ATTRACTIVITY FOR DIFFERENTIAL EQUATIONS SYSTEMS OF FRACTIONAL ORDER

J. VANTERLER DA C. SOUSA [1], MOUFFAK BENCHOHRA [2,3], AND GASTON M. N'GUEREKATA [4]

Abstract. This paper investigates the overall solution attractivity of the fractional differential equation introduced by the ψ-Hilfer fractional derivative and the Krasnoselskii's fixed point theorem. We highlight some particular cases of the result investigated here, especially involving the Riemann-Liouville and Katugampola fractional derivative, elucidating the fundamental property of the ψ-Hilfer fractional derivative, that is, the broad class of particular cases of fractional derivatives that consequently apply to the results investigated herein.

1. Introduction

Why investigate the existence, uniqueness, stability and attractivity of solutions of fractional differential equations? Over the decades, the theory of fractional differential equations has gained prominence and attention by numerous researchers, due to its theoretical importance and applicability [6, 16, 17, 21, 25, 27, 29, 34, 35, 36]. However, it is not an easy and simple task to know which fractional best operator to use to propose a fractional differential equation and to attack for example the existence and uniqueness of solutions. One way to overcome such a problem is to work with more general fractional operators, especially such as the ψ-Hilfer fractional operator (differentiation) and the Riemann-Liouville fractional operator with respect to another function (integration) [19, 23, 24, 28]. What has been noted is the increasing number of works published in the area of fractional differential equations in recent years, due to the fact that the fractional calculation is well consoled, which consequently allows them to be used in other areas, namely: differential equations [1, 2, 5, 7, 8, 9, 10, 11, 12, 14, 15, 32].

Recently, Sousa e Oliveira, through the ψ-Hilfer fractional operator, has obtained interesting and important results of the existence, uniqueness, stability and attractivity of solutions of fractional differential and integrodifferential equations, by means of the fixed point technique and Gronwall inequality [22, 26, 29, 30, 31]. In 2012, Chen et al. [14], investigated the attractivity of solutions of fractional differential equations towards the fractional derivative of Caputo and Riemann-Liouville. In 2018, Zhou et al. [35], performed excellent work on the existence and attractivity of fractional evolution equations with Riemann-Liouville fractional derivative solutions. They establish sufficient conditions for the global attractivity of mild solutions for the Cauchy problems in the case that semigroup is compact. In the

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same year Zhou [36], did another work on attractivity of solutions for fractional evolution equations with almost sectorial operators, establishing sufficient conditions to obtain the results in cases that semigroup is compact as well as non-compact. We suggest further work for further reading [3, 4, 20].

However, it is also clear that although there is a growing number of scientific papers publications, there are few works in the area involving the attractivity of differential equation solutions, particularly involving the ψ-Hilfer fractional derivative, since an important property worth mentioning is the broad class of particular cases they hold.

So, in order to contribute to the theory of fractional differential equations, in this paper we will consider the nonlinear fractional differential equation.

$$\left\{\begin{array}{rcl} {}^{\mathbf{H}}\mathscr{D}_{t_0+}^{p,q;\psi}\xi_1(t) &=& \mathcal{G}(t,\xi_1(t)),\; t\geq t_0 \\ I_{t_0+}^{1-r;\psi}\xi_1(t_0) &=& \xi_0 \end{array}\right. \tag{1.1}$$

where ${}^{\mathbf{H}}\mathscr{D}_{t_0+}^{p,q;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative of order $0<p<1$, type $0\leq q\leq 1$, and $I_{t_0+}^{1-r;\psi}(\cdot)$ is the $\psi-$Riemann-Liouville fractional integral of order $0<1-r<1$, where $r=p+q(1-p)$ and $\mathcal{G}:[t_0,\infty)\times\mathbb{R}\to\mathbb{R}$.

The main motivation for the elaboration of this paper comes from the above highlighted articles on the attractivity of solutions of fractional differential equations and to provide more general and global results. So, in this sense, we aim to investigate the attractivity of the global solution to the nonlinear fractional differential equation in a Banach space.

This article is written as follows: In Section 2, we present fundamental concepts of weighted continuous function spaces, and their respective norm. In this sense, we present the definitions of integral in the Riemann-Liouville sense with respect to another function and ψ-Hilfer fractional derivative. To close the section, some important results that will help throughout the article are presented. In Section 3, we will investigate the main result of the paper, that is, the attractiveness of solutions to the problem of the nonlinear fractional differential equation. On the other hand, we present some particular cases during the section, in order to highlight the fundamental property that the ψ-Hilfer fractional derivative holds, the broad class of particular cases of fractional derivatives, which in this context investigated the results on attractiveness, will also be valid.

2. Preliminaries

In this section, we will present the space of continuous weight functions, as well as their respective norm. In this sense, we present the fundamental concepts of the Riemann-Liouville of fractional integral with respect to another function and the ψ-Hilfer fractional derivative. On the other hand, some fundamental results and the idea of an attractive global solution, concludes the section.

Denoted by $C^n(J,\Omega)$ the space of functions n-times continuously differentiable on the interval $J:=[a,b]$, with values in Ω a Banach space. The space of weighted functions $C_{1-r,\psi}(J,\Omega)$ of ξ over $J'=(a,b]$, are defined by [27, 29]

$$C_{1-r;\psi}(J,\Omega)=\left\{\xi:J'\to\Omega;\;\;(\psi(t)-\psi(0))^{1-r}\xi(t)\in C(J,\Omega)\right\},\; 0\leq r<1,$$

denoted by the norm

$$\left\|\xi\right\|_{C_{1-r;\psi}}=\left\|\left(\psi\left(t\right)-\psi\left(0\right)\right)^{1-r}\xi\left(t\right)\right\|_{C}=\max_{t\in[a,b]}\left|\left(\psi\left(t\right)-\psi\left(0\right)\right)^{1-r}\xi\left(t\right)\right|.$$

Obviously the $C_{1-r;\psi}(J,\Omega)$ space is a Banach space.

Let (a,b) $(-\infty\leq a<b\leq\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $p>0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). The left-sided fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by [23, 24, 28]

$$\mathcal{I}_{a+}^{p;\psi}\xi\left(t\right)=\frac{1}{\Gamma\left(p\right)}\int_{a}^{t}\psi'\left(s\right)\left(\psi\left(t\right)-\psi\left(s\right)\right)^{p-1}\xi\left(s\right)ds. \tag{2.1}$$

Similarly defined, Riemann-Liouville fractional integral right-sided with respect another function.

On the other hand, let $n-1<p<n$ with $n\in\mathbb{N}$, $I=[a,b]$ is the interval such that $-\infty\leq a<b\leq\infty$ and $\xi,\psi\in C^{n}([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t)\neq0$, for all $t\in I$. The ψ-Hilfer fractional derivative left-sided ${}^{\mathbf{H}}\mathscr{D}_{a+}^{p,q;\psi}\left(\cdot\right)$ of function of order p and type $0\leq q\leq1$, is defined by [23, 24, 28]

$${}^{\mathbf{H}}\mathscr{D}_{a+}^{p,q;\psi}\xi\left(x\right)=\mathcal{I}_{a+}^{q(n-p);\psi}\left(\frac{1}{\psi'\left(t\right)}\frac{d}{dt}\right)^{n}\mathcal{I}_{a+}^{(1-q)(n-p);\psi}\xi\left(t\right). \tag{2.2}$$

Similarly defined, the ψ-Hilfer fractional derivative right-sided.

Theorem 2.1. [23] *If $\xi\in C_{r,\psi}^{n}[a,b]$, $n-1<p<n$ and $0\leq q\leq1$, then*

$$\mathcal{I}_{a+}^{p;\psi}\;{}^{\mathbf{H}}\mathscr{D}_{a+}^{p,q;\psi}\xi\left(t\right)=\xi\left(t\right)-\sum_{k=1}^{n}\frac{\left(\psi\left(t\right)-\psi\left(a\right)\right)^{r-k}}{\Gamma\left(r-k+1\right)}\xi_{\psi}^{[n-k]}\mathcal{I}_{a+}^{(1-q)(n-p);\psi}\xi\left(t\right).$$

Definition 2.2. The zero solution $\xi_{1}\left(t\right)$ of system (1.1) is globally attractive if every solution of (1.1) tends to zero as $t\to\infty$.

The following fixed point theorems, including the improvement of a fixed point theorem of Krasnoselskii **(due to Burton)** and Schauder's fixed point theorem, will be needed in the text.

Theorem 2.3. [13] *Let $\mathbb{S}$ be a nonempty, closed, convex and bounded subset of the Banach space Ω and $\mathcal{A}:\Omega\to\Omega$ and $\mathcal{B}:\mathbb{S}\to\Omega$ be two operators such that*

(1) *$\mathcal{A}$ is a contraction with constant $L<1$;*
(2) *$\mathcal{B}$ is continuous, $\mathcal{B}\mathbb{S}$ resides in a compact subset of Ω;*
(3) *$[\xi_{3}=\mathcal{A}\xi_{1}+\mathcal{B}\xi_{2},\ \xi_{1},\xi_{2}\in\mathbb{S}]\Longrightarrow\xi_{3}\in\mathbb{S}$;*

Then the operator equation $\xi_{1}=\mathcal{A}\xi_{1}+\mathcal{B}\xi_{1}$, has a solution in $\mathbb{S}$.

Theorem 2.4. [18] (Schauder Fixed Point Theorem) *If U is a nonempty closed, bounded convex subset of a Banach space Ω and $T:U\to U$ is completely continuous, then T has a fixed point.*

3. Global attractivity with ψ-Hilfer fractional derivative

In this section, we will address the main result of the article, namely, to discuss the attractivity of solutions for the system (1.1) introduced through the ψ-Hilfer fractional derivative. In addition, we will clarify the importance of investigating properties of fractional differential equation solutions with the ψ-Hilfer fractional operator, since they hold a wide class of particular cases preserving their properties.

Before we start investigating the main results of the article, let us assume that $f(t,\xi_1)$ satisfies the following condition:

(H_0) $f(t,\xi_1(t))$ is Lebesgue measurable with respect to t on $[t_0,\infty)$, **and** $\exists p_1 \in (0,p)$ **(p_1 constant) such that** $\displaystyle\int_{t_0}^{h} |f(t,\xi_1(t))|^{1/p_1}\,dt < \infty$ **for all** $t_0 < h < \infty$ **and** $f(t,\xi_1(t))$ **is** continuous with respect to ξ_1 on $[t_0,\infty)$.

Note that, by means of condition (H_0), the equivalent fractional integral equation of (1.1) is

$$\xi_1(t) = \mathcal{N}^{r,\psi}(t,t_0)\xi_0 + \frac{1}{\Gamma(p)}\int_{t_0}^{t}\Psi^{p,\psi}(t,s) f(s,\xi_1(s))\,ds,\ \ t > t_0. \tag{3.1}$$

where

$$\Psi^{p,\psi}(t,s) := \psi'(s)(\psi(t)-\psi(s))^{p-1}$$

and

$$\mathcal{N}^{r,\psi}(t,t_0) := \frac{(\psi(t)-\psi(t_0))^{r-1}}{\Gamma(r)}.$$

For article development, we will define the following operators

$$\Lambda\xi_1(t) = \Lambda_1\xi_1(t) + \Lambda_2\xi_1(t), \tag{3.2}$$

where

$$\Lambda_1\xi_1(t) = \mathcal{N}^{r,\psi}(t,t_0)\xi_0 \tag{3.3}$$

and

$$\Lambda_2\xi_1(t) = \frac{1}{\Gamma(p)}\int_{t_0}^{t}\Psi^{p,\psi}(t,s) f(s,\xi_1(s))\,ds. \tag{3.4}$$

It is clear that $\xi_1(t)$ is a solution of system (1.1) if it is a fixed point of the operator Λ, and the operator Λ_1 is a contraction with constant 0.

Before investigating the first result about global attractivity, let's investigate the Lemma 3.1, Lemma 3.2 and Lemma 3.3 and present other Lemmas as direct consequences.

Lemma 3.1. *Assume that the function $f(t,\xi_1(t))$ satisfies condition (H_0) and $(\psi'(t))^{\frac{1}{1-p_1}} \le \psi'(t)$, with $t \in (t_0,\infty)$ and $p_1 \in (0,p)$. Consider the following condition:*

(H_1) $|f(t,\xi_1(t))| \le \mu(\psi(t)-\psi(t_0))^{-q_1}$ *for* $t \in (t_0,\infty)$ *and* $\xi_1(t) \in C_{1-r;\psi}((t_0,\infty),\mathbb{R})$, $\mu \ge 0$ *and* $p < q_1 < 1$.

Then the operator Λ_2 is continuous and $\Lambda_2\mathbb{S}_{1,\psi}$ resides in a compact subset of $\mathbb{R}$ for $t \ge t_0 + T$,where

$$\mathbb{S}_{1,\psi} = \left\{\xi(t)/\xi(t) \in C_{1-r;\psi}((t_0,\infty),\mathbb{R}), |\xi(t)| \le (\psi(t)-\psi(t_0))^{-r_1}\ \ \text{for } t \ge t_0+T_1\right\} \tag{3.5}$$

$r_1 = \frac{1}{2}(q_1 - p)$, *and $\psi(T_1)$ satisfies that*

$$|\xi_0|\frac{\psi(T_1)^{\frac{1}{2}(r-1)}}{\Gamma(r)} + \frac{\mu\Gamma(1-q_1)}{\Gamma(1+p-q_1)}\psi(T_1)^{-\frac{1}{2}(q-p)} \le 1. \tag{3.6}$$

Proof. $\Lambda : \mathbb{S}_{1,\psi} \to \mathbb{S}_{1,\psi}$, for $t \ge t_0 + T_1$. From the above assumption of $\mathbb{S}_{1\psi}$, it is easy to know that $\mathbb{S}_{1,\psi}$ is a closed, bounded and convex subset of $\mathbb{R}$.

Applying condition (H_1), for $t \geq t_0$, we have

$$
\begin{aligned}
& \left|\Lambda_2 \xi_2(t)\right| \\
\leq\ & \frac{1}{\Gamma(p)} \int_{t_0}^{t} \Psi^{p,\psi}(t,s) \left|f(s,\xi_2(s))\right| ds \\
\leq\ & \frac{\mu}{\Gamma(p)} \int_{t_0}^{t} \Psi^{p,\psi}(t,s) \left(\psi(s)-\psi(t_0)\right)^{-q_1} ds \\
\leq\ & \frac{\mu}{\Gamma(p)} \int_{0}^{\psi(t)-\psi(t_0)} \left(\psi(t)-\psi(t_0)\right)^{p-1} \left(1-\frac{u}{\psi(t)-\psi(t_0)}\right)^{p-1} u^{-q_1}\, du \\
\leq\ & \frac{\mu\left(\psi(t)-\psi(t_0)\right)^{p-1}}{\Gamma(p)} \int_{0}^{1} (1-\kappa)^{p-1} \kappa^{-q_1} \left(\psi(t)-\psi(t_0)\right) d\kappa \\
=\ & \frac{\mu\Gamma(1-q_1)\left(\psi(t)-\psi(t_0)\right)^{p}}{\Gamma(p-q_1+1)} \\
\leq\ & \frac{\mu\Gamma(1-q_1)\left(\psi(t)-\psi(t_0)\right)^{p-q_1}}{\Gamma(p-q_1+1)} \qquad (3.7)
\end{aligned}
$$

and the only restriction for the above inequality is the integrability of $\left(\psi(t)-\psi(t_0)\right)^{-q_1}$, namely $q_1 < 1$.

The procedure to obtain the inequality (3.7), that is, the changes of variables, will be used several times during this paper.

Note that for $t \geq t_0 + T_1$, inequality (3.6) and $q_1 > p$ yield that
$\frac{\mu\Gamma(1-q_1)}{\Gamma(p-q_1+1)} \left(\psi(t)-\psi(t_0)\right)^{-\frac{1}{2}(q_1-p)} \leq \frac{\mu\Gamma(1-q_1)}{\Gamma(p-q_1+1)} \psi(T_1)^{-\frac{1}{2}(q_1-p)} \leq 1.$

Then, for $t \geq t_0 + T_1$, we obtain

$$
\begin{aligned}
& \left|\Lambda_2 \xi_2(t)\right| \\
\leq\ & \frac{\mu\Gamma(1-q_1)\left(\psi(t)-\psi(t_0)\right)^{p-q_1}}{\Gamma(p-q_1+1)} \\
=\ & \left[\frac{\mu\Gamma(1-q_1)\left(\psi(t)-\psi(t_0)\right)^{-\frac{1}{2}(q_1-p)}}{\Gamma(p-q_1+1)}\right] \left(\psi(t)-\psi(t_0)\right)^{-\frac{1}{2}(q_1-p)} \\
\leq\ & \left(\psi(t)-\psi(t_0)\right)^{-\frac{1}{2}(q_1-p)} \\
=\ & \left(\psi(t)-\psi(t_0)\right)^{-r_1} \qquad (3.8)
\end{aligned}
$$

which implies that $\Lambda_2 \mathbb{S}_{1,\psi} \subset \mathbb{S}_{1,\psi}$, for $t \geq t_0 + T_1$.

Λ_2 is continuous. For any $\xi_{2,m}(t), \xi_2(t) \in \mathbb{S}_{1,\psi}$, $m = 1, 2, \ldots$ with

$$\lim_{m\to\infty} \left|\xi_{2,m}(t) - \xi_2(t)\right| = 0,$$

we get $\lim_{m\to\infty} \xi_{2,m}(t) = \xi_2(t)$ and

$$\lim_{m\to\infty} f(t, \xi_{2,m}(t)) = f(t, \xi_2(t)) \text{ for } t \geq t_0 + T_1.$$

Now, let $\varepsilon > 0$ be given, fixed $T \geq t_0 + T_1$ such that

$$\frac{\mu\Gamma(1-q_1)\left(\psi(T)-\psi(t_0)\right)^{-(q_1-p)}}{\Gamma(p-q_1+1)} < \frac{\varepsilon}{2}. \qquad (3.9)$$

On the other hand, let $\nu = \dfrac{p-1}{1-p_1}$, then $1+\nu > 0$ since $p_1 \in (0,p)$. So, for $t_0 + T_1 \leq t \leq T$, we get

$$\begin{aligned}
&\left|\Lambda_2 \xi_{2,m}(t) - \Lambda_2 \xi_2(t)\right| \\
\leq\; & \frac{1}{\Gamma(p)} \int_{t_0}^{t} \Psi^{p,\psi}(t,s) \left| f(s,\xi_{2,m}(s)) - f(s,\xi_2(s)) \right| ds \\
\leq\; & \frac{1}{\Gamma(p)} \left(\int_{t_0}^{t} \psi'(s) \left(\psi(t) - \psi(s)\right)^{\frac{p-1}{1-p_1}} ds \right)^{1-p_1} \times \\
& \times \left(\int_{t_0}^{t} \left| f(s,\xi_{2,m}(s)) - f(s,\xi_2(s)) \right|^{\frac{1}{p_1}} ds \right)^{p_1} \\
\leq\; & \frac{1}{\Gamma(p)} \left(\frac{\left(\psi(T) - \psi(t_0)\right)^{1+\nu}}{1+\nu} \right)^{1-p_1} \left\| f(\cdot,\xi_{2,m}(\cdot)) - f(\cdot,\xi_2(\cdot)) \right\|_{C_{1-r,\psi}} \times \\
& \times \left(\int_{t_0}^{t} \left(\psi(t) - \psi(s)\right)^{r} ds \right)^{p_1} \\
\leq\; & \frac{1}{\Gamma(p)} \left(\frac{\left(\psi(T) - \psi(t_0)\right)^{1+\nu}}{1+\nu} \right)^{1-p_1} \frac{\left(\psi(T) - \psi(t_0)\right)^{r+1}}{\psi'(T)(1+r)} \times \\
& \times \left\| f(\cdot,\xi_{2,m}(\cdot)) - f(\cdot,\xi_2(\cdot)) \right\|_{C_{1-r,\psi}} \to 0 \qquad (3.10)
\end{aligned}$$

as $m \to \infty$.

For $t > T$, making changes of variables in the (3.7), we have

$$\begin{aligned}
&\left|\Lambda_2 \xi_{2,m}(t) - \Lambda_2 \xi_2(t)\right| \\
\leq\; & \frac{1}{\Gamma(p)} \int_{t_0}^{t} \Psi^{p,\psi}(t,s) \left| f(s,\xi_{2,m}(s)) - f(s,\xi_2(s)) \right| ds \\
\leq\; & \frac{2\mu}{\Gamma(p)} \int_{t_0}^{t} \Psi^{p,\psi}(t,s) \left(\psi(s) - \psi(t_0)\right)^{-q_1} ds \\
\leq\; & \frac{2\mu \Gamma(1-q_1)}{\Gamma(1+p-q_1)} \left(\psi(T) - \psi(t_0)\right)^{-(q_1-p)} < \varepsilon. \qquad (3.11)
\end{aligned}$$

Then, for $t \geq t_0 + T$, $\Lambda_2 \xi_{2,m}(t) - \Lambda_2 \xi_2(t) \to 0$ as $m \to \infty$, which implies that $\Lambda_2 \mathbb{S}_{1,\psi}$ is equicontinuous.

Finally, we prove that $\Lambda_2 \mathbb{S}_{1,\psi}$ is continuous. Let $\varepsilon > 0$ be given. Since $\lim\limits_{t \to \infty} \left(\psi(t) - \psi(t_0)\right)^{-r_1} = 0$, there is a $T' > t_0 + T_1$, such that $\left(\psi(t) - \psi(t_0)\right)^{-r_1} < \varepsilon/2$ for $t > T'$. Let, $t_1, t_2 \geq t_0 + T_1$ and $t_2 > t_1$. If $t_1, t_2 \in [t_0 + T_1, T']$, $\displaystyle\int_0^{T'} \left| f(s,\xi_1(s)) \right|^{1/p_1} ds$ exists by condition (H_0), then

$$
\begin{aligned}
&\left|\Lambda_2\xi_1(t_2)-\Lambda_2\xi_1(t_1)\right| \\
\leq\ & \frac{1}{\Gamma(p)}\int_{t_0}^{t_2}\Psi^{p,\psi}(t_2,s)\left|f(s,\xi_1(s))\right|ds \\
&+\frac{1}{\Gamma(p)}\int_{t_0}^{t_1}\Psi^{p,\psi}(t_1,s)\left|f(s,\xi_1(s))\right|ds \\
&+\frac{1}{\Gamma(p)}\int_{t_0}^{t_1}\Psi^{p,\psi}(t_2,s)\left|f(s,\xi_1(s))\right|ds \\
&-\frac{1}{\Gamma(p)}\int_{t_0}^{t_1}\Psi^{p,\psi}(t_2,s)\left|f(s,\xi_1(s))\right|ds \\
\leq\ & \frac{1}{\Gamma(p)}\left(\frac{1}{1+\nu}\right)^{1-p_1}\left[\begin{array}{c}(\psi(t_1)-\psi(t_0))^{1+\nu}-(\psi(t_2)-\psi(t_0))^{1+\nu}\\ +(\psi(t_2)-\psi(t_1))^{1+\nu}\end{array}\right]^{1-p_1}\times \\
&\times\left[\int_{t_0}^{T'}\left|f(s,\xi_1(s))\right|^{\frac{1}{p_1}}ds\right]^{p_1}+\frac{1}{\Gamma(p)}\left(\frac{1}{1+\nu}\right)^{1-p_1} \\
&\times\left[(\psi(t_2)-\psi(t_1))^{1+\nu}\right]^{1-p_1}\left[\int_{t_0}^{T'}\left|f(s,\xi_1(s))\right|^{\frac{1}{p_1}}ds\right]^{p_1} \\
\leq\ & \frac{1}{\Gamma(p)}\left(\frac{1}{1+\nu}\right)^{1-p_1}\left[\int_{t_0}^{T'}\left|f(s,\xi_1(s))\right|^{\frac{1}{p_1}}ds\right]^{p_1}(\psi(t_2)-\psi(t_1))^{p-p_1}\to 0
\end{aligned}
\tag{3.12}
$$

as $t_2\to t_1$.

If $t_1,t_2>T'$, and making changes of variables in the (3.7), we have

$$
\begin{aligned}
&\left|\Lambda_2\xi_1(t_2)-\Lambda_2\xi_1(t_1)\right| \\
\leq\ & \frac{1}{\Gamma(p)}\int_{t_0}^{t_2}\Psi^{p,\psi}(t_2,s)\left|f(s,\xi_1(s))\right|ds \\
&+\frac{1}{\Gamma(p)}\int_{t_0}^{t_1}\Psi^{p,\psi}(t_1,s)\left|f(s,\xi_1(s))\right|ds \\
\leq\ & \frac{M}{\Gamma(p)}\int_{t_0}^{t_2}\Psi^{p,\psi}(t_2,s)(\psi(s)-\psi(t_0))^{-q_1}ds \\
&+\frac{M}{\Gamma(p)}\int_{t_0}^{t_1}\Psi^{p,\psi}(t_1,s)(\psi(s)-\psi(t_0))^{-q_1}ds \\
\leq\ & (\psi(t_1)-\psi(t_0))^{-\frac{1}{2}(q_1-p)}+(\psi(t_2)-\psi(t_0))^{-\frac{1}{2}(q_1-p)} \\
=\ & (\psi(t_1)-\psi(t_0))^{-r_1}+(\psi(t_2)-\psi(t_0))^{-r_1} \\
<\ & \varepsilon
\end{aligned}
\tag{3.13}
$$

as $t_2\to t_1$.

If $t_0+T_1\leq t_1<T'<t_2$, note that if $t_2\to t_1$, then $t_2\to T'$ and $T'\to t_1$, according to the above discussion in a compact subset of $\mathbb{R}$ for $t\geq t_0+T_1$. □

Lemma 3.2. *Assume that the function $f(t,\xi_1(t))$ satisfies condition (H_0) and $t^{\frac{q}{1-p_1}} \leq t^q$ $(q>0)$, with $t \in (t_0,\infty)$ and $p_1 \in (0,p)$. Consider the following condition:*

(F_1) $|f(t,\xi_1(t))| \leq \mu (t^q - t_0^q)^{-q_1}$ for $t \in (t_0,\infty)$ and $\xi_1(t) \in C_{1-r;t^q}((t_0,\infty),\mathbb{R})$, $\mu \geq 0$ and $p < q_1 < 1$.

Then the operator Λ_2 is continuous and $\Lambda_2 \mathbb{S}_{1,t^q}$ resides in a compact subset of $\mathbb{R}$ for $t \geq t_0 + T$, where

$$\mathbb{S}_{1,t^q} = \left\{ \xi(t) / \xi(t) \in C_{1-r;t^q}((t_0,\infty),\mathbb{R}), |\xi(t)| \leq (t^q - t_0^q)^{-r_1} \text{ for } t \geq t_0 + T_1 \right\}$$

$r_1 = \frac{1}{2}(q_1 - p)$, and T_1^q satisfies that

$$|\xi_0| \frac{T_1^{\frac{q}{2}(r-1)}}{\Gamma(r)} + \frac{\mu \Gamma(1-q_1)}{\Gamma(1+p-q_1)} T_1^{-\frac{q}{2}(q-p)} \leq 1.$$

Proof. It follows straight from Lemma 3.1. □

Lemma 3.3. *Assume that the function $f(t,\xi_1(t))$ satisfies condition (H_0) and $t^{\frac{1}{1-p_1}} \leq t$, with $t \in (t_0,\infty)$ and $p_1 \in (0,p)$. Consider the following condition:*

(F_2) $|f(t,\xi_1(t))| \leq \mu (t - t_0)^{-q_1}$ for $t \in (t_0,\infty)$ and $\xi_1(t) \in C_{1-r;t}((t_0,\infty),\mathbb{R})$, $\mu \geq 0$ and $p < q_1 < 1$.

Then the operator Λ_2 is continuous and $\Lambda_2 \mathbb{S}_{1,t}$ resides in a compact subset of $\mathbb{R}$ for $t \geq t_0 + T$,where

$$\mathbb{S}_{1,t} = \left\{ \xi(t) / \xi(t) \in C_{1-r;t}((t_0,\infty),\mathbb{R}), |\xi(t)| \leq (t - t_0)^{-r_1} \text{ for } t \geq t_0 + T_1 \right\}$$

$r_1 = \frac{1}{2}(q_1 - p)$, and T_1 satisfies that

$$|\xi_0| \frac{T_1^{\frac{1}{2}(r-1)}}{\Gamma(r)} + \frac{\mu \Gamma(1-q_1)}{\Gamma(1+p-q_1)} T_1^{-\frac{1}{2}(q-p)} \leq 1.$$

Proof. It follows straight from Lemma 3.1. □

Note that by choosing the $\psi(\cdot)$ function in the condition of Lemma 3.1, we get some particular cases.

Lemma 3.4. *Assume that conditions (H_0) and (H_1) hold, then a solution of system (1.1) is in $\mathbb{S}_{1,\psi}$ for $t \geq t_0 + T_1$.*

Proof. Note that if $\xi_1(t)$ is a fixed point of Λ if is a solution of system (1.1). To prove this, it remains to show that, for fixed $\xi_2 \in \mathbb{S}_{1,\psi}$ and for all $\xi_1 \in C_{1-r;\psi}((t_0,\infty),\mathbb{R})$, $\xi_1 = \Lambda_1 \xi_1 + \Lambda_2 \xi_2 \Longrightarrow \xi_1 \in \mathbb{S}_{1,\psi}$ holds. If $\xi_1 = \Lambda_1 \xi_1 + \Lambda_2 \xi_2$, apply condition (H_1) and using the same procedure gives (3.7), we have

$$\begin{aligned} |\xi_1(t)| &= |\Lambda_1 \xi_1 + \Lambda_2 \xi_2| \\ &\leq \mathcal{N}^{r,\psi}(t,t_0) |\xi_0| + \frac{M}{\Gamma(p)} \int_{t_0}^t \Psi^{p,\psi}(t,s) (\psi(s) - \psi(t_0))^{-q_1} ds \\ &\leq \mathcal{N}^{r,\psi}(t,t_0) |\xi_0| + \frac{\mu \Gamma(1-q_1)}{\Gamma(p-q_1+1)} (\psi(t) - \psi(t_0))^{-(q_1-p)}. \end{aligned} \tag{3.14}$$

Now, for $t \geq t_0 + T_1$ from inequality (3.6) and $0 < p < q_1 < 1$, we get

$$\begin{aligned} & \mathcal{N}^{r,\psi}_{1/2}(t,t_0)\left|\xi_0\right| + \frac{\mu\Gamma\left(1-q_1\right)}{\Gamma\left(p-q_1+1\right)}\left(\psi\left(t\right)-\psi\left(t_0\right)\right)^{-\frac{1}{2}\left(q_1-p\right)} \\ \leq \; & \frac{\psi\left(T_1\right)^{\frac{1}{2}\left(r-1\right)}}{\Gamma\left(r\right)} + \frac{M\Gamma\left(1-q_1\right)}{\Gamma\left(p-q_1+1\right)}\psi\left(T_1\right)^{-\frac{1}{2}\left(q_1-p\right)} \leq 1, \end{aligned} \tag{3.15}$$

where $\mathcal{N}^{r,\psi}_{1/2}(t,t_0) := \dfrac{\left(\psi\left(t\right)-\psi\left(t_0\right)\right)^{\frac{1}{2}\left(r-1\right)}}{\Gamma\left(r\right)}$.

Then, for $t \geq t_0 + T_1$, we obtain

$$\begin{aligned} \left|\xi_1\left(t\right)\right| \; & \leq \; \left[\mathcal{N}^{r,\psi}_{1/2}(t,t_0)\left|\xi_0\right| + \frac{M\Gamma\left(1-q_1\right)}{\Gamma\left(p-q_1+1\right)}\left(\psi\left(t\right)-\psi\left(t_0\right)\right)^{-\frac{1}{2}\left(q_1-p\right)}\right] \\ & \quad \times\left(\psi\left(t\right)-\psi\left(t_0\right)\right)^{-r} \\ & \leq \; \left(\psi\left(t\right)-\psi\left(t_0\right)\right)^{-r} \end{aligned} \tag{3.16}$$

logo $\xi_1(t) \in \mathbb{S}_{1,\psi}$ for $t \geq t_0 + T_1$. By means of t_0 Theorem 2.1 and Lemma 3.1, there exists a $\xi_2 \in \mathbb{S}_{1,\psi}$ such that $\xi_2 = \Lambda_1\xi_2 + \Lambda_2\xi_2$, i.e., H has a fixed point in $S_{1,\psi}$ which is a solution of system (1.1) for $t \geq t_0 + T_1$. □

As a direct consequence of Lemma 3.4, we have the following Lemmas.

Lemma 3.5. *Assume that conditions* (H_0) *and* (F_1) *hold, then a solution of system* (1.1) *is in* $\mathbb{S}_{1,\psi}$ *for* $t \geq t_0 + T_1$.

Proof. Is follows straight from Lemma 3.4. □

Lemma 3.6. *Assume that conditions* (H_0) *and* (F_2) *hold, then a solution of system* (1.1) *is in* $\mathbb{S}_{1,\psi}$ *for* $t \geq t_0 + T_1$.

Proof. Is follows straight from Lemma 3.4. □

Theorem 3.7. *Assume that conditions* (H_0) *and* (H_1) *hold, then the zero solution of system* (1.1) *is globally attractive.*

Proof. Assume that Lemma 3.4, for $t \geq t_0 + T_1$, the solution of (1.1) exists and is in $\mathbb{S}_{1,\psi}$. All functions in $\mathbb{S}_{1,\psi}$ tend 0 as $t \to \infty$, then the solution of (1.1) tends to zero as $t \to \infty$. □

Theorem 3.8. *Assume that conditions* (H_0) *and* (F_1) *hold, then the zero solution of system* (1.1) *with* $q \to 0$ *is globally attractive in the Katugampola fractional derivative sense.*

Proof. Is follows straight from Theorem 3.7. □

Theorem 3.9. *Assume that conditions* (H_0) *and* (F_2) *hold, then the zero solution of system* (1.1) *with* $q \to 0$ *is globally attractive in the Riemann-Liouville fractional derivative sense.*

Proof. Is follows straight from Theorem 3.7. □

Theorem 3.10. *Assume that the function* $f(t,\xi_1)$ *satisfies condition* (H_0) *and* (H_2) $\left|f\left(t,\xi_1\left(t\right)\right)\right| \leq \lambda\left(\psi\left(t\right)-\psi\left(t_0\right)\right)^{-q_2}\left|\xi_1\left(t\right)\right|$ *for* $t \in (t_0,\infty)$ *and* $\xi_1 \in C_{1-r,\psi}\left((t_0,\infty),\mathbb{R}\right)$ *and* $p < q_2 < \frac{1}{2}(1+p)$. *Then the zero solution of system* (1.1) *is globally attractive.*

Proof. Let
(3.17)
$$\mathbb{S}_{2,\psi}=\left\{\xi(t)/\xi(t)\in C_{1-r,\psi}((t_0,\infty),\mathbb{R}),\ \ |\xi(t)|\leq(\psi(t)-\psi(t_0))^{-r_2}\ \text{ for } t\geq t_0+T_1\right\}$$
where $r_2=\frac{1}{2}(1-p)$, and T_2 satisfies that

$$\frac{\psi(T_2)^{\frac{1}{2}(r-1)}}{\Gamma(r)}|\xi_0|+\frac{Lr(1-q_2-r_2)}{\Gamma(1+p-q_2-r_2)}\psi(T_2)^{-(q_1-p)}\leq 1. \tag{3.18}$$

For fixed $\xi_2\in\mathbb{S}_{2,\psi}$ and for all $\xi\in\mathbb{R}$, $\xi_1=\Lambda_1\xi_1+\Lambda_2\xi_2\Longrightarrow\xi\in\mathbb{S}_{2,\psi}$, holds. If $\xi_1=\Lambda_1\xi_1+\Lambda_2\xi_2$, from condition (H_2) and making the same change of variable from (3.7), we have

$$\begin{aligned}|\xi_1(t)| &= |\Lambda_1\xi_1+\Lambda_2\xi_2|\\ &\leq \mathcal{N}^{r,\psi}(t,t_0)|\xi_0|+\frac{L}{\Gamma(p)}\int_{t_0}^{t}\Psi^{p,\psi}(t,s)(\psi(s)-\psi(t_0))^{-q_2}|x(s)|\,ds\\ &\leq \mathcal{N}^{r,\psi}(t,t_0)|\xi_0|+\frac{L}{\Gamma(p)}\int_{t_0}^{t}\Psi^{p,\psi}(t,s)(\psi(s)-\psi(t_0))^{-q_2-r_2}\,ds\\ &\leq \mathcal{N}^{r,\psi}(t,t_0)|\xi_0|+\frac{L\Gamma(1-q_2-r_2)}{\Gamma(p+1-q_2-r_2)}(\psi(t)-\psi(t_0))^{-(q_2+r_2-p)}.\end{aligned}$$
(3.19)

Note that $(\psi(t)-\psi(s))^{-q_2-r_2}$ the inequality (3.19) is integrable by means of $q_2<\frac{1}{2}(1+p)$ and $r_2=\frac{1}{2}(1-p)$. For $t\geq t_0+T_2$, from inequality (3.18) and $0<p<q_2<\frac{1}{2}(1+p)<1$, we have

$$\begin{aligned}&\mathcal{N}^{r,\psi}_{1/2}(t,t_0)|\xi_0|+\frac{\lambda r(1-q_2-r_2)}{\Gamma(p+1-q_2-r_2)}(\psi(t)-\psi(t_0))^{-(q_2+r_2-p)}\\ \leq\ &\frac{\psi(T_2)^{\frac{1}{2}(r-1)}}{\Gamma(r)}|\xi_0|+\frac{\lambda\Gamma(1-q_2-r_2)}{\Gamma(p+1-q_2-r_2)}\psi(T_2)^{-(q_2+r_2-p)}\leq 1.\end{aligned} \tag{3.20}$$

Thus, for

$$\begin{aligned}|\xi_1(t)| &\leq \left[\begin{array}{c}\mathcal{N}^{r,\psi}_{1/2}(t,t_0)|\xi_0|\\ +\dfrac{\lambda\Gamma(1-q_2-r_2)}{\Gamma(p+1-q_2-r_2)}(\psi(t)-\psi(t_0))^{-(q_2+r_2-p)}\end{array}\right](\psi(t)-\psi(t_0))^{-r_2}\\ &\leq (\psi(t)-\psi(t_0))^{-r_2}\end{aligned} \tag{3.21}$$

logo $\xi_1(t)\in\mathbb{S}_{2,\psi}$ for $t\geq t_0+T_2$. Meanwhile, from inequalities (3.19) and (3.21) also implies that $|\Lambda_2\xi(t)|\leq(\psi(t)-\psi(t_0))^{-r_2}$ which leads to $\Lambda_2\mathbb{S}_{2,\psi}\subset\mathbb{S}_{2,\psi}$ for $t\geq t_0+T_2$.

Note that, similar to the Lemma 3.1 proof, it is clear that the operator Λ_2 is continuous and $\Lambda_2\mathbb{S}_{2,\psi}$ resides in a compact subset of $\mathbb{R}$ for $t\geq t_0+T_2$. By Theorem 2.1 the revision of Krasnoselskiis Theorem, there exists a $y\in\mathbb{S}_{2,\psi}$ such that $\xi_2=\Lambda_1\xi_2+\Lambda_2\xi_2$ i.e., Λ has a fixed point in $\mathbb{S}_{2,\psi}$, which is a solution of system (1.1). Moreover, all function in $\mathbb{S}_{2,\psi}\to 0$ as $t\to\infty$, which shows that the zero solutions of system (1.1) of globally attractive. □

Theorem 3.11. *Assume that the function $f(t,\xi_1)$ satisfies condition* (H_0) *and* (F_3) $|f(t,\xi_1(t))|\leq\lambda(t^q-t_0^q)^{-q_2}|\xi_1(t)|$ *$(q>0)$ for $t\in(t_0,\infty)$ and $\xi_1\in C_{1-r,t^q}((t_0,\infty),\mathbb{R})$*

and $p < q_2 < \frac{1}{2}(1+p)$*. Then the zero solution of* system (1.1) *with* $q \to 0$ *is globally attractive in the Katugampola fractional derivative sense.*

Proof. Is follows straight from Theorem 3.10. □

Theorem 3.12. *Assume that the function* $f(t,\xi_1)$ *satisfies condition* (H_0) *and* (F_4) $|f(t,\xi_1(t))| \leq \lambda(t-t_0)^{-q_2}|\xi_1(t)|$ *for* $t \in (t_0,\infty)$ *and* $\xi_1 \in C_{1-r,t}((t_0,\infty),\mathbb{R})$ *and* $p < q_2 < \frac{1}{2}(1+p)$*. Then the zero solution of system* (1.1) *with* $q \to 0$ *is globally attractive Riemann-Liouville fractional derivative sense.*

Proof. Is follows straight from Theorem 3.10. □

Corollary 3.13. *Admit that the functions* $f(t,\xi_1)$ *satisfies conditions* (H_0) *and* $(H_{2'})$ $|f(t,\xi_1(t)) - f(t,\xi_2(t))| \leq \lambda(\psi(t)-\psi(t_0))^{-q_2}|\xi_1(t)-\xi_2(t)|$ *for* $t \in (t_0,\infty)$ *and* $\xi_1(t), \xi_2(t) \in C_{1-r,\psi}((t_0,\infty),\mathbb{R})$*,* $\lambda \geq 0$ *and* $p < q_2 < \frac{1}{2}(1+p)$, $f(t,0) \equiv 0$.

Then the zero solution of system (1.1) *is globally attractive.*

Proof. Using the condition $(H_{2'})$, we obtain

$$|f(t,\xi_1(t))| = |f(t,\xi_1(t)) - f(t,0)| \leq L(\psi(t)-\psi(t_0))^{-q_2}|\xi_1|$$

which implies that condition $(H_{2'})$ holds. The global attractive result can directly be obtained by Theorem 3.10. □

Corollary 3.14. *Admit that the functions* $f(t,\xi_1)$ *satisfies conditions* (H_0) *and* (F_5) $|f(t,\xi_1(t)) - f(t,\xi_2(t))| \leq L(t^q - t_0^q)^{-q_2}|\xi_1(t)-\xi_2(t)|$ *for* $t \in (t_0,\infty)$ *and* $\xi_1(t), \xi_2 \in C_{1-r,t^q}((t_0,\infty),\mathbb{R})$*,* $L \geq 0$ *and* $p < q_2 < \frac{1}{2}(1+p)$, $f(t,0) \equiv 0$.

Then the zero solution of system (1.1) *with* $q \to 0$ *is globally attractive Katugampola fractional derivative sense.*

Proof. Is follows straight from Corollary 3.13. □

Corollary 3.15. *Admit that the functions* $f(t,\xi_1)$ *satisfies conditions* (H_0) *and* (F_6) $|f(t,\xi_1(t)) - f(t,\xi_2(t))| \leq \lambda(t-t_0)^{-q_2}|\xi_1(t)-\xi_2(t)|$ *for* $t \in (t_0,\infty)$ *and* $\xi_1(t), \xi_2(t) \in C_{1-r,t}((t_0,\infty),\mathbb{R})$*,* $\lambda \geq 0$ *and* $p < q_2 < \frac{1}{2}(1+p)$, $f(t,0) \equiv 0$.

Then the zero solution of system (1.1) *with* $q \to 0$ *is globally attractive in the Riemann-Liouville fractional derivative sense.*

Proof. Is follows straight from Corollary 3.13. □

Theorem 3.16. *Assume that the function* $f(t,\xi_1)$ *satisfies conditions* (H_0) *and* (H_3) $|f(t,\xi_1(t))| \leq k(\psi(t)-\psi(t_0))^{-q_3}|\xi_1(t)|^{\eta}$ *for* $t \in (t_0,\infty)$ *and* $\xi_1(t) \in C_{1-r,\psi}((t_0,\infty),\mathbb{R})$*,* $k \geq 0, \eta \geq 0$ *and* $p < q_3 < \frac{2+\eta p}{2+\eta}$.

Then the zero solution of system (1.1) *is globally attractive.*

Proof. Let

$$\mathbb{S}_{3,\psi} = \left\{\xi_2(t)/\xi_2(t) \in C((t_0,\infty),\mathbb{R}), \ |\xi_2(t)| \leq (\psi(t)-\psi(t_0))^{-r_3} \text{ for } t \geq t_0 + T_3\right\}$$

where, $r_3 = \frac{1}{2}(q_3 - p)$, and $\psi(T_3) \geq 1$ and satisfies

$$\frac{|\xi_0|}{\Gamma(r)}\psi(T_3)^{\frac{1}{2}(r-1)} + \frac{k\Gamma(1-q_3-r_3\eta)}{\Gamma(1+p-q_3-r_3\eta)}\psi(T_3)^{-\frac{1}{2}(q_3-p)} \leq 1. \tag{3.22}$$

First, we prove that condition (c) of Theorem 2.1 holds. If $\xi_1 = \mathcal{A}\xi_1 + \Lambda_2\xi_2$, from conditions (H_3) and making the same change of variable from (3.7), we have

$$\begin{aligned} |\xi_1(t)| &= |\Lambda_1\xi_1 + \Lambda_2\xi_2| \\ &\leq \mathcal{N}^{r,\psi}(t,t_0)|\xi_0| + \frac{1}{\Gamma(p)}\int_{t_0}^{t}\Psi^{p,\psi}(t,s)|f(s,\xi_2(s))|\,ds \\ &\leq \mathcal{N}^{r,\psi}(t,t_0)|\xi_0| + \frac{k}{\Gamma(p)}\int_{t_0}^{t}\Psi^{p,\psi}(t,s)(\psi(s)-\psi(t_0))^{-q_3}|\xi_2(s)|^{\eta}\,ds \\ &\leq \mathcal{N}^{r,\psi}(t,t_0)|\xi_0| + \frac{k}{\Gamma(p)}\int_{t_0}^{t}\Psi^{p,\psi}(t,s)(\psi(s)-\psi(t_0))^{-q_3-\eta r_3}\,ds \\ &\leq \mathcal{N}^{r,\psi}(t,t_0)|\xi_0| \\ &+ \frac{k\Gamma(1-q_3-\eta r_3)}{\Gamma(p+1-q_3-\eta r_3)}(\psi(t)-\psi(t_0))^{-(q_3+r_3\eta-p)}. \end{aligned} \tag{3.23}$$

Note that $(\psi(t)-\psi(t_0))^{-q_3-r_3\eta}$ in inequality (3.23) is integrable because $q_3 < \dfrac{2+\eta p}{2+\eta}$ and $r_3 = \dfrac{1}{2}(q_3-p)$ lead to $q_3 + r_3\eta < 1$.

For $t \geq t_0 + T_3$, from inequality (3.22) and $0 < p < q_3 < \dfrac{2+\eta p}{2+\eta} < 1$, we get

$$\begin{aligned} &\mathcal{N}^{r,\psi}_{1/2}(t,t_0)|\xi_0| + \frac{k\Gamma(1-q_3-\eta r_3)}{\Gamma(p+1-q_3-\eta r_3)}(\psi(t)-\psi(t_0))^{-\frac{1}{2}(q_3-p)} \\ \leq\; &\frac{\psi(T_3)^{\frac{1}{2}(r-1)}}{\Gamma(r)}|\xi_0| + \frac{k\Gamma(1-q_3-\eta r_3)}{\Gamma(p+1-q_3-\eta r_3)}\psi(T_3)^{-\frac{1}{2}(q_3-p)} \leq 1. \end{aligned}$$

Thus, for $t \geq t_0 + T_3$,

$$\begin{aligned} |\xi_1(t)| &\leq \mathcal{N}^{r,\psi}_{1/2}(t,t_0)|\xi_0| + \frac{k\Gamma(1-q_3-\eta r_3)}{\Gamma(p+1-q_3-\eta r_3)}(\psi(t)-\psi(t_0))^{-(q_3+r_3\eta-p)} \\ &\leq \mathcal{N}^{r,\psi}_{1/2}(t,t_0)|\xi_0| + \frac{k\Gamma(1-q_3-\eta r_3)}{\Gamma(p+1-q_3-\eta r_3)}(\psi(t)-\psi(t_0))^{-(q_3-p)} \\ &\leq \left[\mathcal{N}^{r,\psi}_{1/2}(t,t_0)|\xi_0| + \frac{k\Gamma(1-q_3-\eta r_3)}{\Gamma(p+1-q_3-\eta r_3)}(\psi(t)-\psi(t_0))^{-(q_3-p)}\right] \\ &\quad\times(\psi(t)-\psi(t_0))^{-r_3} \\ &\leq (\psi(t)-\psi(t_0))^{-r_3} \end{aligned} \tag{3.24}$$

which implies that $\xi_1(t) \in \mathbb{S}_{3,\psi}$ for $t \geq t_0 + T_3$. Meanwhile combining (3.23) and (3.24) also implies that $|\Lambda_2\xi_2(t)| \leq (\psi(t)-\psi(t_0))^{-r_3}$ which leads to that $\Lambda_2\mathbb{S}_{3,\psi} \subset \mathbb{S}_{3,\psi}$ for $t \geq t_0 + T_3$.

Since the rest of the test is similar to Theorem 3.10, and we omit it. □

Theorem 3.17. *Assume that the function $f(t,\xi_1)$ satisfies conditions (H_0) and (F_7) $|f(t,\xi_1(t))| \leq k(t^q - t_0^q)^{-q_3}|\xi_1(t)|^{\eta}$ for $t \in (t_0,\infty)$ and $\xi_1(t) \in C_{1-r,t^q}((t_0,\infty),\mathbb{R})$, $k \geq 0, \eta \geq 0$ and $p < q_3 < \frac{2+\eta p}{2+\eta}$.*

Then the zero solution of system (1.1) with $q \to 0$ is globally attractive in the Katugampola fractional derivative sense.

Proof. Is follows straight from Theorem 3.16. □

Theorem 3.18. *Assume that the function* $f(t,\xi_1)$ *satisfies conditions* (H_0) *and* (F_7) $|f(t,\xi_1(t))| \leq k(t-t_0)^{-q_3}|\xi_1(t)|^{\eta}$ *for* $t\in(t_0,\infty)$ *and* $\xi_1(t)\in C_{1-r,t}((t_0,\infty),\mathbb{R})$, $k\geq 0, \eta\geq 0$ *and* $p<q_3<\frac{2+\eta p}{2+\eta}$.

Then the zero solution of system (1.1) *with* $q\to 0$ *is globally attractive in the Riemann-Liouville fractional derivative sense.*

Proof. Is follows straight from Theorem 3.16. □

From the proof of Theorem 3.7, we find that the term $r_3\eta$ is actually out of work in the proof, the attractive result many be attained if we consider a weaker condition than condition (H_3). Then it follows the next theorem.

Theorem 3.19. *Assume that the function* $f(t,\xi_1)$ *satisfies conditions* (H_0) *and* (H_4) $|f(t,\xi_1(t))| \leq k(\psi(t)-\psi(t_0))^{-r_3}|\xi_1(t)|^{\eta}$ *for* $t\in(t_0,\infty)$ *and* $\xi_1(t)\in C_{1-r,\psi}((t_0,\infty),\mathbb{R})$, $k\geq 0, \eta>1$ *and* $p-(\eta-1)(1-p)<q_3<p$.

Then the zero solution of system (1.1) *is globally attractive.*

Proof. Let

$$\mathbb{S}'_{3,\psi}=\left\{\xi(t)/\xi(t)\in C((t_0,\infty),\mathbb{R}),\ |\xi(t)|\leq(\psi(t)-\psi(t_0))^{-r'_3}\text{ for }t\geq t_0+T'_3\right\}$$

where $\frac{1}{\eta-1}(p-q_3)<r'_3<1-p$, $\psi(T'_3)$ and satisfies that

$$\frac{|x_0|}{\Gamma(r)}\psi(T'_3)^{r-1+r'_3}+\frac{k\Gamma(1-q_3-r'_3\eta)}{\Gamma(1+p-q_3-r'_3\eta)}\psi(T'_3)^{-q_3+p-(\eta-1)r'_3}\leq 1.$$

Since $\eta>1$, for $t\geq t_0+T'_3$, similar to (3.24) we have

$$\begin{aligned}
&|\xi_1(t)|\\
\leq\ &\left[\frac{|\xi_0|}{\Gamma(r)}(\psi(t)-\psi(t_0))^{r-1+r'_3}+\frac{k\Gamma(1-q_3-r'_3\eta)}{\Gamma(1+p-q_3-r'_3\eta)}(\psi(t)-\psi(t_0))^{-q_3+p-(\eta-1)r'_3}\right]\\
&\times(\psi(t)-\psi(t_0))^{-r'_3}\\
\leq\ &\left[\frac{|\xi_0|}{\Gamma(r)}\psi(T'_3)^{r-1+r'_3}+\frac{k\Gamma(1-q_3-r'_3\eta)}{\Gamma(1+p-q_3-r'_3\eta)}\psi(T'_3)^{-q_3+p-(\eta-1)r'_3}\right]\\
&\times(\psi(t)-\psi(t_0))^{-r'_3}\\
\leq\ &(\psi(t)-\psi(t_0))^{-r'_3}
\end{aligned}$$

which implies that $\xi_1(t)\in\mathbb{S}'_{3,\psi}$ for $t\geq t_0+T'_3$. The remaining part of the proof is similar to that of Theorem 3.16, and we omit it.

□

Theorem 3.20. *Assume that the function* $f(t,\xi_1)$ *satisfies conditions* (H_0) *and* (F_8) $|f(t,\xi_1(t))| \leq k(t^q-t_0^q)^{-r_3}|\xi_1(t)|^{\eta}$ *for* $t\in(t_0,\infty)$ *and* $\xi_1(t)\in C_{1-r,t^q}((t_0,\infty),\mathbb{R})$, $k\geq 0, \eta>1$ *and* $p-(\eta-1)(1-p)<q_3<p$.

Then the zero solution of system (1.1) *with* $q\to 0$ *is globally attractive in the Katugampola fractional derivative sense.*

Proof. Is follows straight from Theorem 3.19. □

Theorem 3.21. *Assume that the function* $f(t,\xi_1)$ *satisfies conditions* (H_0) *and* (F_8) $|f(t,\xi_1(t))| \leq k(t-t_0)^{-r_3}|\xi_1(t)|^{\eta}$ *for* $t\in(t_0,\infty)$ *and* $\xi_1(t)\in C_{1-r,t}((t_0,\infty),\mathbb{R})$, $k\geq 0, \eta>1$ *and* $p-(\eta-1)(1-p)<q_3<p$.

Then the zero solution of system (1.1) *with* $q \to 0$ *is globally attractive in the Riemann-Liouville fractional derivative sense.*

Proof. Is follows straight from Theorem 3.19. □

Note that when investigating the overall attractive of system solutions (1.1), we always seek to highlight two particular cases of the a priory results investigated at any given time, highlighting the importance of the ψ-Hilfer fractional operator, as well as the conservation of their properties. Moreover, it should be noted that by choosing other functions $\psi(\cdot)$, it is possible to obtain all the results investigated here.

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[1] Department of Applied Mathematics, Imecc-State University of Campinas, 13083-859, Campinas, SP, Brazil
E-mail address: vanterler@ime.unicamp.br

[2] Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, P. O. Box 89, 22000, Sidi Bel-Abbes, Algeria

[3] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: benchohra@yahoo.com

Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA
E-mail address: Gaston.N'Guerekata@morgan.edu