

# STRONGLY S-CLOSED SPACES AND FIRMLY CONTRA-CONTINUOUS FUNCTIONS

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## Abstract

In the present paper, we offer a new form of firm continuity, called firm contra-continuity, by which we characterize strongly S-closed spaces. Moreover, we investigate the basic properties of firmly contra-continuous functions. We also introduce and investigate the notion of locally contra-closed graphs.

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## 1 Introduction

Kupka [8] has used firm continuity to investigate compactness. Recently Caldas, et al. have used firm semi-continuity to study semi-compactness. In this note we continue this line of investigation by introducing a form of firm continuity, which we call firm contra-continuity, and using it to study strongly S-closed spaces. Dontchev [6] introduced strongly S-closed spaces and showed that contra-continuous images of strongly S-closed spaces are compact. Baker [2] extended this result by showing that subcontra-continuous images of strongly S-closed spaces are compact. Quite recently, Ganster et al. [7] further investigated, among others, the notion of strongly S-closedness. Our purpose in this note is to characterize strongly S-closed spaces in terms of firm contra-continuity and subcontra-continuity. In particular, we show that a space  $X$  is strongly S-closed if and only if for every space  $Y$  every subcontra-continuous function  $f : X \rightarrow Y$  is firmly contra-continuous. Moreover, some of the basic properties of firmly contra-continuous functions are investigated. For example, we show that firm contra-continuity implies slight continuity. Finally, we introduce the notion of locally contra-closed graphs and present some of its fundamental properties.

## 2 Preliminaries

The symbols  $X$  and  $Y$  represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set  $A$  are signified by  $\text{Cl}(A)$  and  $\text{Int}(A)$ , respectively. A set  $A$  is regular open if  $A = \text{Int}(\text{Cl}(A))$ . A set  $A$  is semiopen [9] (respectively, preopen [10],  $\beta$ -open [1]) provided that  $A \subseteq \text{Cl}(\text{Int}(A))$  (respectively,  $A \subseteq \text{Int}(\text{Cl}(A))$ ,  $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$ ). A set  $A$  is regular closed (respectively, semiclosed, preclosed,  $\beta$ -closed) if the complement of  $A$  is regular open (respectively, semiopen, preopen,  $\beta$ -open). We denote the intersection of all semiclosed sets containing  $A$  by  $sCl(A)$ . Recall that a set  $A \subset X$  is called a semi-generalized closed set (briefly  $sg$ -closed set) [3] if  $sCl(A) \subset U$  whenever  $A \subset U$  and  $U$  is semi-open. The complement of an  $sg$ -closed set is called  $sg$ -open.

**Definition 1** A space  $X$  is said to be  $S$ -closed [12] (respectively, almost compact [6]) if every semiopen cover (respectively, open cover) of  $X$  has a finite subfamily, the closures of whose members cover  $X$ .

**Definition 2** 1) A space  $X$  is said to be strongly  $S$ -closed [6] if every closed cover of  $X$  has a finite subcover.

2) Let  $A$  be a subset of  $X$ . We say that  $A$  is strongly  $S$ -closed relative to  $X$  if every cover of  $A$  by closed sets of  $X$  has a finite subcover.

Observe that if  $X$  is regular and strongly  $S$ -closed then the weight of  $X$  does not exceed  $2^{|A|}$ , where  $A$  is the finite dense subset of  $X$ . Recall that the least cardinal of a base for the space  $X$  is called the weight of  $X$ .

**Remark 2.1** Dontchev [6] showed that strongly  $S$ -closedness and compactness are independent of each other. For example Hilbert cube is compact but not strongly  $S$ -closed. But the real line with a topology in which non-empty open sets are the one containing the origin is an example of a strongly  $S$ -closed space which is not compact (see [6], Remark 3.1). He also noticed that a set is regular closed if and only if it is both closed and  $sg$ -open. It follows that a topological space  $X$  is  $S$ -closed if and only if it is strongly  $S$ -closed and  $sg$ -compact. Recall that a topological space  $X$  is called  $sg$ -compact [4] if every cover of  $X$  by  $sg$ -open sets has a finite subcover.

**Definition 3** A function  $f : X \rightarrow Y$  is said to be contra continuous [6] if  $f^{-1}(V)$  is closed for every open subset  $V$  of  $Y$ .

**Definition 4** A function  $f : X \rightarrow Y$  is said to be subcontra-continuous [2] provided there is an open base  $\mathcal{B}$  for  $Y$  such that  $f^{-1}(V)$  is closed for every  $V \in \mathcal{B}$

**Definition 5** A function  $f : X \rightarrow Y$  is said to be  $\beta$ -continuous [1] (respectively, precontinuous [10]) if  $f^{-1}(V)$  is  $\beta$ -open (respectively, preopen) for every open subset  $V$  of  $Y$ .

**Definition 6** A function  $f : X \rightarrow Y$  is said to be firmly continuous [8] if for every open cover  $\Lambda$  of  $Y$  there exists a finite open cover  $\Gamma$  of  $X$  such that for every  $U \in \Gamma$  there exists  $V \in \Lambda$  such that  $f(U) \subseteq V$

### 3 Characterization of strongly S-closed spaces

**Definition 7** A function  $f : X \rightarrow Y$  is said to have property  $\varphi$  [8] provided that for every open cover  $\Lambda$  of  $Y$  there exists a finite cover (the members of which need not be open)  $\{A_1, A_2, \dots, A_n\}$  of  $X$  such that for each  $i \in \{1, 2, \dots, n\}$  there exists  $V \in \Lambda$  for which  $f(A_i) \subseteq V$ .

**Definition 8** A function  $f : X \rightarrow Y$  is said to be firmly contra-continuous if for every open cover  $\Lambda$  of  $Y$  there exists a finite closed cover  $\mathcal{F}$  of  $X$  such that for every  $F \in \mathcal{F}$  there exists  $V \in \Lambda$  such that  $f(F) \subseteq V$

The following examples show that firm contra-continuity is independent of firm continuity.

**Example 3.1** Let  $X = \{a, b, c\}$  have the topology  $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$  and let  $f : X \rightarrow X$  be the identity mapping. Since  $f$  is continuous and  $X$  is finite,  $f$  is obviously firmly continuous. Since any closed cover of  $X$  must contain  $X$ , we see that  $f$  is not firmly contra-continuous.

**Example 3.2** Let  $X = \{a, b, c\}$  have the topologies  $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$  and  $\sigma = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ , and let  $f : (X, \tau) \rightarrow (X, \sigma)$  be the identity mapping. Since any  $\tau$ -open cover of  $X$  must contain  $X$ , it follows that  $f$  is not firmly continuous. However, since  $f$  is contra-continuous and  $X$  is finite,  $f$  is firmly contra-continuous.

If  $(X, \tau)$  is a topological space, then the topology on  $X$  with a base consisting of the  $\tau$ -closed sets will be denoted by  $\tau_c$ .

**Theorem 3.3** For a space  $(X, \tau)$  the following properties are equivalent:

- (a)  $(X, \tau)$  is strongly S-closed;
- (b) For every space  $Y$ , every subcontra-continuous function  $f : X \rightarrow Y$  is firmly contra-continuous;
- (c) The identity function  $f : (X, \tau) \rightarrow (X, \tau_c)$  is firmly contra-continuous;

(d) *The identity function  $f : (X, \tau) \rightarrow (X, \tau_c)$  has property  $\varphi$ ;*

(e) *For every space  $Y$ , every subcontra-continuous function  $f : X \rightarrow Y$  has property  $\varphi$ .*

*Proof.* (a)  $\Rightarrow$  (b) Assume  $X$  is strongly S-closed and that  $f : X \rightarrow Y$ , where  $Y$  is an arbitrary space, is subcontra-continuous with respect to the base  $\mathcal{B}$  for  $Y$ . Let  $\Lambda$  be an open cover of  $Y$ . Therefore for each  $y \in f(X)$  there exists  $V_y \in \Lambda$  such that  $y \in V_y$  and there exists  $B_y \in \mathcal{B}$  such that  $y \in B_y \subseteq V_y$ . Then  $\{B_y : y \in f(X)\}$  is an open cover of  $f(X)$ . Since the subcontra-continuous image of a strongly S-closed space is compact [2], we have that  $f(X)$  is compact. Therefore there is a finite subcover  $\{B_{y_i} : i = 1, 2, \dots, n\}$  which covers  $f(X)$ . If we let  $F_i = f^{-1}(B_{y_i})$  for every  $i \in \{1, 2, \dots, n\}$ , then  $\{F_i : i = 1, 2, \dots, n\}$  is a finite closed cover of  $X$  for which  $f(F_i) \subseteq B_{y_i} \subseteq V_{y_i}$  for every  $i \in \{1, 2, \dots, n\}$ . Hence  $f$  is firmly contra-continuous.

(b)  $\Rightarrow$  (c) Clear, since the identity function  $f : (X, \tau) \rightarrow (X, \tau_c)$  is subcontra-continuous with respect to the base consisting of the  $\tau$ -closed sets.

(c)  $\Rightarrow$  (d) Clear, since firm contra-continuity obviously implies property  $\varphi$ .

(d)  $\Rightarrow$  (a) Assume the identity function  $f : (X, \tau) \rightarrow (X, \tau_c)$  has property  $\varphi$ . Let  $\mathcal{F}$  be a closed cover of  $(X, \tau)$ . Then  $\mathcal{F}$  is an open cover of  $(X, \tau_c)$ . Since the identity function  $f : (X, \tau) \rightarrow (X, \tau_c)$  has property  $\varphi$ , there exists a finite cover  $\{A_1, A_2, \dots, A_n\}$  of  $(X, \tau)$  such that for each  $i \in \{1, 2, \dots, n\}$  there exists  $F_i \in \mathcal{F}$  for which  $A_i = f(A_i) \subseteq F_i$ . Obviously  $\{F_i : i = 1, 2, \dots, n\}$  is a finite subcover of  $\mathcal{F}$ , which proves that  $(X, \tau)$  is strongly S-closed.

(b)  $\Rightarrow$  (e) Clear, since firm contra-continuity implies property  $\varphi$ .

(e)  $\Rightarrow$  (d) Clear, since the identity function  $f : (X, \tau) \rightarrow (X, \tau_c)$  is subcontra-continuous with respect to the base  $\mathcal{B}$  consisting of the  $\tau$ -closed sets. ■

Since a subcontra-continuous,  $\beta$ -continuous image of an S-closed space is compact [2], we have the following version of the implication (a)  $\Rightarrow$  (b) in Theorem 3.3.

**Theorem 3.4** *If  $X$  is an S-closed space, then for every space  $Y$ , every subcontra-continuous,  $\beta$ -continuous function  $f : X \rightarrow Y$  is firmly contra-continuous.*

Similarly, since a subcontra-continuous, precontinuous image of an almost compact space is compact [2], we have the following result.

**Theorem 3.5** *If  $X$  is almost compact, then for every space  $Y$ , every subcontra-continuous, precontinuous function  $f : X \rightarrow Y$  is firmly contra-continuous.*

## 4 Additional properties of firmly contra-continuous functions

All of the results in this section are special cases of the following theorem.

**Theorem 4.1** *Let  $f : X \rightarrow Y$  be firmly contra-continuous. If  $V$  is an open subset of  $Y$  and  $A$  is a closed subset of  $Y$  such that  $A \subseteq V$ , then  $\text{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$ .*

*Proof.* Let  $x \in f^{-1}(A)$ . Since  $\{V, Y - A\}$  is an open cover of  $Y$ , there exists a finite closed cover  $\mathcal{F}$  of  $X$  such that for every  $F \in \mathcal{F}$ , we have  $f(F) \subseteq V$  or  $f(F) \subseteq Y - A$ . Let  $F_x \in \mathcal{F}$  such that  $x \in F_x$ . Then  $f(F_x) \subseteq V$  and hence  $f^{-1}(A) \subseteq \cup_{x \in f^{-1}(A)} F_x \subseteq f^{-1}(V)$ . Since  $\mathcal{F}$  is finite,  $\cup_{x \in f^{-1}(A)} F_x$  is a finite union of closed sets and hence closed. Therefore  $\text{Cl}(f^{-1}(A)) \subseteq \cup_{x \in f^{-1}(A)} F_x \subseteq f^{-1}(V)$ . ■

Kupka [8] observed that a firmly continuous function need not be continuous. The following example shows that a firmly contra-continuous function need not be subcontra-continuous, even when the domain is strongly S-closed. In particular, the requirements that  $f$  be subcontra-continuous and firmly contra-continuous in Theorem 3.3(b) cannot be interchanged.

**Example 4.2** *Let  $X = [0, 3]$  have the topology  $\sigma = \{U \subseteq X : 3 \in U\} \cup \{\emptyset\}$  and let  $Y$  be the real numbers with the topology  $\tau = \{(a, +\infty) : a \in Y\} \cup \{Y, \emptyset\}$ . Finally let  $f : (X, \sigma) \rightarrow (Y, \tau)$  be the inclusion mapping. To see that  $f$  is firmly contra-continuous, note that every open cover of  $Y$  must contain either  $Y$  or a set of the form  $(a, +\infty)$ , where  $a < 0$ , and that both of these sets contain  $f(X)$ . To see that  $f$  is not subcontra-continuous, let  $\mathcal{B}$  be an open base for  $Y$ . Then there exists  $B \in \mathcal{B}$  such that  $3 \in B \subseteq (2, +\infty)$ . Then  $B = (a, +\infty)$  where  $2 \leq a < 3$  and hence  $f^{-1}(B) = (a, 3]$ , which is not closed in  $X$ . Finally note that  $X$  is strongly S-closed.*

Recall that a space is called zero dimensional provided it has a clopen base.

**Corollary 4.3** *If  $f : X \rightarrow Y$  is firmly contra-continuous and  $Y$  is zero dimensional, then  $f$  is subcontra-continuous.*

*Proof.* Assume  $\mathcal{B}$  is a clopen base for  $Y$  and let  $B \in \mathcal{B}$ . Then by Theorem 4.1, if we let  $A = V = B$ , we have  $\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(B)$ , which proves that  $f^{-1}(B)$  is closed. Therefore  $f$  is subcontra-continuous with respect to the base  $\mathcal{B}$ . ■

**Definition 9** *A function  $f : X \rightarrow Y$  is said to be slightly continuous [11] provided that for every  $x \in X$  and for every clopen subset  $V$  of  $Y$  containing  $f(x)$ , there exists an open subset  $U$  of  $X$  containing  $x$  such that  $f(U) \subseteq V$ .*

The following characterizations of slight continuity will be useful.

**Theorem 4.4** *For a function  $f : X \rightarrow Y$  the following properties are equivalent:*

- (a)  $f$  is slightly continuous;
- (b) [11] The inverse image of every clopen subset of  $Y$  is an open subset of  $X$ ;
- (c) The inverse image of every clopen subset of  $Y$  is a closed subset of  $X$ ;
- (d) [11] The inverse image of every clopen subset of  $Y$  is a clopen subset of  $X$ .

The proof of the following corollary is analogous to that of Corollary 4.3.

**Corollary 4.5** *If  $f : X \rightarrow Y$  is firmly contra-continuous, then  $f$  is slightly continuous.*

If the codomain of a function is either  $T_1$  or regular, then Theorem 4.1 can be used to prove that firm contra-continuity implies a local version of contra-continuity.

**Definition 10** *A function  $f : X \rightarrow Y$  is said to be locally contra-continuous provided that for every  $x \in X$  and for every open subset  $V$  of  $Y$  containing  $f(x)$ , there exists a closed subset  $F$  of  $X$  containing  $x$  such that  $f(F) \subseteq V$ .*

**Example 4.6** *The identity mapping on the real numbers with the usual topology is locally contra-continuous, but not contra-continuous. Actually the identity function on any regular or  $T_1$  space with an open non closed set has this property.*

**Corollary 4.7** *If  $f : X \rightarrow Y$  is firmly contra-continuous and  $Y$  is either regular or  $T_1$ , then  $f$  is locally contra-continuous.*

*Proof.* Assume  $Y$  is regular. Let  $x \in X$  and let  $V$  be an open subset of  $Y$  containing  $f(x)$ . Then there exists an open subset  $U$  of  $Y$  such that  $f(x) \in U \subseteq \text{Cl}(U) \subseteq V$ . By Theorem 4.1  $x \in \text{Cl}(f^{-1}(\text{Cl}(U))) \subseteq f^{-1}(V)$ . Thus, if  $F = \text{Cl}(f^{-1}(\text{Cl}(U)))$ , then  $F$  is a closed set containing  $x$  for which  $f(F) \subseteq V$  and therefore  $f$  is locally contra-continuous.

The proof for the case where  $Y$  is  $T_1$  is analogous if  $\{f(x)\}$  is used in place of  $U$ . ■

## 5 Locally contra-closed graphs

Recall that for a function  $f : X \rightarrow Y$ , the subset  $\{(x, f(x)) \mid x \in X\} \subset X \times Y$  is called the graph of  $f$  and is denoted by  $G(f)$ .

**Definition 11** *A function  $f : X \rightarrow Y$  has a locally contra-closed graph if for each  $(x, y) \in (X \times Y) \setminus G(f)$ , there exist a closed subset  $D$  of  $X$  containing  $x$  and an open subset  $V$  of  $Y$  containing  $y$  such that  $(D \times V) \cap G(f) = \emptyset$ .*

**Lemma 5.1** *A function  $f : X \rightarrow Y$  has a locally contra-closed graph if and only if for each  $(x, y) \in (X \times Y) \setminus G(f)$ , there exist a closed subset  $D$  of  $X$  containing  $x$  and an open subset  $V$  of  $Y$  containing  $y$  such that  $f(D) \cap V = \emptyset$ .*

*Proof.* It is an immediate consequence of Definition 11 and the fact that for any subsets  $D \subset X$  and  $V \subset Y$ ,  $(D \times V) \cap G(f) = \emptyset$  if and only if  $f(D) \cap V = \emptyset$ . ■

**Theorem 5.2** *If  $f : X \rightarrow Y$  is locally contra-continuous and  $Y$  is Hausdorff, then  $G(f)$  is locally contra-closed in  $X \times Y$ .*

*Proof.* Let  $(x, y) \in (X \times Y) \setminus G(f)$ . Then  $y \neq f(x)$ . Since  $Y$  is Hausdorff, there exist open subsets  $V_1$  and  $V_2$  of  $Y$  containing  $y$  and  $f(x)$ , respectively, such that  $V_1 \cap V_2 = \emptyset$ . Since  $f$  is locally contra-continuous, there exists a closed set  $D$  of  $X$  containing  $x$  such that  $f(D) \subset V_2$ . This means that  $f(D) \cap V_1 = \emptyset$ . It follows that  $G(f)$  is locally contra-closed in  $X \times Y$ . ■

**Corollary 5.3** *If  $f : X \rightarrow Y$  is firmly contra-continuous and  $Y$  is Hausdorff, then  $G(f)$  is locally contra-closed in  $X \times Y$ .*

**Theorem 5.4** *If  $f : X \rightarrow Y$  has a locally contra-closed graph,  $f(K)$  is closed in  $Y$  for each subset  $K$  strongly  $S$ -closed relative to  $X$ .*

*Proof.* Suppose that  $y$  is a point in  $Y \setminus f(K)$ . We have  $(x, y) \notin G(f)$  for each  $x \in K$ . Since  $G(f)$  is locally contra-closed, there exists a closed subset  $D_x$  of  $X$  containing  $x$  and an open set  $V_x$  of  $Y$  containing  $y$  such that  $f(D_x) \cap V_x = \emptyset$ . The family  $\{D_x \mid x \in K\}$  is a cover of  $K$  by closed sets of  $X$ . Then, there exists a finite subset  $K_0$  of  $K$  such that  $K \subset \bigcup\{D_x \mid x \in K_0\}$ . Set  $V = \bigcap\{V_x \mid x \in K_0\}$ . Now we have

$$f(K) \cap V \subset \bigcup_{x \in K_0} (f(D_x) \cap V) \subset \bigcup_{x \in K_0} (f(D_x) \cap V_x) = \emptyset.$$

This shows that  $y \notin Cl(f(K))$  and hence  $f(K)$  is closed in  $Y$ . ■

**Corollary 5.5** *If  $f : X \rightarrow Y$  is a surjection with a locally contra-closed graph, then  $Y$  is  $T_1$ .*

*Proof.* Suppose that  $q$  is a point of  $Y$ . Since  $f$  is surjective, there exists a point  $d \in X$  such that  $f(d) = q$ . The singleton  $\{d\}$  is strongly  $S$ -closed relative to  $X$ . By Theorem 5.3,  $\{q\}$  is closed in  $Y$ . Since the singleton sets in  $Y$  are closed,  $Y$  is  $T_1$ . ■

**Theorem 5.6** *If  $f : X \rightarrow Y$  is a injection with a locally contra-closed graph, then  $X$  is  $T_1$ .*

*Proof.* Let  $x$  and  $y$  be two distinct points of  $X$ . Then  $f(x) \neq f(y)$ . Since  $f$  has a locally contra-closed graph, there exist a closed set  $D$  in  $X$  containing  $x$  and an open set  $V$  in  $Y$  containing  $f(y)$  such that  $f(D) \cap V = \emptyset$ . This means that  $y \notin D$  and therefore  $X$  is  $T_1$ . ■

**Corollary 5.7** *If  $f : X \rightarrow Y$  is a bijection with a locally contra-closed graph, then both  $X$  and  $Y$  are  $T_1$ .*

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