ON ρ-HOMEOMORPHISMS IN TOPOLOGICAL SPACES

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Abstract

In this paper, we first introduce a new class of closed map called ρ-closed map. Moreover, we introduce a new class of homeomorphism called a ρ-homeomorphism.We also introduce another new class of closed map called ρ^*-closed map and introduce a new class of homeomorphism called a ρ^*-homeomorphism and prove that the set of all ρ^*-homeomorphisms forms a group under the operation of composition of maps.

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1 Introduction

In the course of generalizations of the notion of homeomorphism, Maki et al. [24]introduced g-homeomorphisms and gc-homeomorphisms in topological spaces. Devi et al. [6,7] studied semi-generalized homeomorphisms and generalized semi-homeomorphisms and also they have introduced α-homeomorphisms in topological spaces. In this paper, We first introduce ρ-closed maps in topological spaces and then we introduce and study ρ-homeomorphism. We also introduce ρ^*-closed map and ρ^*-homeomorphism. It turns out that the set of all ρ^*-homeomorphisms forms a group under the operation of composition of maps.

2 preliminaries

Throughout this paper (X,τ), (Y,σ) and (Z,η) will always denote topological spaces on which no separation axioms are assumed,unless otherwise mentioned. when A is a subset of (X,τ), cl(A) and int(A) denote the closure and the interior of the set A, respectively.

we recall the following definitions and some results, which are used in the sequel.

Definition 2.1. Let (X,τ) be a topological space. A subset A of a space (X,τ) is called:

1. preopen[20] if A⊆int(cl(A)) and preclosed if cl(int(A))⊆A.

2. semiopen[18] if A⊆cl(int(A)) and semiclosed if int(cl(A))⊆A.

3. semipreopen[1] if A⊆cl(int(cl(A))) and semipreclosed if int(cl(int(A)))⊆A.

Definition 2.2. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called:

1. generalized closed(briefly g-closed)[19] if cl(A)⊆U whenever A⊆U and U is open in (X, τ).

2. generalized preclosed(briefly gp-closed)[25] if pcl(A)⊆U whenever A⊆U and U is open in (X, τ).

3. generalized preregular closed(briefly gpr-closed)[11] if pcl(A)⊆U whenever A⊆U and U is regularopen in (X, τ).

4. gp-closed [27] if pcl (A)⊆ U whenever A⊆ U and U is -open in X.

5. -closed [32] if cl(A) ⊆U whenever A⊆ U and U is semi open in X.

6. ĝ-closed [33] if cl(A) ⊆U whenever A⊆ U and U is semi open in X.

7. *g-closed [36] if cl(A)⊆ U whenever A⊆ U and U is ĝ-open in X.

8. #g- semi closed (briefly #gs-closed)[35] if scl(A) ⊆U whenever A ⊆U and U is *g-open in X.

9. g̃-closed set [15] if cl(A) ⊆U whenever A ⊆U and U is #gs-open in X.

10. ρ-closed set [16] if pcl (A) ⊆Int(U) whenever A ⊆U and U is g̃-open in (X, τ).

11. π-open [37] if it is a finite union of regular open sets. The complement of a π-open set is said to be π-closed.

The complements of the above mentioned sets are called their respective open set.

Definition 2.3. A function f: $(X, \tau)\rightarrow$ (Y,σ) is called

1. Semi-continuous [18] if f^{-1}(V) is semiopen in (X, τ) for every open set V in (Y,σ).

2. Pre-continuous [20] if f^{-1}(V) is Preclosed in (X, τ) for every closed set V in (Y, σ).

3. g-continuous [4] if f^{-1}(V) is g-closed in (X, τ) for every closed set V in (Y, σ).

4. ω-continuous [32] if f^{-1}(V) is ω-closed in (X, τ) for every closed set V in (Y, σ).

5. gp-continuous [2] if $f^{-1}(V)$ is gp-closed in (X, τ) for every closed set V in (Y, σ).

6. gpr-continuous [12] if $f^{-1}(V)$ is gpr-closed in (X, τ) for every closed set V in (Y, σ).

7. gp-continuous [28] if $f^{-1}(V)$ is gp-closed in (X, τ) for every closed set V in (Y, σ).

8. #g-semicontinuous [35] if $f^{-1}(V)$ is #gs-closed in (X, τ) for every closed set V in (Y, σ).

9. $\tilde{g}$-continuous [30] if $f^{-1}(V)$ is $\tilde{g}$-closed in (X, τ) for every closed set V in (Y, σ).

10. Contra-continuous [9] if $f^{-1}(V)$ is closed in (X, τ) for every open set V in (Y, σ).

11. $\tilde{g}$-irresolute [30] if $f^{-1}(V)$ is $\tilde{g}$-closed in (X, τ) for every $\tilde{g}$-closed set V in (Y, σ).

12. M-Preclosed [22] if f(V) is Preclosed in (Y, σ) for every preclosed set V in (X, τ).

13. M-precontinuous[20] if $f^{-1}(V)$ is Preclosed in (X, τ) for every preclosed set V in (Y, σ).

14. RC-continuous [10] if $f^{-1}(V)$ is regular closed in (X, τ) for every open set V in (Y, σ).

15. ρ-continuous [17] if $f^{-1}(V)$ is ρ-closed in (X, τ) for every closed set V in (Y, σ).

16. ρ-irresolute [17] if $f^{-1}(V)$ is ρ-closed in (X, τ) for every ρ-closed set V in (Y, σ).

17. contra-open [5] if f(V) is closed in (Y, σ) for every open set V in (X, τ).

18. preclosed [25] if f(V) is preclosed in (Y, σ) for every closed set V in (X, τ).

19. ω-closed [32] if f(V) is ω-closed in (Y, σ) for every closed set V in (X, τ).

20. g-closed [21] if f(V) is g-closed in (Y, σ) for every closed set V in (X, τ).

21. gp-closed [25] if f(V) is gp-closed in (Y, σ) for every closed set V in (X, τ).

22. gpr-closed [26] if f(V) is gpr-closed in (Y, σ) for every closed set V in (X, τ).

23. πgp-closed if f(V) is πgp-closed in (Y, σ) for every closed set V in (X, τ).

24. gs-closed if f(V) is gs-closed in (Y, σ) for every closed set V in (X, τ).

25. $\tilde{g}$-closed [14] if f(V) is $\tilde{g}$-closed in (Y, σ) for every closed set V in (X, τ).

Definition 2.4. A space (X, τ) is called

1. a $T_{1/2}$ space [19] if every g-closed set is closed.
2. a T_ω space [32] if every ω-closed set is closed.
3. a gs$T^{\#}1/2$ space [35] if every #g-semi-closed set is closed.
4. a T$\tilde{g}$ -space [30] if every $\tilde{g}$ -closed set is closed.
5. a ρ-$T_s$space [16] if every ρ_s-closed set is closed.

Definition 2.5. A bijective function f: $(X, \tau) \to (Y, \sigma)$ is called a

1. homeomorphism if f is both open and continuous.
2. generalized homeomorphism (briefly g-homeomorphism) [24] if f is both g-open and g-continuous.
3. semi-homeomorphism [6] if f is both continuous and semi-open.
4. pre-homeomorphism [23] if f is both M-precontinuous and M-preopen.
5. gp-homeomorphism if f is both gp-continuous and gp-open.
6. gpr-homeomorphism if f is both gpr-continuous and gpr-open.
7. πgp-homeomorphism if f is both πgp-continuous and πgp-open.

Definition 2.6. (i) Let (X, τ) be a topological space and A $\subseteq$X. We define the ρ-closure of A [16] (briefly ρ-cl(A)) to be the intersection of all ρ-closed sets containing A.

(ii) Let (X, τ) be a topological space and A $\subseteq$X. We define the ρ-interiorof A [16] (briefly ρ-int(A)) to be the union of all ρ-open sets contained in A.

(iii) A topological space (X, τ) is ρ-compact [17] if every ρ-open cover of X has a finite subcover.

(iv) Let (X, τ) be a topological space. Let x be a point of (X, τ) and V be a subset of X. Then V is called a ρ-open neighbourhood(simply ρ-neighbourhood) [17] of x in (X, τ) if there exists a ρ-open set U of (X, τ) such that x$\in$U $\subseteq$V.

Proposition 2.7. *[16] Let (X, τ) be a topological space and A $\subseteq$X. The following properties are hold:*

(i) ρ-cl(A) is the smallest ρ-closed set containing A.

(ii) If A is ρ-closed then ρ-cl(A) = A. Converse not true.

(iii) ρ-int(A) is the largest ρ-open set contained in A.

(iv) If A $\subset$B then ρ-cl(A)$\subset$$\rho$-cl(B).

3 ρ-closed maps

Definition 3.1. A map f: $(X, \tau) \rightarrow (Y, \sigma)$ is said to be ρ-closed if the image of every closed set in (X, τ) is ρ-closed in (Y, σ).

Example 3.2. (i) Let X = Y = $\{a, b, c, d, e\}$,$\tau = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\sigma = \{\emptyset, \{b\}, \{d, e\} \{b, d, e\}, \{a, c, d, e\}, Y\}$ Define a map f:$(X, \tau) \rightarrow (Y, \sigma)$ by f(a) = d, f(b) = e, f(c)=b, f(d)=c, f(e)=a. Then f is a ρ-closed map.

(ii) Let X = Y = $\{a, b, c, d, e\}$, $\tau = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\sigma = \{\emptyset, \{b\}, \{d, e\} \{b, d, e\}, \{a, c, d, e\}, Y\}$Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is not a ρ-closed map. Since for the closed set V = $\{e\}$in (X, τ), f(V) = $\{e\}$,Which is not a ρ-closed set in (Y, σ).

Theorem 3.3. *Every Contra-closed map and Preclosed map f:$(X, \tau) \rightarrow (Y, \sigma)$ is ρ-closed map.*

Proof. **:**Let V be a closed set in (X, τ). Then f(V) is open and preclosed in (Y, σ). Hence by Theorem 3.2[16], f(V) is ρ-closed in (Y, σ). Therefore f is a ρ-closed map. □

Converse of this theorem need not be true as seen from the following example.

Example 3.4. As in Example 3.2(i), f is a ρ-closed map but neither contra-closed map nor preclosed map. Since for the closed set V = $\{a, b, e\}$in (X, τ), f(V) = $\{a, d, e\}$ is neither preclosed nor open in (Y, σ).

Theorem 3.5. *Every ρ-closed map f:$(X, \tau) \rightarrow (Y, \sigma)$ is a gp-closed (resp.gpr-closed, πgp-closed) map.*

Proof. **:** Let V be a closed set in (X, τ). Then f(V) is a ρ-closed set in (Y, σ). By Theorem 3.4[16], f(V) is gp-closed in (Y, σ) (resp. By Theorem 3.6[16], f(V) is gpr-closed in (Y, σ), By Theorem 3.10[16], f(V) is πgp-closed in (Y, σ)). Hence f is a gp-closed (resp.gpr-closed, πgp-closed) map. □

Converse of this theorem need not be true as seen from the following examples.

Example 3.6. (i)Let X = Y = $\{a, b, c, d, e\}$,τ ={ϕ,{a,b},{a,b,d},{a,b,c,d}, {a,b,d,e},X}, $\sigma = \{\emptyset, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, Y\}$ .Define f :$(X, \tau) \rightarrow (Y, \sigma)$ by f(a) = c; f(b) = e; f(c) = a; f(d) = b; f(e) = d. Then the function f is a gp-closed map but not ρ-closed map. Since for the closed set V = $\{e\}$in (X, τ), f(V) = $\{d\}$, is a gp-closed set but not a ρ-closed set in (Y, σ).

(ii) Let X = Y = $\{a, b, c, d, e\}$,τ = {ϕ,{a,b},{a,b,d},{a,b,c,d},{a,b,d,e},X}, $\sigma = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, Y\}$ .Define f as in Example3.6(i), the function f is gpr-closed map but not ρ-closed map. Since for all the closed sets in (X, τ), its images are all gpr-closed sets in (X,σ) but no one is ρ-closed set in (Y, σ).

(iii)As in Example 3.6(i), Define f :$(X, \tau) \rightarrow (Y, \sigma)$ by f(a) = c; f(b) = b; f(c) = a; f(d) = e; f(e) = d. Then the function f is aπgp-closed map but not ρ-closed map. Since for the closed set V =$\{a, d\}$is πgp-closed set but not a ρ-closed set in (Y, σ).

Remark 3.7. The following examples show that closed map is independent of ρ-closed map.

Example 3.8. (i)As in Example 3.2(i), f is a ρ-closed map but not a closed map. since for the closed set v = $\{e\}$ in (X, τ), f(V) = $\{a\}$is ρ-closed but not closed in (Y, σ).

(ii)As in Example 2.30[17], [0,$\frac{1}{4}$] is closed in $[0, 1]$, f([0,$\frac{1}{4}$])= [0,$\frac{1}{2}$] is closed in $[0, 2]$but it is not ρ-closed in $[0, 2]$. since [0,$\frac{1}{2}$]$\subseteq$[0,1), open in [0,2] and hence $\tilde{g}$ -open in [0,2] but [0,$\frac{1}{2}$] is not contained in (0,1).

Remark 3.9. The following examples show that g-closed map is independent of ρ-closed map.

Example 3.10. (i)Let X = Y = $\{a, b, c\}$,τ = $\{\phi, \{a, c\}, X\}$, σ = $\{\phi$,{a},{a,b}, Y}. Define a map f :$(X, \tau)\rightarrow(Y, \sigma)$ by f(a) = c; f(b) = b; f(c) = a. Then f is a ρ-closed map but not g-closed map. since for the closed set V = $\{b\}$in (X, τ), f(V) = $\{b\}$ is ρ-closed but not g-closed in (Y, σ).

(ii) consider [0,1] and [0,2] with usual topology. Define f :[0,1]$\rightarrow$[0,2] by f(x) = 2x. Let [0,$\frac{1}{4}$] be closed in [0,1]. Then f([0,$\frac{1}{4}$])= [0,$\frac{1}{2}$] is g-closed in [0,2] but not ρ-closed in [0,2]. Hence f is g-closed but not ρ-closed.

Remark 3.11. The following example shows that the composition of two ρ-closed maps need not be ρ-closed.

Example 3.12. Let X = Y = Z = $\{a, b, c\}$,τ = $\{\phi, \{a\}, \{b, c\}, X\}$, σ = $\{\phi, \{a, c\}, Y\}$, η= $\{\phi, \{a\}, \{a, b\}, Z\}$. Define f :$(X, \tau)\rightarrow(Y, \sigma)$ by f(a) = f(b) = b; f(c) = a and define g: $(Y, \sigma)\rightarrow$(Z,η) by g(a) = c; g(b) = b ; g(c) = a. Then both f and g are ρ-closed maps but their composition gf: $(X, \tau)\rightarrow$(Z,η)is not a ρ-closed map. since for the closed set V = $\{b, c\}$ in (X, τ), gf(V) = $\{a, b\}$, Which is not a ρ-closed set in (Z,η).

Theorem 3.13. *If f: $(X, \tau)\rightarrow(Y, \sigma)$ is ρ-closed, g: $(Y, \sigma)\rightarrow$(Z,η) is ρ-closed and (Y, σ) is ρ-$T_{1/2}$ space then their composition gf : $(X, \tau)\rightarrow$(Z,η)is ρ-closed.*

Proof. **:**Let V be a closed set in (X, τ). Then f(V) is a ρ-closed set in (Y, σ). Since (Y, σ) is ρ-$T_{1/2}$, then f(V) is a closed set in (Y, σ).Hence g(f(V))=(gf)(V) is a ρ-closed in (Z,η). Therefore gf is a ρ-closed map. □

Theorem 3.14. *If f :$(X, \tau)\rightarrow(Y, \sigma)$ is a $\tilde{g}$-closed (resp.g-closed, ω-closed, gs-closed) map, g:$(Y, \sigma)\rightarrow$(Z,η) is a ρ-closed map and Y is $T\tilde{g}$ -space(resp.$T_{1/2}$space, $T_\omega$space ,gs$T^{\#}{}_{1/2}$space) then their composition gf: $(X, \tau)\rightarrow$(Z,η) is a ρ-closed map.*

Proof. :Let V be a closed set in (X, τ).Then f(V) is a $\tilde{g}$-closed (resp. g-closed,ω-closed, gs-closed) set in(Y, σ). Since (Y, σ) is a $T\tilde{g}$ -space (resp.$T_{1/2}$space, $T_\omega$space, gs$T^{\#}{}_{1/2}$space), therefore f(V) is a closed set in (Y, σ). Since g is ρ-closed, g($f(V)$)= $(gf)(V)$is ρ-closed in (Z,η).Therefore gf is a ρ-closed map. □

Theorem 3.15. *If f : $(X,\tau)\to(Y,\sigma)$ is a $\tilde{g}$-closed and Contra-closed map, g:$(Y,\sigma)\to$(Z,η) is a M-Preclosed and open map then their composition gf : $(X,\tau)\to$(Z,η) is ρ-closed map.*

Proof. Let V be a closed set in (X,τ). Then f(V) is $\tilde{g}$-closed and open in (Y,σ). Since every $\tilde{g}$-closed is Preclosed and g is M-preclosed and open, hence g$(f(V))$ = $(gf)(V)$is preclosed and open in (Z,η). By Theorem 3.2 [16], $(gf)(V)$is ρ-closed in (Z,η). Therefore gf is a ρ-closed map. □

Theorem 3.16. *Let f : $(X,\tau)\to(Y,\sigma)$ be a closed map and g:$(Y,\sigma)\to$(Z,η) be a ρ-closed map then their composition gf : $(X,\tau)\to$(Z,η) is ρ-closed.*

Proof. Let V be a closed set in (X,τ). Then f(V)is a closed set in (Y,σ). Hence g$(f(V))$= $(gf)(V)$is ρ-closed set in (Z,η). Therefore gf is a ρ-closed map. □

Remark 3.17. If f is ρ-closed map and g is closed, then their composition need not be a ρ-closed map as seen from the following example.

Example 3.18. Let X = Y = Z = $\{a,b,c\}$,τ = $\{\phi,\{a\},\{b,c\},X\}$, σ = $\{\phi,\{a,c\},Y\}$, η= $\{\phi,\{c\},\{a,c\},Z\}$. Define f :$(X,\tau)\to(Y,\sigma)$ be f(a) = f(b) = c; f(c) = b and define g: $(Y,\sigma)\to$(Z,η) be the identity map. Then f is a ρ-closed map and g is a closed map. But their composition gf: $(X,\tau)\to$(Z,η) is not a ρ-closed map. Since for the closed set V = $\{a\}$in (X,τ),$(gf)(V)$= g$(f(V))$ = g(c)= $\{c\}$, which is not is ρ-closed set in (Z,η).

Theorem 3.19. *If f :$(X,\tau)\to(Y,\sigma)$ is a ρ-closed, g:$(Y,\sigma)\to$(Z,η) is M-Preclosed and $\tilde{g}$-irresolute map then gf : $(X,\tau)\to$(Z,η) is ρ-closed.*

Proof. Let V be a closed set in (X,τ). Then f(V) is a ρ-closed set in (Y,σ). Hence by Theorem 3.16[17], g$(f(V))$= $(gf)(V)\rho$-closed in (Z,η). Therefore gf is a ρ-closed map. □

Theorem 3.20. *Let f : $(X,\tau)\to(Y,\sigma)$ and g:$(Y,\sigma)\to$(Z,η) be two mappings such that their composition gf : $(X,\tau)\to$(Z,η) be a ρ-closed mapping. Then the following statements are true if:*

1. f is continuous and surjeetive then g is ρ-closed.
2. g is ρ-irresolute, injeetive then f is ρ-closed
3. f is $\tilde{g}$ -continuous, surjeetive and (X,τ) is a T$\tilde{g}$ -space, then g is ρ-closed.
4. f is g-continuous, surjeetive and (X,τ) is a $T_{1/2}$ space then g is ρ-closed.
5. f is ρ-continuous, surjetive and (X,τ) is a ρ-T_s space then g is ρ-closed.

Proof. 1. Let A be a closed set in (Y,σ). Since f is continuous, $f^{-1}(A)$is closed in (X,τ) and since gf is ρ-closed, $(gf)\big(f^{-1}(A)\big)$= g(A)is a ρ-closed in (Z,η), since f is surjeetive. Therefore, g is a ρ-closed map.

2. Let A be a closed set in (X,τ). Since gf is ρ-closed, then $(gf)(A)$is ρ-closed in (Z,η). Since g is ρ-irresolute, then $g^{-1}(gf)(A)$ is ρ-closed in (Y,σ), since g is injeetive. Thus, f is a ρ-closed map.

3. Let A be a closed set of (Y,σ). Since f is $\tilde{g}$-continuous, $f^{-1}(A)$ is $\tilde{g}$-closed in (X,τ). Since (X,τ) is a T$\tilde{g}$ -space, $f^{-1}(A)$ is closed in (X,τ), since gf is ρ-closed, $(gf)\big(f^{-1}(A)\big)$= g$(A)$is ρ-closed in (Z,η), since f is surjeetive. Thus g is a ρ-closed map.

4. Let A be a closed set of (Y,σ). Since f is g-continuous, $f^{-1}(A)$ is g-closed in (X,τ). Since (X,τ) is a $T_{1/2}$-space, $f^{-1}(A)$ is closed in (X,τ), since gf is ρ-closed, $(gf)\big(f^{-1}(A)\big)$= g$(A)$is ρ-closed in (Z,η), since f is surjeetive. Thus g is a ρ-closed map.

5. Let A be a closed set (Y,σ). Since f is ρ-continuous, f^{-1}(A) is ρ-closed in (X,τ). Since (X,τ) is a ρ-$T_s$space and by Theorem 3.33 [15], f^{-1}(A) is closed in (X,τ). Since gf is ρ-closed,$(gf)f^{-1}\,(A)$= g(A) is ρ-closed in (Z,η). Since f is surjeetive. Thus, g is a ρ-closed map. □

As for the restriction f_A of a map f :$(X,\tau)\to(Y,\sigma)$ to a subset A of (X,τ), we have the following.

Theorem 3.21. *Let (X,τ) and (Y,σ) be any topological spaces, Then if :*

1. f :$(X,\tau)\to(Y,\sigma)$ is ρ-closed and A is a closed subset of (X,τ) then f_A: $(A,\tau_A)\to$ (Y,σ) is ρ-closed.
2. f :$(X,\tau)\to(Y,\sigma)$ is ρ-closed and A = f^{-1}(B),for some closed set B of (Y,σ), then f_A: $(A,\tau_A)\to$ (Y,σ) is ρ-closed.

Proof. 1. Let B be a closed set of (A,τ_A). Then B = A$\bigcap$F for some closed set F of (X,τ) and so B is closed in (X,τ). Since f is ρ-closed, then f(B) is ρ-closed in (Y,σ). But f(B)= $f_A(B)$and therefore $f_A$is a ρ-closed map.

2. Let F be a closed set of (A,τ_A). Then F = A$\bigcap$H for some closed set H of (X,τ). Now $f_A\,(F)$= f(F)= f$(A\bigcap H)$ = f$\big(f^{-1}(B)\bigcap H\big)$= B$\bigcap$f(H).Since f is ρ-closed, f(H) is ρ-closed in (Y,σ) and so B∩f(H) is ρ-closed in (Y,σ). Therefore $f_A$is a ρ-closed map. □

Theorem 3.22. *A map f :$(X,\tau)\to(Y,\sigma)$ is ρ-closed if and only if for each subset S of (Y,σ) and for each open set U containing f^{-1}(S) there is a ρ-open set V of (Y,σ) such that S⊂ V and f^{-1}(V)⊂ U.*

Proof. Suppose that f is a ρ-closed map. Let S⊂Y and U be an open subset of (X,τ) such that f^{-1}(S)⊂U. Then V = $(f(U^c))^c$is a ρ-open set containing S such that f^{-1}(V)⊂U. For the converse, Let S be a closed set of (X,τ). Then $f^{-1}\left((f(s))^c\right)$ ⊂S^c and $S^c$is open. By assumption, there exists a ρ-open set V of (Y,σ) such that$(f(S)^c)$⊂V and $f^{-1}(V)$⊂S^c and so S $\subset\big(f^{-1}(V)\big)^c$. Hence V^c ⊂f(S)⊂f$\big(f^{-1}(V)^c\big)$⊂V^cwhich implies f(S)= V^c. since $V^c$is ρ-closed in (Y,σ), f$(S)$is ρ-closed in (Y,σ) and therefore f is ρ-closed. □

Theorem 3.23. *If a mapping f :$(X,\tau)\to(Y,\sigma)$ is ρ-closed then ρ-cl(f(A))⊆f(cl(A)) for every subset A of (X,τ).*

Proof. Suppose that f is ρ-closed and A$\subseteq X$, Then f($cl(A)$) is ρ-closed in (Y, σ). Hence by Theorem4.22[16],ρ-cl($f(cl(A))$= f($cl(A)$) .Also f(A)$\subseteq$f(cl(A)), and by Proposition2.7(iv), we have, ρ-cl($f(A)$)$\subseteq\rho$-cl($f(cl(A))$ = f($cl(A)$) . □

Converse of this theorem need not be true as seen from the following example.

Example 3.24. Let X = $\{a, b, c\}$= Y, $\tau = \{\emptyset, \{a\}, \{b, c\}\ X\}$, $\sigma = \{\phi$,{a},{a,b}, Y}. Define f :$(X, \tau)\to(Y, \sigma)$ by f(a) = c; f(b) = a; f(c) = b. For every subset A of X,we have ρ-cl($f(A)$)$\subseteq$f($cl(A)$). But f is not a ρ-closed map.Since for the closed set V = $\{b, c\}$in (X, τ), f(V) =$\{a, b\}$ is not a ρ-closed set in (Y, σ).

4 ρ-Open maps

Definition 4.1. A map f :$(X, \tau)\to(Y, \sigma)$ is said to ρ-open map if the image f(A) is ρ-open in (Y, σ) for every open set A in (X, τ).

Theorem 4.2. *For any bijection f :$(X, \tau)\to(Y, \sigma)$, the following statements are equivalent.*

1. f^{-1}: $(Y, \sigma)\to(X, \tau)$ is ρ-continuous
2. f is a ρ-open map and
3. f is a ρ-closed map.

Proof. (1)→(2) Let U be an open set of (X, τ). By assumption, $\left(f^{-1}\right)^{-1}(U)$= f($U$)is ρ-open in (Y, σ) and so f is a ρ-open map.

(2)→(3)Let V be a closed set of (X, τ). Then V^c is open in (X, τ). BY assumption f(V^c)= $(f(V))^c$is ρ-open in (Y, σ) and therefore f(V)is ρ-closed in (Y, σ). Hence f is a ρ-closed map.

(3)→(1) Let V be a closed set of (X, τ). By assumption f(V) is ρ-closed in (Y, σ). But f(V)= $\left(f^{-1}\right)^{-1}(V)$and therefore f^{-1} is ρ-continuous on (Y, σ). □

Theorem 4.3. *Let f :$(X, \tau)\to(Y, \sigma)$ be mapping. If f is a ρ-open mapping then for a subset A of (X, τ), f($int(A)$)$\subset$ ρ-int($f(A)$)*

Proof. Suppose f is ρ-open. Let A$\subset X$. since int(A)is open in (X, τ) and f is ρ-open, then f($int(A)$)is ρ-open in (Y, σ). Now f($int(A)$)$\subset f(A)$and by Proposition 2.7(iii), we have, f($int(A)$)$\subset \rho - int(f(A))$. □

Converse of this theorem need not be true as seen from the following example.

Example 4.4. Let X = $\{a, b, c, d, e\}$ = Y, $\tau = \{\phi$,{b,c,d},{a,b,c,d},{b,c,d,e},X}, $\sigma = \{\emptyset, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, d, e\}, Y\}$. Define f :$(X, \tau)\to(Y, \sigma)$ by f(a) = a; f(b) = c; f(c) = d; f(d) = e; f(e) = b. For a subset A of X, f($int(A)$)$\subset\rho$-int($f(A)$)but f is not a ρ-open map. Since for a subset A = $\{a, b, c, d\}$of X, $f(int(A)) = \{a, c, d, e\}$, f($A$)= $\{a, c, d, e\}$, clearly f($int(A)$)$\subseteq\rho$-int($f(A)$) but f(A) is not ρ-open in (Y, σ).

Theorem 4.5. *Let f :$(X,\tau)\rightarrow(Y,\sigma)$ be mapping. If f is a ρ-open mapping, then for each xϵX and for each neighbourhood U of x in (X,τ), there exists a ρ-neighbourhood W of f(x) in (Y,σ) such that $W \subset f(U)$.*

Proof. Let xϵX and U be an arbitrary neighbourhood of x. Then there exists an open set V in (X,τ) such that xϵV$\subseteq$U. By assumption, f(V)is a ρ-open set in (Y,σ). Further, f(x) ϵf(V) $\subseteq$f(U), clearly f(U) is a ρ-neighbourhood of f(x) in (Y,σ) and so the theorem holds, by taking W = f(V). □

Converse of this theorem need not be true as seen from the following example.

Example 4.6. As in example 4.4, Let U = $\{a,b,c,d\}$be an open set in (X,τ) and f(a) = a. Then aϵU and for each a = f(a) ϵ f(U) = $\{a,c,d,e\}$, by assumption, there exists a ρ-neighbourhood W_a= $\{a,c,d,e\}$of a in (Y,σ) such that $W_a\subseteq$f(U). But f(U) is not a ρ-open set in (Y,σ).

Theorem 4.7. *A function f :$(X,\tau)\rightarrow(Y,\sigma)$ is ρ-open if and only if for any subet B of (Y,σ) and for any closed set S containing f^{-1}(B), there exsts a ρ-closed set A of (Y,σ) containing B such that f^{-1}(A)$\subseteq$S.*

Proof. Similar to theorem 3.22. □

5 ρ-Homeomorphisms

Definition 5.1. A bijection f :$(X,\tau)\rightarrow(Y,\sigma)$ is called ρ-homeomorphism if f is both ρ-continuous and ρ-open.

Example 5.2. Let X=Y=$\{a,b,c\}$,$\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$, σ = {ϕ,{b},{b,c}, Y}. Define a map f :$(X,\tau)\rightarrow(Y,\sigma)$ by f(a) = c, f(b) = b, f(c) = a. Then f is a ρ-homeomorphism

Theorem 5.3. *Every ρ-homeomorphism is a gp-homeomorphism(resp.$gpr-homeomorphism, \pi gp-homeomorphism$).*

Proof. By Theorem 2.5[17], every ρ-continuous map is gp-continuous (resp.by Theorem2.7[17], gpr-continuous,by Theorem2.11[17] πgp-continuous) and also by Theorem3.4[16],every ρ-open map is gp-open(resp.by Theorem3.4[16], gpr-open,by theorem3.10[16],πgp-open),the proof follows. □

Converse of the above theorem need not be true as seen from the following example.

Example 5.4. (i) Let X=Y=$\{a,b,c\}$, $\tau = \{\emptyset, \{a,b\}, X\}$, σ = {ϕ,{a},{a,b}, Y}. Define f :$(X,\tau)\rightarrow(Y,\sigma)$ by f(a) = c, f(b) = a, f(c) = b.Then f is gp-homeomorphism but not ρ-homeomorphism. Since for the closed set V = $\{c\}$in (Y,σ), $f^{-1}(V)$ = $\{a\}$is not ρ-closed in (X,τ), that is f is not ρ-continuous.

(ii) Let X=Y=$\{a,b,c\}$, $\tau = \{\emptyset, \{a\}, \{a,b\}, \{c,a\}, X\}$, σ = {ϕ,{b},{c,a}, Y}. Let f :$(X,\tau)\rightarrow(Y,\sigma)$ be an identity map.Then f is πgp-homeomorphism but not ρ-homeomorphism. Since for the closed set V = $\{c.a\}$in (Y,σ), f$^{-1}(V)$ = $\{c,a\}$is not ρ-closed in (X,τ), that is f is not ρ-continuous.

(iii) Let X=Y=$\{a,b,c\}$, $\tau = \{\emptyset, \{a\}, \{b,c\}, X\}$, $\sigma = \{\emptyset, \{c\}, \{a,b\}, Y\}$. Define f :$(X,\tau)\to(Y,\sigma)$ by f(a) = c, f(b) = a, f(c) = b.Then f is gpr-homeomorphism but not ρ-homeomorphism. Since for the closed set V = $\{c\}$in (Y,σ), f$^{-1}(V)$ = $\{a\}$is not ρ-closed in (X,τ), that is f is not ρ-continuous.

Theorem 5.5. *Let f :$(X,\tau)\to(Y,\sigma)$ be both contra-open and contra-continuous functions. If f is a gp-homeomorphism ,then f is a ρ-homeomorphism.*

Proof. Let U be open in (X,τ). Then f(U) is gp-open in (Y,σ). Hence Y-f(U) is gp-closed in (Y,σ). Since f is contra-open ,then f(U) is closed in (Y,σ) and so Y-f(U) is open in (Y,σ). By Theorem 2.2[29],Y-f(U) is preclosed in (Y,σ) and by Theorem 3.2[16], Y-f(U) is ρ-closed in (Y,σ), that is f(U) is ρ-open in (Y,σ). Hence f is ρ-open. Let V be closed in (Y,σ). Then f$^{-1}(V)$ is gp-closed in (X,τ). Since f is contra-continuous, then f$^{-1}(V)$ is open in (X,τ).By Theorem2.2[29] and by Theorem 3.2[16], f$^{-1}(V)$is ρ-closed in (X,τ). Hence f is ρ-continuous. Since f is ρ-continuous and ρ-open, therefore f is ρ-homeomorphism. □

Definition 5.6. A function f :$(X,\tau)\to(Y,\sigma)$ is called

1. contra-π-open(resp.regular-contra-open), if f(U) is π-closed(resp.regular closed) in (Y,σ) for every open set U in (X,τ).
2. contra-π-continuous, if f^{-1}(V) is π-open in (X,τ) for every closed set V in (Y,σ).

Theorem 5.7. *Let f :$(X,\tau)\to(Y,\sigma)$ be both contra-π-open and contra-π-continuous functions. If f is a πgp-homeomorphism ,then f is a ρ-homeomorphism.*

Proof. Let U be open in (X,τ). Then f(U) is πgp-open in (Y,σ). Hence Y-f(U) is πgp-closed in (Y,σ). Since f is contra-π-open ,then f(U) is π-closed in (Y,σ) and so Y-f(U) is π-open in (Y,σ). By Theorem 2.4[27],Y-f(U) is preclosed in (Y,σ) and since every π-open is open and by Theorem 3.2[16], Y-f(U) is ρ-closed in (Y,σ), that is f(U) is ρ-open in (Y,σ). Hence f is ρ-open. Let V be closed in (Y,σ). Then f$^{-1}(V)$ is πgp-closed in (X,τ). Since f is contra-π-continuous, then f$^{-1}(V)$ is π-open in (X,τ).By Theorem2.4[27] and since every π-open is open and by Theorem 3.2[16], f$^{-1}(V)$is ρ-closed in (X,τ). Hence f is ρ-continuous. Since f is ρ-continuous and ρ-open, therefore f is ρ-homeomorphism. □

Theorem 5.8. *Let f :$(X,\tau)\to(Y,\sigma)$ be both contra-regular open and RC-continuous functions. If f is a gpr-homeomorphism ,then f is a ρ-homeomorphism.*

Proof. Let U be open in (X,τ). Then f(U) is gpr-open in (Y,σ). Hence Y-f(U) is gpr-closed in (Y,σ). Since f is contra-regular open, then f(U) is regular closed in (Y,σ) and so Y-f(U) is regular open in (Y,σ). By Theorem 3.10[11],Y-f(U) is preclosed in (Y,σ) and since every regular open is open and by Theorem 3.2[16], Y-f(U) is ρ-closed in (Y,σ), that is f(U) is ρ-open in (Y,σ). Hence f is ρ-open. Let V be closed in (Y,σ). Then f$^{-1}(V)$ is gpr-closed in (X,τ). Since f is completely contra-continuous, then $f^{-1}(V)$ is regularopen in (X,τ).By Theorem 3.10[11] and since every regular open is open and by Theorem 3.2[16], $f^{-1}(V)$is ρ-closed in (X,τ). Hence f is ρ-continuous. Since f is ρ-continuous and ρ-open, therefore f is ρ-homeomorphism. □

Theorem 5.9. *Let f :$(X,\tau)\to(Y,\sigma)$ be both contra-open and contra-continuous functions. If f is pre-homeomorphism, then f is a ρ-homeomorphism.*

Proof. Let U be open in (X,τ). Then f(U) is preopen in (Y,σ). Hence Y-f(U) is preclosed in (Y,σ). Since f is contra-open, then f(U) is closed in (Y,σ) and so Y-f(U) is open in (Y,σ). By Theorem 3.2[16], Y-f(U) is ρ-closed in (Y,σ), that is f(U) is ρ-open in (Y,σ). Hence f is ρ-open. Let V be closed in (Y,σ). Then $f^{-1}(V)$ is preclosed in (X,τ). Since f is contra-continuous, then $f^{-1}(V)$ is open in (X,τ).By Theorem 3.2[16], $f^{-1}(V)$is ρ-closed in (X,τ). Hence f is ρ-continuous. Since f is ρ-continuous and ρ-open, therefore f is ρ-homeomorphism. □

Remark 5.10. ρ-homeomorphism and homeomorphism are independent as can be seen from the following examples.

Example 5.11. (i) Let X=$\{a,b,c,d\}$=Y, $\tau=\{\phi,\{a,b\},\{a,b,c\},X\}$,$\sigma=\{\phi$,{a,b}, {a,b,d},Y}. If f :$(X,\tau)\to(Y,\sigma)$ is an identity function, then f is a ρ-homeomorphism but not homeomorphism. Since f is neither continuous nor open.

(ii) Let X=$\{a,b,c\}$=Y, $\tau=\{\phi,\{a\},\{b,c\},X\}$,$\sigma=\{\phi,\{c\},\{a,b\},Y\}$. Define f :$(X,\tau)\to(Y,\sigma)$ by f(a)=c, f(b)=a, f(c)=b. Then f is a homeomorphism but not ρ-homeomorphism.Since for the closed set V=$\{c\}$,f^{-1}(V)=$\{a\}$is not ρ-closed in (X,τ), that is f is not ρ-continuous.

Remark 5.12. ρ-homeomorphism and g-homeomorphism are independent as can be seen from the following examples.

Example 5.13. (i) Let X=$\{a,b,c\}$=Y, $\tau=\{\phi,\{a\},\{a,b\},\{a,c\},X\}$,$\sigma=\{\phi$,{b}, {a,b},Y}. Define f :$(X,\tau)\to(Y,\sigma)$ by f(a)=b, f(b)=a, f(c)=c. Then f is a ρ-homeomorphism but not g-homeomorphism.Since for the open set V=$\{a,c\}$in (X,τ), f(V)=$\{b,c\}$is not g-open in (Y,σ).

(ii) Consider $[0,1]$and $[0,2]$with usual topology. Define f : $[0,1]\to[0,2]$ by f(x) = 2x. Also f^{-1}(x) = x/2. Then f is a g-homeomorphism but not ρ-homeomorphism. Since for the closed set V =$[0,\frac{1}{2}]$ in $[0,2]$, f^{-1}(V) = $[0,\frac{1}{4}]$ is g-closed in $[0,1]$ but not ρ-closed in $[0,1]$, that is f is not ρ-continuous.

Remark 5.14. ρ-homeomorphism and semi-homeomorphism are independent as can be seen from the following examples.

Example 5.15. (i) Let X=Y=$\{a,b,c\}$,$\tau=\{\phi,\{a,b\},X\}$,$\sigma=\{\phi,\{a\},\{a,b\},Y\}$. Let f :$(X,\tau)\to(Y,\sigma)$ be an identity map. Then f is a ρ-homeomorphism.But f is not a semi-homeomorphism. Since for the closed set V=$\{b,c\}$in (Y,σ), f^{-1}(V)=$\{b,c\}$,Which is not closed in (X,τ). Therefore f is not a continuous map.

(ii) Let X=Y=$\{a,b,c\}$,$\tau=\{\phi,\{a\},\{b,c\},X\}$,$\sigma=\{\phi,\{c\},\{a,b\},Y\}$. Define f :$(X,\tau)\to(Y,\sigma)$ by f(a)=c, f(b)=a, f(c)=b. Then f is a semi-homeomorphism.

But f is not a ρ-homeomorphism. Since for the closed set V=$\{c\}$in (Y, σ), f^{-1}(V)=$\{a\}$,Which is not ρ-closed in (X, τ). Therefore f is not a ρ-continuous map.

Remark 5.16. ρ-homeomorphism and pre-homeomorphism are independent as can be seen from the following examples.

Example 5.17. (i) Let X=Y=$\{a, b, c\}$,τ =$\{\phi, \{a\} \{a, b\}, X\}$,$\sigma$ =$\{\phi$,{a},{b}, {a,b},{b,c},Y$\}$. Define f :$(X, \tau) \to (Y, \sigma)$ by f(a)=b, f(b)=a, f(c)=c. Then f is a ρ-homeomorphism.But f is not a pre-homeomorphism. Since for the closed set V=$\{b, c\}$in (Y, σ), f^{-1}(V)=$\{c, a\}$,Which is not preclosed in (X, τ). Therefore f is not a pre-continuous map.

(ii) Let X=Y=$\{a, b, c\}$,τ =$\{\phi, \{a, b\}, X\}$,σ =$\{\phi, \{a\}, \{a, b\}, Y\}$. Define f :$(X, \tau) \to (Y, \sigma)$ by f(a)=c, f(b)=a, f(c)=b. Then f is a pre-homeomorphism. But f is not a ρ-homeomorphism. Since for the closed set V=$\{c\}$in (Y, σ), f^{-1}(V)=$\{a\}$,Which is not ρ-closed in (X, τ). Therefore f is not a ρ-continuous map.

Theorem 5.18. *Let f :$(X, \tau) \to (Y, \sigma)$ be a bijection ρ-continuous map. Then the following statements are equivalent.*

1. f is a ρ-open map.
2. f is a ρ-homeomorphism.
3. f is a ρ-closed map.

Proof. (1)$\to$(2) By hypothesis and by assumption, proof is obvious.

(2)$\to$(3) Let V be a closed set in (X, τ). Then V^{c}is open in (X, τ). By hypothesis, f(V^c) = (f(V))^c is ρ-open in (Y, σ).That is, f(V) is ρ-closed in (Y, σ). Therefore f is a ρ-closed map.

(3)$\to$(1) Let V be a open set in (X, τ). Then V^{c}is closed in (X, τ). By hypothesis, f(V^c) = (f(V))^c is ρ-closed in (Y, σ).That is, f(V) is ρ-open in (Y, σ). Therefore f is a ρ-open map. □

Remark 5.19. The composition of two ρ-homeomorphism maps need not be a ρ-homeomorphism as can be seen from the following example.

Example 5.20. Let X=Y=Z=$\{a, b, c\}$,τ =$\{\phi, \{a, b\}, X\}$,σ =$\{\phi, \{a\}, \{a, b\}, Y\}$, η =$\{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Z\}$. Let f :$(X, \tau) \to (Y, \sigma)$ be an identity map and define g :$(Y, \sigma) \to$(Z,η) by g(a)=b, g(b)=a, g(c)=c. Then both f and g are ρ-homeomorphisms, but their composition gf :$(X, \tau) \to$(Z,η) is not a ρ-homeomorp hism. Since for the closed set V=$\{a\}$in (Z,η), (gf)$^{-1}$(V)=$\{b\}$, Which is not a ρ-closed set in (X, τ). Therefore gf is not a ρ-continuous map and so gf is not a ρ-homeomorphism.

Theorem 5.21. *Let f :$(X, \tau) \to (Y, \sigma)$ be a ρ-homeomorphism. Let A be an open ρ-closed subset of X and let B be a closed subset of Y such that f(A)=B.Assume that $\rho C(X, \tau)$ (the class of all ρ-closed sets of (X, τ)) be closed under finite intersections. Then the restriction f_A:(A,τ_A)$\to$(B,σ_B) is a ρ-homeomorphism.*

Proof. We have to show that f_A is a bijection, f_A is a ρ-open map and f_A is a ρ-continuous map.

(i) Since f is one-one, f_A is also one-one. Also since f(A)=B we have f_A(A)=B so that f_A is onto and hence f_A is a bijection.

(ii) Let U be an open set of (A,τ_A). Then U = A$\bigcap$H, for some open set H in (X,τ).Since f is one-one ,then f(U)=f(A$\bigcap$H)=f(A)$\bigcap$f(H)=B$\bigcap$f(H).Since f is ρ-open and H is an open set in (X,τ), then f(H) is a ρ-open set in (Y,σ). Therefore f(U) is a ρ-open set in (B,σ_B), Hence f_A is a ρ-open map.

(iii) Let V be a closed set in (B,σ_B). Then V=B$\bigcap$K, for some closed set K in (Y,σ).Since B is a closed set in (Y,σ),then V is a closed set in (Y,σ).By hypothesis and assumption, f^{-1}(V)$\bigcap$A=H_1(say) is a ρ-closed set in (X,τ).Since f_A^{-1}(V)=H_1, it is sufficient to show that $H_1$is a ρ-closed set in (A,τ_A).Let G_1be $\tilde{g}$-open in (A,τ_A) such that $H_1 \subseteq G_1$. Then by hypothesis and by Lemma3.21[17],G_1 is $\tilde{g}$-open in X. Since H_1 is a ρ-closed set in (X,τ),we have $Pcl_X(H_1) \subseteq Int(G_1)$. Since A is open and by Lemma 2.10[12],$Pcl_A(H_1) = Pcl_X(H_1) \bigcap A \subseteq Int(G_1) \bigcap A = Int(G_1) \bigcap Int(A) = Int(G_1 \bigcap A) \subseteq Int(G_1)$ and so $H_1 = f_A^{-1}(V)$ is ρ-closed set in (A,τ_A).There fore $f_A$is a ρ-continuous map. Hence $f_A$is a ρ-homeomorphism. □

Definition 5.22. A topological space (X,τ) is called a ρ-hausdorff if for each pair x,y of distinct points of X, there exists ρ-open neighbourhoods $U_1$and $U_2$of x and y,respectively,that are disjoint.

Theorem 5.23. *Let (X,τ) be a topological space and let (Y,σ) be a ρ-hausdorff space. Let f :$(X,\tau) \rightarrow (Y,\sigma)$ be a one-one ρ-irresolute map. Then (X,τ) is also a ρ-hausdorff space.*

Proof. Let x_1,x_2 be any two distinct points of X. Since f is one-one, $x_1 \neq x_2$implies $f(x_1) \neq f(x_2)$. Let $y_1 = f(x_1)$, $y_2 = f(x_2)$ so that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Then y_1, $y_2 \in Y$ such that $y_1 \neq y_2$. Since (Y,σ) is ρ-hausdorff, then there exists ρ-open sets $U_1$and $U_2$of (Y,σ) such that $y_1 \in U_1$,$y_2 \in U_2$and $U_1 \cap U_2 = \phi$. Since f is ρ-irresolute, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are ρ-open sets of (X,τ). Now $f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2) = f^{-1}(\phi) = \phi$, and $y_1 \in U_1$implies $f^{-1}(y_1) \in f^{-1}(U_1)$ implies $x_1 \in f^{-1}(U_1)$, $y_2 \in U_2$ implies $f^{-1}(y_2) \in f^{-1}(U_2)$ implies $x_2 \in f^{-1}(U_2)$. Thus it is shown that for every pair of distinct points x_1,x_2 of X, there exists disjoint ρ-open sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ such that $x_1 \in f^{-1}(U_1)$ and $x_2 \in f^{-1}(U_2)$. Accordingly, the space (X,τ) is a ρ-hausdorff space. □

Theorem 5.24. *Every ρ-compact subset A of a ρ-hausdorff space X is ρ-closed. Assume that $\rho O(X,\tau)$ (the class of all ρ-open sets of (X,τ)) be closed under finite intersections.*

Proof. We shall show that X-A is ρ-open. let x∈X-A,Since X is hausdorff, for every y∈A, there exists disjoint ρ-open neighbourhoods $U_y$and $V_y$of x and y such that $U_y \bigcap V_y = \phi$. Now the collection $\{V_y / y \in A\}$ is a ρ-open cover of A,since A is compact ,there exists a finite subcover $\{y_i$,i=1,...,n$\}$ such that $A \subset \cup\{V_{y_i}$,i=1,...,n$\}$.Let U = $\cap\{U_{y_i}$, i=1,...,n$\}$ and V = $\cup\{V_{y_i}$,i=1,...,n$\}$. Then, by assumption, U is an ρ-open neighbourhood of x. clearly U $\cap$V =ϕ,hence U $\cap$A = ϕ,thus U $\subset$X-A ,which means X-A is ρ-open,therefore A is ρ-closed. □

Theorem 5.25. *Let (X, τ) a topological space and let (Y, σ) be a ρ-hausdorff space. Assume that $\rho O(X, \tau)$ (the class of all ρ-open sets of (X, τ)) be closed under finite intersections. If f , g are ρ-irresolute maps of X into Y,then the set $A = \{x \in X : f(x) = g(x)\}$is a ρ-closed subset of (X, τ).*

Proof. We shall show that X-A is an ρ-open subset of (X, τ).Now X-A = {x∈X: f(x) ≠g(x)}.Let p ∈X-A. Set y_1= f(p), y_2= g(p). By the defnition of X-A, we have $y_1 \neq y_2$. Thus y_1,y_2 are two distinct points of Y. Since $(Y, \sigma)$is a ρ-hausdorff space,there exists ρ-open sets U_1, U_2 of (Y, σ) such that y_1= f(p)∈U_1, y_2= g(p) ∈U_2 and $U_1 \cap U_2 = \phi$.Therefore p ∈$f^{-1}(U_1)$, p ∈$g^{-1}(U_2)$, so that p ∈ $f^{-1}(U_1) \cap g^{-1}(U_2)$ = W(say).Since f and g are ρ-irresolute maps, $f^{-1}(U_1)$ and $g^{-1}(U_2)$ are ρ-open sets of (X, τ) and by assumption W is a ρ-open set containing p.We will now show that W⊂X-A.Let y∈W, since $U_1 \cap U_2 = \phi$, then f(y) ≠g(y) and hence from the defnition of X-A, y ∈X-A. Therefore W ⊂X-A, which means X-A is a ρ-open set. It follows that A is a ρ-closed subset of (X, τ). □

We define another new class of maps called ρ*-closed maps.

Definition 5.26. A map f :$(X, \tau) \rightarrow (Y, \sigma)$ is said to be a ρ*-closed map if the image f(A) is ρ-closed in (Y, σ) for every ρ-closed set A in (X, τ).

Example 5.27. As in example 3.2, f is a ρ*-closed map.

Theorem 5.28. *If f :$(X, \tau) \rightarrow (Y, \sigma)$ is $\tilde{g}$-irresolute and M-preclosed functions then f is a ρ*-closed map.*

Proof. By Theorem 3.16[17], the theorem follows. □

Theorem 5.29. *Every ρ-closed map is a ρ*-closed map if (X, τ) is ρ-$T_S$space.*

Proof. Let f :$(X, \tau) \rightarrow (Y, \sigma)$ be a ρ-closed map and V be a ρ-closed set in (X, τ). Since (X, τ) is a ρ-$T_S$space, then V is a closed set in (X, τ) and since f is ρ-closed, then f(V) is a ρ-closed set in (Y, σ). Hence f is a ρ*-closed map. □

We next introduce a new class of maps called ρ*-homeomorphisms. This class of maps is closed under composition of maps.

Definition 5.30. A bijection f :$(X, \tau) \rightarrow (Y, \sigma)$ is said to be ρ*-homeomorphism if both f and $f^{-1}$are ρ-irresolute.

Example 5.31. Let X=Y=$\{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$,$\sigma = \{\phi, \{a, b\}, Y\}$. Let f :$(X, \tau) \rightarrow (Y, \sigma)$ be an identity map. Then f is a ρ*-homeomorphism.

Theorem 5.32. *A bijective ρ-irresolute map of a ρ-compact space X onto a ρ-hausdorff space Y is a ρ*-homeomorphism.*

Proof. Let (X, τ) be a ρ-compact space and (Y, σ) be a ρ-hausdorff space.Let f :$(X, \tau) \rightarrow (Y, \sigma)$ be a bijective ρ-irresolute map.We have to show that f is a ρ*-homeomorphism.We need only to show that $f^{-1}$is a ρ-irresolute map.Let F be a

ρ-closed set in (X,τ).Since (X,τ) is a ρ-compact space, then by Theorem5.6[17], F is a ρ-compact subset of (X,τ).Since f is irresolute and by Theorem5.7[17],f(F) is a ρ-compact subset of (Y,σ).Since (Y,σ) is a ρ-hausdorff space, then by Theorem5.24, f(F) is a ρ-closed set in (Y,σ).Hence f is a ρ*-homeomorphism. □

Theorem 5.33. *If f :$(X,\tau)\to(Y,\sigma)$ and g :$(Y,\sigma)\to$(Z,η) are ρ*-homeomorphisms then their composition* gf *:$(X,\tau)\to$(Z,η) is also ρ*-homeomorphism.*

Proof. Let V be a ρ-closed set in (Z,η).Now $(\text{gf})^{-1}$(V)= $\text{f}^{-1}(\text{g}^{-1}$(V)).Since g is a ρ*-homeomorphism, then g^{-1}(V) is a ρ-closed set in (Y,σ) and Since f is a ρ*-homeomorphism, then $\text{f}^{-1}(\text{g}^{-1}$(V)) is a ρ-closed set in (X,τ). Therefore gf is ρ-irresolute. Also for a ρ-closed set F in (X,τ), we have (gf)(F)=g(f(F)). Since f is a ρ*-homeomorphism, then f(F) is a ρ-closed set in (Y,σ) and since g is a ρ*-homeomorphism, then g(f(F)) is a ρ-closed set in (Z,η). Therefore $(\text{gf})^{-1}$is ρ-irresolute. Hence gf is a ρ*-homeomorphism. □

Theorem 5.34. *ρ*-homeomorphism is an equivalence relation in the collection of all topological spaces.*

Proof. We have to show that f :$(X,\tau)\to(X,\tau)$ is a ρ*-homeomorphism(reflexive),if f :$(X,\tau)\to(Y,\sigma)$ is a ρ*-homeomorphism then g :$(Y,\sigma)\to(X,\tau)$ is also a ρ*-homeo-morphism(symmetry) and if f :$(X,\tau)\to(Y,\sigma)$ and g :$(Y,\sigma)\to$(Z,η) are ρ*-homeom-orphisms then gf :$(X,\tau)\to$(Z,η) is a ρ*-homeomorphism(transitive).

Reflexive and symmetry are immediate and by theorem 5.33,transitive follows. □

We denote the family of all ρ*-homeomorphism of a topological space (X,τ) onto itself by ρ*-h(X,τ).

Theorem 5.35. *The set ρ*-h(X,τ) is a group under the composition of maps.*

Proof. Define a binary operation Υ: ρ*-h(X,τ) x ρ*-h(X,τ) $\to$ ρ*-h(X,τ) by $\Upsilon(f,g)=$ gf (the composition of f and g) for all f,g$\in\rho$*-h(X,τ). Then by Theorem 5.33, gf$\in\rho$*-h(X,τ) . We know that the composition of maps is associative and the identity map I :$(X,\tau)\to(X,\tau)$ belonging to ρ*-h(X,τ) serves as the identity element. If f$\in\rho$*-h(X,τ) then $\text{f}^{-1}\in$ ρ*-h(X,τ) such that f $\text{f}^{-1}=\text{f}^{-1}$ f = I and so inverse exists for each element of ρ*-h(X,τ). Therefore (ρ*-h(X,τ),) is a group under the operation of composition of maps. □

Theorem 5.36. *Let f :$(X,\tau)\to(Y,\sigma)$ be a ρ*-homeomorphism. Then f induces an isomorphisms from the group ρ*-h(X,τ) onto the group ρ*-h(Y,σ).*

Proof. We define a map κ_f :ρ*-h$(X,\tau)\to\rho$*-h(Y,σ) by $\kappa_f(\theta)=$ f θ f^{-1},for every h$\in$ ρ*-h(X,τ).Where f is a given map. We have to show that κ_f is a bijective homomorphism. Bijection of κ_f is clear. Further, for all θ_1, $\theta_2\in\rho$*-h(X,τ), $\kappa_f(\theta_1\ \theta_2)=$ f $(\theta_1\ \theta_2)$ f^{-1}=(f θ_1 f^{-1}) (f θ_2 f^{-1}) $=\kappa_f(\theta_1)\ \kappa_f(\theta_2)$. Therefore, κ_f is a homomorphism and so it is an isomorphism induced by f. □

Converse of this theorem need not be true as seen from the following example. That is,there exists a map f :$(X,\tau)\to(Y,\sigma)$ which induces an isomorphism κ_f :ρ*-h$(X,\tau)\to\rho$*-h(Y,σ), but not ρ*-homeomorphism.

Example 5.37. As in example 5.17(ii),f is not a ρ*-homeomorphism. But the induced homeomorphism κ_f :ρ*-h$(X,\tau)\to\rho$*-h(Y,σ) is an isomorphism. Since $\kappa_f(\theta_c)$ = f θ_c f^{-1}= θ_a and $\kappa_f(I_x) = I_y$, where θ_c:$(X,\tau)\to(X,\tau)$ and θ_a: $(Y,\sigma)\to(Y,\sigma)$ are defined by θ_c(a) = b,θ_c(b) = a,θ_c(c) = c and θ_a(a) = c,θ_a(b) = b,θ_a(c) = a. Then we have ρ*-h(X,τ) = $\{\theta_c, I_x\}$ and ρ*-h(Y,σ) = $\{\theta_a, I_y\}$, where I_x: $(X,\tau)\to(X,\tau)$ and I_y:$(Y,\sigma)\to(Y,\sigma)$ are identity maps.

Definition 5.38. Let κ_f :ρ*-h$(X,\tau)\to\rho$*-h(Y,σ) be a function defined by $\kappa_f(\theta)$ = f θ f^{-1},for every $\theta\in$ ρ*-h(X,τ). Let κ_f be a homomorphism. Let K = { θ/ $\theta\in\rho$*-h(X,τ) , $\kappa_f(\theta) = I_y$}, where $I_y$is an identity element of ρ*-h(Y,σ). Then K is called the kernel of κ_f and is denoted by kerκ_f.

Theorem 5.39. *Let κ_f be a homomorphism. Then κ_f is one-one if and only if kerκ_f = $\{I_x\}$.*

Proof. suppose κ_f is one-one. Then clearly kerκ_f = $\{I_x\}$.Reverse part is, suppose kerκ_f = $\{I_x\}$,$\kappa_f(\theta_1) = \kappa_f(\theta_2)$ implies f θ_1 f^{-1} = f θ_2 f^{-1}implies (f θ_1 f^{-1}) (f θ_2 $f^{-1})^{-1}$= I_y, hence $\theta_1\theta_2^{-1}\in$kerκ_f=$\{I_x\}$ and so θ_1= θ_2. Therefore κ_f is one-one. □

Theorem 5.40. *Let κ_f :ρ*-h$(X,\tau)\to\rho$*-h(Y,σ) be a homomorphism. Then kerκ_f is a normal subgroup of ρ*-h(X,τ).*

Proof. Since $\kappa_f(I_x) = I_y$, $I_x\in$kerκ_f and hence kerκ_f $\neq\phi$. Now let θ_1, $\theta_2\in$ kerκ_f ,then $\kappa_f(\theta_1) = \kappa_f(\theta_2) = I_y$. Therfore $\kappa_f(\theta_1{\theta_2}^{-1}) = \kappa_f(\theta_1)\ \kappa_f(\theta_2^{-1}) = I_y$. Thus $\theta_1{\theta_2}^{-1}\in$ker$\kappa_f$ and hence kerκ_f is a subgroup of ρ*-h(X,τ). Now let $\theta_1\in$kerκ_f and g $\in\rho$*-h(X,τ), then κ_f(g θ_1 g^{-1}) = $I_y$and so g θ_1 $g^{-1}\in$kerκ_f , therefore kerκ_f is a normal subgroup of ρ*-h(X,τ). □

Theorem 5.41. *Let κ_f:ρ*-h$(X,\tau)\to\rho$*-h(Y,σ) be an epimorphism. Let K be the kernel of K_f. Then ρ*-h(X,τ) / K $\cong\rho$*-h(Y,σ).[Fundamental theorem of homomorphism]*

Proof. Define μ: ρ*-h(X,τ) / K $\to$ ρ*-h(Y,σ) by μ(Ka)=κ_f(a). Clearly $\mu$is a well defined bijection.Now μ(KaKb)=μ(Kab)=κ_f(ab)=κ_f(a) κ_f(b) =μ(Ka)μ(Kb), therefore μ is a homomorphism. Thus κ_f induces an isomorphism μ from ρ*-h(X,τ) / K onto ρ*-h(Y,σ). Hence ρ*-h(X,τ) / K $\cong$ ρ*-h(Y,σ). □

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