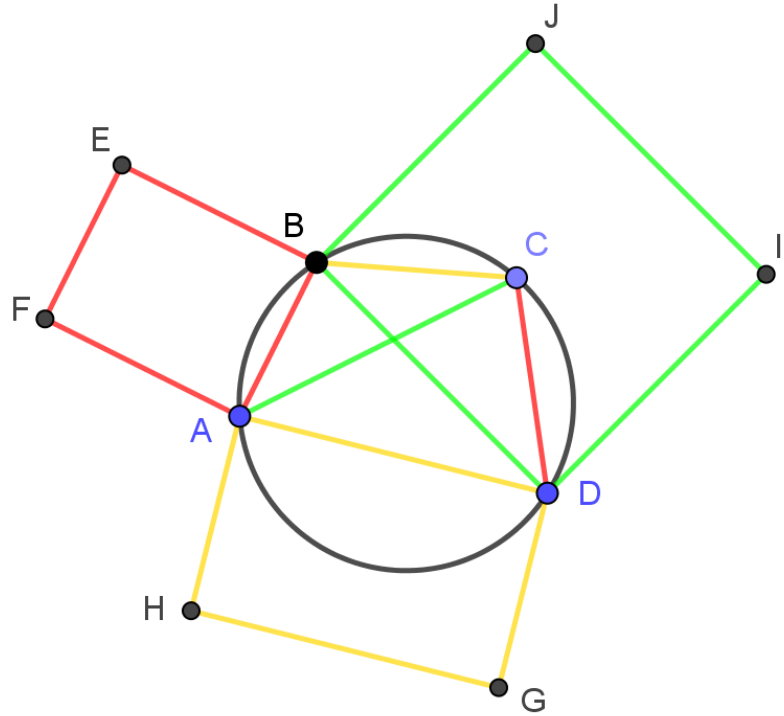


Fig. 2. Rectangle ABEF is constructed with sides AB and AF(=DC).
Rectangle ADGH is constructed with sides AD and AH (=BC).
Rectangle BDIJ is constructed with sides BD and BJ (=AC).

Ptolemy's theorem says that the sum of the areas of the two smaller rectangles ABEF, ADGH is equal to the area of the largest rectangle BDIJ.

It is difficult to see from the Fig. 2 that the sum of the areas of the smaller rectangles is equal to the area of the biggest rectangle. To make the visualization of equality of the areas we need to locate a point P on the diagonal BD that helps divide the rectangle on the diagonal into two rectangles the areas of which are equal to the areas of rectangles on the sides of the quadrilateral.

P is located on BD such that angle BCA is equal to angle PCD (see Fig. 3). We see from Fig. 3 that triangles ABC and PCD are similar; as also triangles BCP and ACD. Therefore, we get, (BP. AC) = (BP . BJ. ), (PD . AC) = (PD . PK).

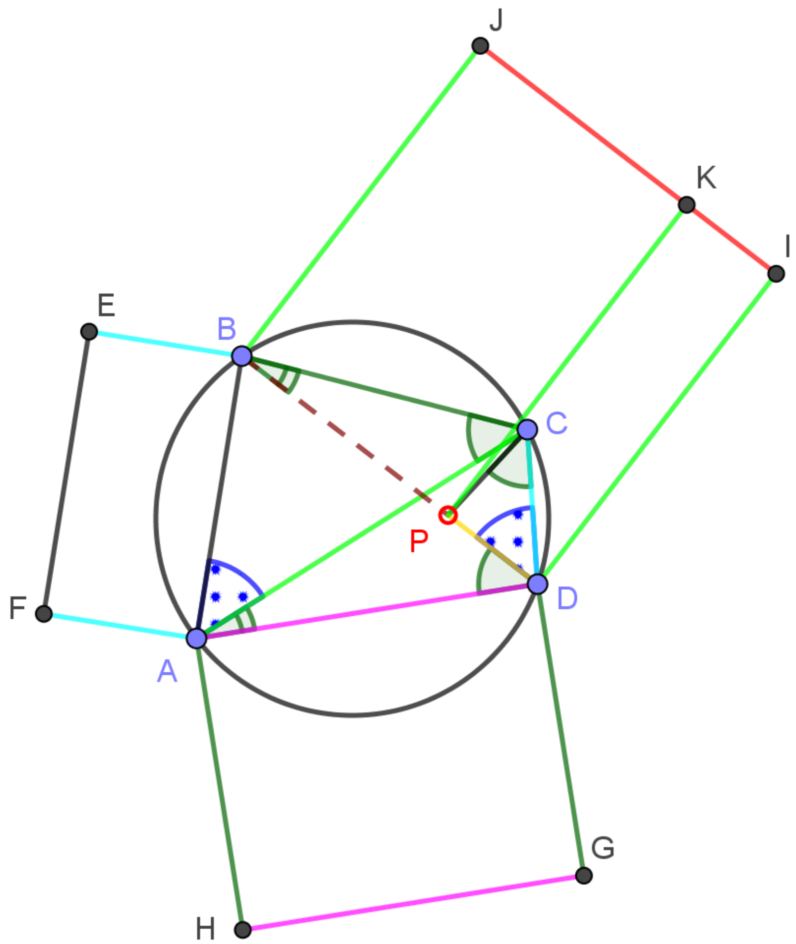


Fig. 3. Point P is chosen on BD such that Angle ABD = Angle PCD.. Rectangles are constructed on sides BP and PD and diagonal BD each of width AC. Area BPKJ = Area ADGH; Area ABEF = Area PDIK.

This difficulty of seeing the equality of areas in Fig. 2 can also be overcome by making the component triangle ABC, of the quadrilateral, an equilateral triangle. When we do this, we can see the equality of areas clearly.

In Fig. 4 we show the component equilateral triangle ABC of the quadrilateral ABCD with its circumcircle and the point D – the fourth vertex of the quadrilateral. We make D a movable point along the circle.

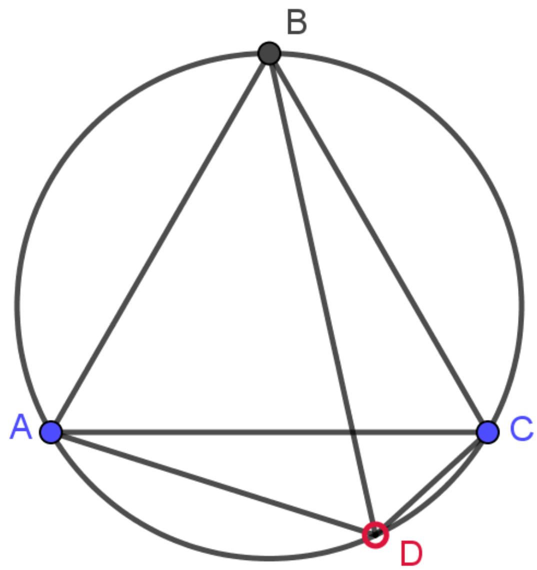


Fig. 4. ABC is an equilateral triangle. ABCD is a cyclic quadrilateral shown with the circumcircle.

We locate P on BD by constructing an equilateral triangle of side DC. P falls on BD[2]. Now we can construct the rectangles on sides AB, AD as also on the diagonal BD (see Fig. 5) as we did earlier. We divide the rectangle BDIJ into two rectangles, BPKJ and PKJD using P. The width of all these rectangles is the same and is equal to the length of the side of triangle ABC. It is this equality of widths that introduces the simplicity in displaying the equality of areas. A theorem on application of Ptolemy's theorem[1,2] shows us that the sum of the sides AD and DC is equal to the length of the diagonal BD.

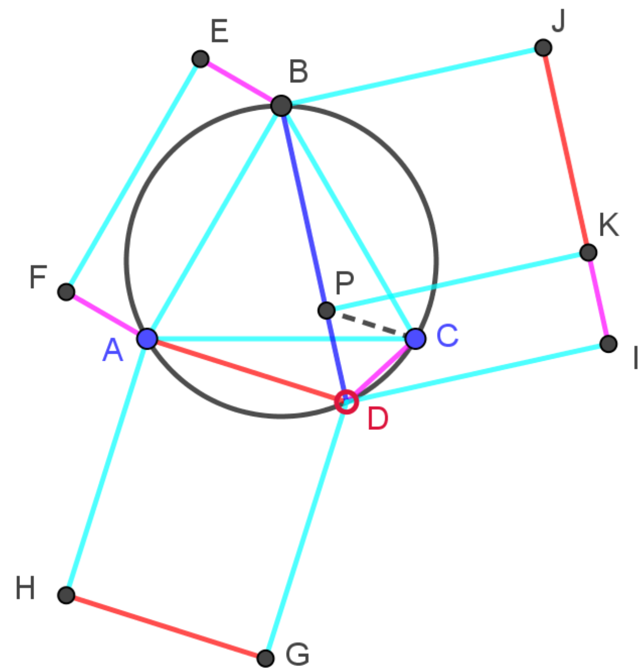


Fig. 5 Rectangles are constructed on sides AB, AD and diagonal BD of the quadrilateral. All these rectangles have one side equal to length of the side of the equilateral triangle ABC. The rectangle on the diagonal is divided into two rectangles of areas equal to the area of rectangles on the sides of the quadrilateral,. DC = DP = BE, AD = BP= JK , (AD + DC = BP + PD = BD).

Since D is a movable point on the circumcircle, we move it to the opposite end of the diameter through B. We get the kite ABCD (see Fig. 6). ABD is a right triangle, so is triangle BCD. The configuration in Fig. 6 is the configuration of maximum area (since BD is maximum when it is diameter).

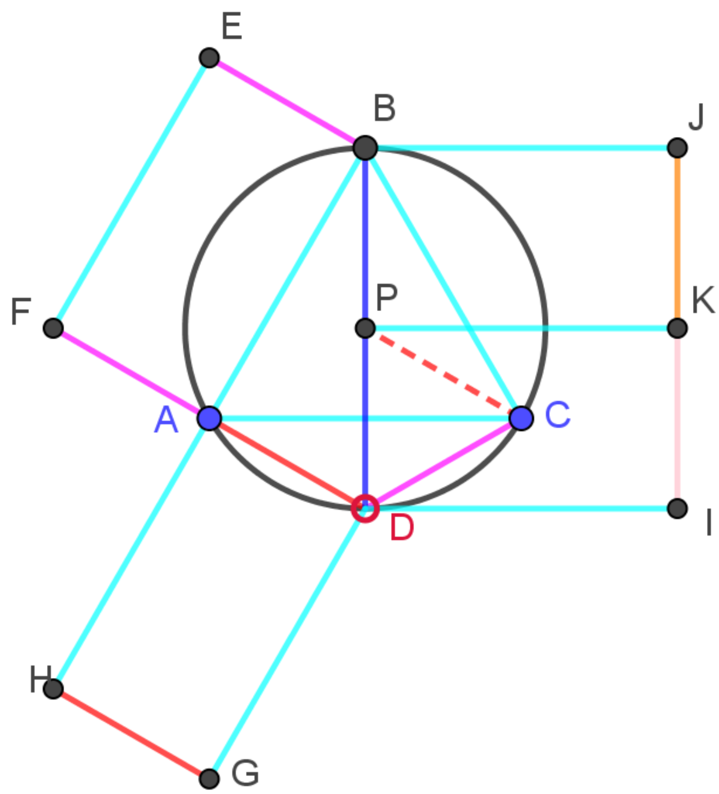


Fig. 6. When BD becomes the diameter, the quadrilateral ABCD assume s the confuguration of a kite.

We redraw the kite in the form of a rectangle ABCD by moving the equal adjacent sides to form the opposite sides (see Fig. 7). We now apply Ptolemy's theorem to the rectangle ABCD.

$$AB \times CD + AD \times BC = AC \times BD \qquad (2)$$

$$AB \times AB + AD \times AD = AC \times AC \qquad (3)$$

$$AB^2 + AD^2 = AC^2 \qquad (4)$$

Equation (4) is a statement of Pythogorus theorem.

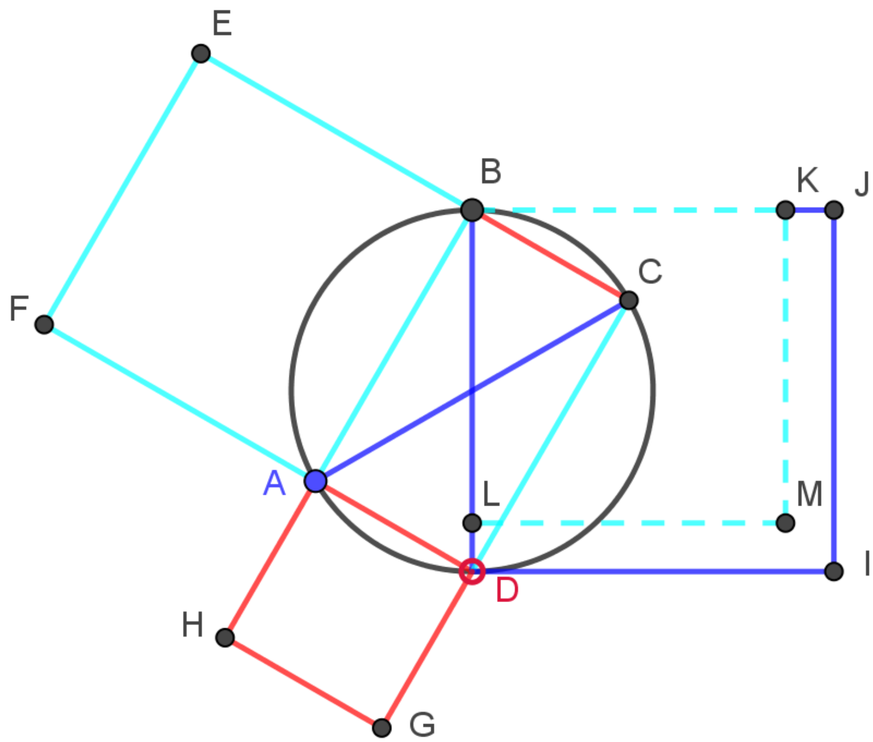


Fig. 7. ABCD is the rectangular configuration of the kite. ABC and ADC are right angled triangles. Ptolemy's theorem applied to the rectangle gives Pythorus theorem.

Thus, we find Pythogorus theorem is a special case of Ptolemy’s theorem. It is difficult to see the equality of areas even in this rectangular configuration whereas it is easy to see that in kite configuration on Fig. 6.

For the sake of completion, we now move D to coincide with an apex of the triangle ABC (see Fig. 8).

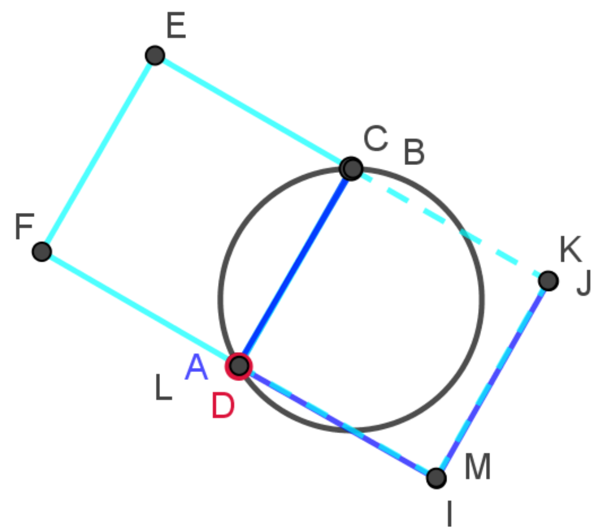


Fig. 8. D is moved to coincide with the apex A of the equilateral triangle ABC. B moves to coincide with C. The cyclic rectangle ABCD becomes a chord of the circle and assumes the minimum area configuration.

The quadrilateral assumes the configuration of the chord AC. This is the configuration of minimum area.

$$AB \times CD + AD \times BC = AC \times BD \qquad (5)$$

$$AB \times AB + 0 = AC \times BK = AC \times AC \qquad (6)$$

$$AB^2 = AC^2 \qquad (7)$$

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**References**

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2. Shailesh Shirali, At Right angles, November (2016)
3. https://www.youtube.com/watch?v=bJOuzqu3MUQ