Division by Zero Calculus, Derivatives and Laurent's Expansion

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Abstract: Based on the preprint survey paper ([25]), we will give a fundamental relation among the basic concepts of division by zero calculus, derivatives and Laurent's expansion as a direct extension of the preprint ([28]) which gave the generalization of the division by zero calculus to differentiable functions. In particular, we will find a new viewpoint and applications to the Laurent expansion, in particular, to residures in the Laurent expansion.

Key Words: Division by zero, division by zero calculus, differentiable, analysis, residure, Laurent expansion, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, (z^n)/n = \log z$ for $n = 0$, $e^{(1/z)} = 1$ for $z = 0$.

AMS Mathematics Subject Classifications: 00A05, 00A09, 42B20, 30E20.

1 Division by zero calculus

In order to state the new results in a self-contained way, we will recall the simple background on the division by zero calculus for differentiable functions based on ([28]). For the basic references on the division by zero and the division by zero calculus, see the references cited in the reference.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{1.1}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as** $1/0 = 0$ **and** $0/0 = 0$**, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense.

In addition, when the function $f(x)$ is not differentiable, by many meanings of zero, we **should define** as

$$\frac{f(x)}{x-a}|_{x=a} = 0,$$

for example, since 0 represents impossibility. In particular, the value of the function $|x|/x$ at $x = 0$ is zero. The value of the function $x/|x|$ at $x = 0$ is also zero in our sense.

We will give its naturality of the definition.

Indeed, we consider the function $F(x) = f(x) - f(a)$ and by the definition, we have

$$\frac{F(x)}{x-a}|_{x=a} = F'(a) = f'(a).$$

Meanwhile, by the definition, we have

$$\lim_{x\to a} \frac{F(x)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a). \tag{1.2}$$

For many applications, see the references cited in the reference.

The identity (1.1) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings.

Note that $f'(a)$ represents the principal variation of order $x-a$ of the function $f(x)$ at $x=a$ which is defined independently of $f(a)$ in (1.2). This is a basic meaning of the division by zero calculus $\frac{f(x)}{x-a}|_{x=a}$.

Following this idea, we can accept the formula, naturally, for also $n=0$ for the general formula.

In the expression (1.1), the value $f'(a)$ in the right hand side is represented by the point a, meanwhile the expression

$$\frac{f(x)}{x-a}|_{x=a} \tag{1.3}$$

in the left hand side, is represented by the dummy variable $x-a$ that represents the property of the function around the point $x=a$ with the sense of the division

$$\frac{f(x)}{x-a}.$$

For $x \neq a$, it represents the usual division.

Of course, by our definition

$$\frac{f(x)}{x-a}|_{x=a} = \frac{f(x)-f(a)}{x-a}|_{x=a}, \tag{1.4}$$

however, here $f(a)$ may be replaced by any constant. This fact looks like showing that the function

$$\frac{1}{x-a}$$

is zero at $x=a$ in a sense. Of course, this result is derived immediately from the definition of the division by zero calculus.

When we apply the relation (1.1) to the elementary formulas for differentiable functions, we can imagine some deep results. For example, in the simple formula

$$(u+v)' = u' + v',$$

we have the result

$$\frac{u(x)+v(x)}{x-a}|_{x=a} = \frac{u(x)}{x-a}|_{x=a} + \frac{v(x)}{x-a}|_{x=a},$$

that is not trivial in our definition.

In the following well-known formulas, we have some **deep meanings** on the division by zero calculus.

$$(uv)' = u'v + uv',$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

and the famous laws

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Note also the logarithm derivative, for $u, v > 0$

$$(\log(uv))' = \frac{u'}{u} + \frac{v'}{v}$$

and for $u > 0$

$$(u^v)' = u^v \left(v' \log u + v\frac{u'}{u}\right).$$

For the second order derivatives, we have the familiar formulas:

$$(uv)'' = u''v + 2u'v' + uv'',$$

$$\frac{d^2 f(g(t))}{dt^2} = f''(g(t))g'(t) + f'(g(t))g''(t),$$

$$\left(\frac{1}{f}\right)'' = \frac{2(f')^2 - ff''}{f^3}$$

and

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}.$$

The representation of the higher order differential coefficients $f^{(n)}(a)$ is very simple and, for example, for the Taylor expansion we have the beautiful representation

$$f(a) = \sum_{n=0}^{\infty} \frac{f(x)}{(x-a)^n}|_{x=a} \cdot (x-a)^n.$$

Further

$$\frac{f(x)}{(x-a)^2}|_{x=a} = \frac{f''(a)}{2}$$

$$= \lim_{x\to 0} \frac{f(a+x)+f(a-x)-2f(a)}{2x^2}.$$

2 Differential coeffcients and residures

We note the basic relation for analytic functions $f(z)$ for the analytic extension of $f(x)$ to complex variable z

$$\frac{f(x)}{(x-a)^n}|_{x=a} = \frac{f^{(n)}(a)}{n!} = Res._{\zeta=a}\left\{\frac{f(\zeta)}{(\zeta-a)^{n+1}}\right\}.$$

We therefore see the basic identities among the division by zero calculus, differential coefficients and residues in the case of analytic functions. Among these basic concepts, the differential coefficients are studied deeply and so, from the results of the differential coefficient properties, we can derive another results for the division by zero calculus and residures. In this viewpoint, in particular, from the differential coefficient properties stated as in the above, we can derive the correspondent properties.

For example, for the product case, we have

$$Res._{\xi=x}\frac{(fg)(\xi)}{(\xi-x)^2}$$

$$= Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}g(x) + f(x)Res._{\xi=x}\frac{g(\xi)}{(\xi-x)^2}.$$

For the rational case

$$Res._{\xi=x}\frac{(f/g)(\xi)}{(\xi-x)^2}$$

$$= \frac{Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}g(x) - f(x)Res._{\xi=x}\frac{g(\xi)}{(\xi-x)^2}}{g(x)^2}.$$

For the inverse function $x = f^{-1}(y)$ for the function $y = f(x)$,

$$Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}Res._{\eta=y}\frac{f^{-1}(\eta)}{(\eta-y)^2} = 1.$$

For the second order derivatives, we have, for $x = g(t)$

$$Res._{\tau=t}\frac{f(g(\tau))}{(\tau-t)^3} = Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^3}Res._{\tau=t}\frac{g(\tau)}{(\tau-t)^2}$$

$$+Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}Res._{\tau=t}\frac{g(\tau)}{(\tau-t)^3}$$

and

$$Res._{\xi=x}\frac{(1/f)(\xi)}{(\xi-x)^3}$$

$$= \frac{2\left(Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}\right)^2 f(x) - f(x)Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^3}}{f(x)^3}.$$

From the formula

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

we obtain

$$Res._{\tau=t}\frac{f(x(\tau),y(\tau))}{(\tau-t)^2} = Res._{\xi=x}\frac{f(\xi,y(t))}{(\xi-x)^2}Res._{\tau=t}\frac{x(\tau)}{(\tau-t)^2}$$

$$+Res._{\eta=y}\frac{f(x(t),\eta)}{(\eta-y)^2}Res._{\tau=t}\frac{y(\tau)}{(\tau-t)^2}.$$

For the implicit function theorem for the function $y = f(x)$ of $F(x,y) = 0$ satisfying $F(x, f(x)) = 0$

$$f'(x) = -\frac{F_x(x,y)}{F_y(x,y)},$$

we obtain

$$Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2} = -Res._{\xi=x}\frac{F(\xi,y)}{(\xi-x)^2}\left(Res._{\eta=y}\frac{F(x,\eta)}{(\eta-y)^2}\right)^{-1}.$$

For the formula

$$f''(x) = -\frac{F_{xx} + 2F_{xy}f' + F_{yy}(f')^2}{F_y},$$

we have the correspondent formula.

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