NEUTROSOPHIC SEMI-CONTINUOUS MULTIFUNCTIONS

R. DHAVASEELAN, S. JAFARI, N. RAJESH AND F. SMARANDACHE

Abstract. In this paper we introduce the concepts of neutrosophic upper and neutrosophic lower semi-continuous multifunctions and study some of their basic properties.

1. introduction

There is no doubt that the theory of multifunctions plays an important role in functional analysis and fixed point theory. It also has a wide range of applications in economic theory, decision theory, non-cooperative games, artificial intelligence, medicine and information sciences. Inspired by the research works of F. Smarandache [[1],[2]], we introduce and study the notions of neutrosophic upper and neutrosophic lower semi-continuous multifunctions in this paper. Further, we present some characterizations and properties of such notions.

2. preliminaries

Throughout this paper, by (X, τ) or simply by X we will mean a topological space in the classical sense, and (Y, τ_1) or simply Y will stand for a neutrosophic topological space as defined by Salama [**?**].

Definition 2.1. [1] *Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $A = < x, \mu_A(x), \sigma_A(x), \gamma_A(x) >$, where $\mu_A(x)$, $\sigma_A(x)$ and $\gamma_A(x)$ which represent the degree of member ship function, the degree of indeterminacy, and the degree of non-membership, respectively of each element $x \in X$ to the set A.*

Definition 2.2. [3] *A neutrosophic topology on a nonempty set X is a family τ of neutrosophic subsets of X which satisfies the following three conditions:*

(1) $0, 1 \in \tau$,
(2) *If $g, h \in \tau$, their $g \wedge h \in \tau$,*
(3) *If $f_i \in \tau$ for each $i \in I$, then $\vee_{i \in I} f_i \in \tau$.*

The pair (X, τ) is called a neutrosophic topological space.

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Definition 2.3. *Members of τ are called neutrosophic open sets, denoted by $NO(X)$, and complement of neutrosophic open sets are called neutrosophic closed sets, where the complement of a neutrosophic set A, denoted by A^c, is $1 - A$.*

Neutrosophic sets in Y will be denoted by $\lambda, \gamma, \delta, \rho$, etc., and although subsets fo X will be denoted by A, B, U, V, etc. A neutrosophic point in Y with support $y \in Y$ and value $\alpha(0 < \alpha \leq 1)$ is denoted by y_α. A neutrosophic set λ in Y is said to be quasi-coincident (q-coincident) with a neutrosophic set μ, denoted by $\lambda q \mu$, if and only if there exists $y \in Y$ such that $\lambda(y) + \mu(y) > 1$. A neutrosophic set λ of Y is called a neutrosophic neighbourhood of a fuzzy point y_α in Y if there exists a neutrosophic open set μ in Y such that $y_\alpha \in \mu \leq \lambda$. The intersection of all neutrosophic closed sets of Y containing λ is called the neutrosophic closure of λ and is denoted by $\mathrm{Cl}(\lambda)$. The union of all neutrosophic open sets contained in λ is called the neutrosophic interior of λ and is denoted by $\mathrm{Int}(\lambda)$. The family of all open sets of a topological space X is denoted by $O(X)$ and $O(X, x)$ denoted the family $\{A \in O(X) | x \in A\}$, where x is a point of X.

Definition 2.4. *Let (X, τ) be a topological space in the classical sense and (Y, τ_1) be an neutrosophic topological space. $F : (X, \tau) \to (Y, \tau_1)$ is called a neutrosophic multifunction if and only if for each $x \in X, F(x)$ is a neutrosophic set in Y.*

Definition 2.5. *For a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$, the upper inverse $F^+(\lambda)$ and lower inverse $F^-(\lambda)$ of a neutrosophic set λ in Y are defined as follows:*
$F^+(\lambda) = \{x \in X | F(x) \leq \lambda\}$ and $F^-(\lambda) = \{x \in X | F(x) q \lambda\}$.

Lemma 2.6. *For a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$, we have $F^-(1 - \lambda) = X - F^+(\lambda)$, for any neutrosophic set λ in Y.*

3. Neutrosophic semicontinuous multifunctions

Definition 3.1. *A neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ is said to be*

(1) *neutrosophic upper semicontinuous at a point $x \in X$ if for each $\lambda \in NO(Y)$ containing $F(x)$ (therefore, $F(x) \leq \lambda$), there exists $U \in O(X, x)$ such that $F(U) \leq \lambda$ (therefore $U \subset F^+(\lambda)$).*
(2) *neutrosophic lower semicontinuous at a point $x \in X$ if for each $\lambda \in NO(Y)$ with $F(x) q \lambda$, there exists $U \in O(X, x)$ such that $U \subseteq F^-(\lambda)$.*
(3) *neutrosophic upper semicontinuous (neutrosophic lower semicontinuous) if it is neutrosophic upper semicontinuous (neutrosophic lower semicontinuous) at each point $x \in X$.*

Theorem 3.2. *The following assertions are equivalent for a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$:*

(1) *F is neutrosophic upper semicontinuous;*
(2) *For each point x of X and each neutrosophic neighbourhood λ of $F(x)$, $F^+(\lambda)$ is a neighbourhood of x;*
(3) *For each point x of X and each neutrosophic neighbourhood λ of $F(x)$, there exists a neighbourhood U of x such that $F(U) \leq \lambda$;*
(4) *$F^+(\lambda) \in O(X)$ for oeach $\lambda \in NO(Y)$;*
(5) *$F^-(\delta)$ is a closed set in X for each neutrosophic closed set δ of Y;*
(6) *$\mathrm{Cl}(F^-(\mu)) \subseteq F^-(\mathrm{Cl}(\mu))$ for each neutrosophic set μ of Y.*

Proof. (1)⇒(2) Let $x \in X$ and μ be a neutrosophic neighbourhood of $F(x)$. Then there exists $\lambda \in NO(Y)$ such that $F(x) \leq \lambda \leq \mu$, By (1), there exists $U \in O(X,x)$ such that $F(U) \leq \lambda$. Therefore $x \in U \subseteq F^+(\mu)$ and hence $F^+(\mu)$ is a neighbourhood of x.
(2)⇒(3) Let $x \in X$ and λ be a neutrosophic neighbourhood of $F(x)$. Put $U = F^+(\lambda)$. Then by (2), U is neighbourhood of x and $F(U) = \bigvee_{x\in U} F(x) \leq \lambda$.
(3)⇒(4) Let $\lambda \in NO(Y)$, we want to show that $F^+(\lambda) \in O(X)$. So let $x \in F^+(\lambda)$. Then there exists a neighbourhood G of x such that $F(G) \leq \lambda$. Therefore for some $U \in O(X,x), U \subseteq G$ and $F(U) \leq \lambda$. Therefore we get $x \in U \subseteq F^+(\lambda)$ and hence $F^+(\lambda) \in O(X)$.
(4)⇒(5) Let δ be a neutrosophic closed set in Y. So, we have $X \backslash F^-(\delta) = F^+(1-\delta) \in O(X)$ and hence $F^-(\delta)$ is closed set in X.
(5)⇒(6) Let μ be any neutrosophic set in Y. Since $\mathrm{Cl}(\mu)$ is neutrosophic closed set in Y, $F^-(\mathrm{Cl}(\mu))$ is closed set in X and $F^-(\mu) \subseteq F^-(\mathrm{Cl}(\mu))$. Therefore, we obtain $\mathrm{Cl}(F^-(\mu)) \subseteq F^-(\mathrm{Cl}(\mu))$.
(6)⇒(1) Let $x \in X$ and $\lambda \in NO(Y)$ with $F(x) \leq \lambda$. Now $F^-(1-\lambda) = \{x \in X | F(x) q (1-\lambda)\}$. So, for x not belongs to $F^-(1-\lambda)$. Then, we must have $F(x) \hbar (1-\lambda)$ and this is implies $F(x) \leq 1-(1-\lambda) = \lambda$ which is true. Therefore $x \notin F^-(1-\lambda)$ by (6), $x \notin \mathrm{Cl}(F^-(1-\lambda))$ and there exists $U \in O(X,x)$ such that $U \cap F^-(1-\lambda) = \emptyset$. Therefore, we obtain $F(U) = \bigvee_{x\in U} F(x) \leq \lambda$. This proves F is neutrosophic upper semicontinuous. □

Theorem 3.3. *The following statements are equivalent for a neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$:*

(1) *F is neutrosophic lower semicontinuous;*
(2) *For each $\lambda \in NO(Y)$ and each $x \in F^-(\lambda)$, there exists $U \in O(X,x)$ such that $U \subseteq F^-(\lambda)$;*
(3) *$F^-(\lambda) \in O(X)$ for every $\lambda \in NO(Y)$.*
(4) *$F^+(\delta)$ is a closed set in X for every neutrosophic closed set δ of Y;*
(5) *$\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(\mu))$ for every neutrosophic set μ of Y;*
(6) *$F(\mathrm{Cl}(A)) \leq \mathrm{Cl}(F(A))$ for every subset A of X;*

Proof. (1)⇒(2) Let $\lambda \in NO(Y)$ and $x \in F^-(\lambda)$ with $F(x)q\lambda$. Then by (1), there exists $U \in O(X, x)$ such that $U \subseteq F^-(\lambda)$.
(2)⇒(3) Let $\lambda \in NO(Y)$ adn $x \in F^-(\lambda)$. Then by (2), there exists $U \in O(X, x)$ such that $U \subseteq F^-(\lambda)$. Therefore, we have $x \in U \subseteq \operatorname{Cl}\operatorname{Int}(U) \subseteq \operatorname{Cl}\operatorname{Int}(F^-(\lambda))$ and hence $F^-(\lambda) \in O(X)$.
(3)⇒(4) Let δ be a neutrosophic closed in Y. So we have $X \backslash F^+(\delta) = F^-(1-\delta) \in O(X)$ and hence $F^+(\delta)$ is closed set in X.
(4)⇒(5) Let μ be any neutrosophic set in Y. Since $\operatorname{Cl}(\mu)$ is neutrosophic closed set in Y, then by (4), we have $F^+(\operatorname{Cl}(\mu))$ is closed set in X and $F^+(\mu) \subseteq F^+(\operatorname{Cl}(\mu))$. Therefore, we obtain $\operatorname{Cl}(F^+(\mu)) \subseteq F^+(\operatorname{Cl}(\mu))$.
(5)⇒(6) Let A be any subset of X. By (5), $\operatorname{Cl}(A) \subseteq \operatorname{Cl} F^+(F(A)) \subseteq F^+(\operatorname{Cl}(F(A)))$. Therefore we obtain $\operatorname{Cl}(A) \subseteq F^+(\operatorname{Cl} F(A))$. This implies that $F(\operatorname{Cl}(A)) \leq \operatorname{Cl} F(A)$.
(6)⇒(5) Let μ be any neutrosophic set in Y. By (6), $F(\operatorname{Cl} F^+(\mu)) \leq \operatorname{Cl}(F(F^+(\mu)))$ and hence $\operatorname{Cl}(F^+(\mu)) \subseteq F^+(\operatorname{Cl}(F(F^+(\mu)))) \subseteq F^+(\operatorname{Cl}(\mu))$. Therefore $\operatorname{Cl}(F^+(\mu)) \subseteq F^+(\operatorname{Cl}(\mu))$.
(5)⇒(1) Let $x \in X$ and $\lambda \in NO(Y)$ with $F(x)q\lambda$. Now, $F^+(1-\lambda) = \{x \in X | F(x) \leq 1-\lambda\}$. So, for x not belongs to $F^+(1-\lambda)$, then we have $F(x) \nleq 1-\lambda$ and this implies that $F(x)q\lambda$. Therefore, $x \notin F^+(1-\lambda)$. Since $1-\lambda$ is neutrosophic closed set in Y, by (5), $x \notin \operatorname{Cl}(F^+(1-\lambda))$ and there exists $U \in O(X, x)$ such that $\emptyset = U \cap F^+(1-\lambda) = U \cap (X \backslash F^-(\lambda))$. Therefore, we obtain $U \subseteq F^-(\lambda)$. This proves F is neutrosophic lower semicontinuous. □

Definition 3.4. *For a given neutrosophic multifunction* $F : (X, \tau) \to (Y, \tau_1)$*, a neutrosophic multifunction* $\operatorname{Cl}(F) : (X, \tau) \to (Y, \tau_1)$ *is defined as* $(\operatorname{Cl} F)(x) = \operatorname{Cl} F(x)$ *for each* $x \in X$*.*

We use $\operatorname{Cl} F$ and the following Lemma to obtain a characterization of lower neutrosophic semicontinuous multifunction.

Lemma 3.5. *If* $F : (X, \tau) \to (Y, \tau_1)$ *is a neutrosophic multifunction, then* $(\operatorname{Cl} F)^-(\lambda) = F^-(\lambda)$ *for each* $\lambda \in NO(Y)$*.*

Proof. Let $\lambda \in NO(Y)$ and $x \in (\operatorname{Cl} F)^-(\lambda)$. This means that $(\operatorname{Cl} F)(x)q\lambda$. Since $\lambda \in NO(Y)$, we have $F(x)q\lambda$ and hence $x \in F^-(\lambda)$. Therefore $(\operatorname{Cl} F)^-(\lambda) \subseteq F^-(\lambda) - - - (*)$. Conversely, let $x \in F^-(\lambda)$ since $\lambda \in NO(Y)$ then $F(x)q\lambda \subseteq (\operatorname{Cl} F)(x)q\lambda$ and hence $x \in (\operatorname{Cl} F)^-(\lambda)$. Therefore $F^-(\lambda) \subseteq (\operatorname{Cl} F)^-(\lambda) - - - - (**)$. From $(*)$ and $(**)$, we get $(\operatorname{Cl} F)^-(\lambda) = F^-(\lambda)$. □

Theorem 3.6. *A neutrosophic multifunction* $F : (X, \tau) \to (Y, \tau_1)$ *is neutrosophic lower semicontinuous if and only if* $\operatorname{Cl} F : (X, \tau) \to (Y, \tau_1)$ *is neutrosophic lower semicontinuous.*

Proof. Suppose F is neutrosophic lower semicontinuous. Let $\lambda \in NO(Y)$ and $F(x)q\lambda$. This means that $x \in F^-(\lambda)$. Then there exists $U \in$

$O(X,x)$ such that $U \subseteq F^-(\lambda)$. Therefore, we have $x \in U \subseteq \mathrm{Int}(U) \subseteq \mathrm{Int}\, F^-(\lambda)$ and hence $F^-(\lambda) \in O(X)$. Then by Lemma 3.5, we have $U \subseteq F^-(\lambda) = (\mathrm{Cl}\, F)^-(\lambda)$ and $(\mathrm{Cl}\, F)^-(\lambda) \in O(X)$, and hence $(\mathrm{Cl}\, F)(x)q\lambda$. Therefore $\mathrm{Cl}\, F$ is fuzy lower semicontinuous. Conversely, suppose $\mathrm{Cl}\, F$ is neutrosophic lower semicontinuous. If for each $\lambda \in NO(Y)$ with $(\mathrm{Cl}\, F)(x)q\lambda$ and $x \in (\mathrm{Cl}\, F)^-(\lambda)$ then there exists $U \in O(X,x)$ such that $U \subseteq (\mathrm{Cl}\, F)^-(\lambda)$. By Lemma 3.5 and Theorem 3.3 (3), we have $U \subseteq (\mathrm{Cl}\, F^-(\lambda)) = F^-(\lambda)$ and $F^-(\lambda) \in O(X)$. Therefore F is neutrosophic lower semicontinuous. □

Definition 3.7. *Given a family $\{F_i : (X,\tau) \to (Y,\sigma) : i \in I\}$ of neutrosophic multifunctions, we define the union $\underset{i\in I}{\vee} F_i$ and the intersection $\underset{i\in I}{\wedge} F_i$ as follows: $\underset{i\in I}{\vee} F_i : (X,\tau) \to (Y,\sigma)$, $(\underset{i\in I}{\vee} F_i)(x) = \underset{i\in I}{\vee} F_i(x)$ and $\underset{i\in I}{\wedge} F_i : (X,\tau) \to (Y,\sigma)$, $(\underset{i\in I}{\wedge} F_i)(x) = \underset{i\in I}{\wedge} F_i(x)$.*

Theorem 3.8. *If $F_i : X \to Y$ are neutrosophic upper semi-continuous multifunctions for $i = 1, 2, ..., n$, then $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic upper semi-continuous multifunction.*

Proof. Let A be a neutrosophic open set of Y. We will show that $(\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A) = \{x \in X : \overset{n}{\underset{i\in I}{\vee}} F_i(x) \subset A\}$ is open in X. Let $x \in (\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$. Then $F_i(x) \subset A$ for $i = 1, 2, ..., n$. Since $F_i : X \to Y$ is neutrosophic upper semi-continuous multifunction for $i = 1, 2, ..., n$, then there exists an open set U_x containing x such that for all $z \in U_x$, $F_i(z) \subset A$. Let $U = \overset{n}{\underset{i\in I}{\cup}} U_x$. Then $U \subset (\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$. Thus, $(\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$ is open and hence $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic upper semi-continuous multifunction. □

Lemma 3.9. *Let $\{A_i\}_{i\in I}$ be a family of neutrosophic sets in a neutrosophic topological space X. Then a neutrosophic point x is quasi-coincident with $\vee A_i$ if and only if there exists an $i_0 \in I$ such that xqA_{i_0}.*

Theorem 3.10. *If $F_i : X \to Y$ are neutrosophic lower semi-continuous multifunctions for $i = 1, 2, ..., n$, then $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic lower semi-continuous multifunction.*

Proof. Let A be a neutrosophic open set of Y. We will show that $(\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A) = \{x \in X : (\overset{n}{\underset{i\in I}{\vee}} F_i)(x)qA\}$ is open in X. Let $x \in (\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$. Then $(\overset{n}{\underset{i\in I}{\vee}} F_i)(x)qA$ and hence $F_{i0}(x)qA$ for an i_0. Since $F_i : X \to Y$ is neutrosophic lower semi-continuous multifunction, there exists an open set U_x containing x such that for all $z \in U$, $F_{i0}(z)qA$.

Then $(\overset{n}{\underset{i\in I}{\vee}} F_i)(z)qA$ and hence $U \subset (\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$. Thus, $(\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$ is open and hence $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic lower semi-continuous multifunction. □

Theorem 3.11. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*

(1) *$F_i = F_{|U_i}$ is a neutrosophic lower semi-continuous multifunction for all $i \in I$,*
(2) *F is neutrosophic lower semi-continuous.*

Proof. (1) ⇒ (2): Let $x \in X$ and A be a neutrosophic open set in Y with $x \in F^-(A)$. Since $\{U_i : i \in I\}$ is an open cover for X, then $x \in U_{i0}$ for an $i_0 \in I$. We have $F(x) = F_{i0}(x)$ and hence $x \in F^-_{i0}(A)$. Since $F_{|U_i0}$ is neutrosophic lower semi-continuous, there exists an open set $B = G \cap U_{i0}$ in U_{i0} such that $x \in B$ and $F^-(A) \cap U_{i_0} = F_{|U_i}(A) \supset B = G \cap U_{i0}$, where G is open in X. We have $x \in B = G \cap U_{i0} \subset F^-_{|U_i0}(A) = F^-(A) \cap U_{i0} \subset F^-(A)$. Hence, F is neutrosophic lower semi-continuous.
(2) ⇒ (1): Let $x \in X$ and $x \in U_i$. Let A be a neutrosophic open set in Y with $F_i(x)qA$. Since F is lower semi-continuous and $F(x) = F_i(x)$, there exists an open set U containing x such that $U \subset F^-(A)$. Take $B = U_i \cap U$. Then B is open in U_i containing x. We have $B \subset F^-i(A)$. Thus F_i is a neutrosophic lower semi-continuous. □

Theorem 3.12. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*

(1) *$F_i = F_{|U_i}$ is a neutrosophic upper semi-continuous multifunction for all $i \in I$,*
(2) *F is neutrosophic upper semi-continuous.*

Proof. It is similar to that of Theorem 3.11. □

Remark 3.13. *A subset A of a topological space (X, τ) can be considered as a neutrosophic set with characteristic function defined by*

$$A(x) = \begin{cases} 1 & \textit{if } x \in A \\ 0 & \textit{if } x \notin A. \end{cases}$$

Let (Y, σ) be a neutrosophic topological space. The neutrosophic sets of the form $A \times B$ with $A \in \tau$ and $B \in \sigma$ form a basis for the product neutrosophic topology $\tau \times \sigma$ on $X \times Y$, where for any $(x, y) \in X \times Y$, $(A \times B)(x, y) = min\{A(x), B(y)\}$.

Definition 3.14. *For a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$, the neutrosophic graph multifunction $G_F : X \to X \times Y$ of F is defined by $G_F(x) = x_1 \times F(x)$ for every $x \in X$.*

Theorem 3.15. *If the neutrosophic graph multifunction G_F of a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$ is neutrosophic lower semi-continuous, then F is neutrosophic lower semi-continuous.*

Proof. Suppose that G_F is neutrosophic lower semi-continuous and $x \in X$. Let A be a neutrosophic open set in Y such that $F(x)qA$. Then there exists $y \in Y$ such that $(F(x))(y) + A(y) > 1$. Then $(G_F(x))(x, y)+(X\times A)(x, y) = (F(x))(y)+A(y) > 1$. Hence, $G_F(x)q(X\times A)$. Since G_F is neutrosophic lower semi-continuous, there exists an open set B in X such that $x \in B$ and $G_F(b)q(X \times A)$ for all $b \in B$. Let there exists $b_0 \in B$ such that $F(b_0)qA$. Then for all $y \in Y$, $(F(b_0))(y)+A(y) < 1$. For any $(a, c) \in X \times Y$, we have $(G_F(b_0))(a, c) \subset (F(b_0))(c)$ and $(X \times A)(a, c) \subset A(c)$. Since for all $y \in Y$, $(F(b_0))(y) + A(y) < 1$, $(G_F(b_0))(a, c) + (X \times A)(a, c) < 1$. Thus, $G_F(b_0)q(X \times A)$, where $b_0 \in B$. This is a contradiction since $G_F(b)q(X \times A)$ for all $b \in B$. Hence, F is neutrosophic lower semi-continuous. □

Theorem 3.16. *If the neutrosophic graph multifunction G_F of a neutrosophic multifunction $F : X \to Y$ is neutrosophic upper semi-continuous, then F is neutrosophic upper semi-continuous.*

Proof. Suppose that G_F is neutrosophic upper semi-continuous and let $x \in X$. Let A be neutrosophic open in Y with $F(x) \subset A$. Then $G_F(x) \subset X\times A$. Since G_F is neutrosophic upper semi-continuous, there exists an open set B containing x such that $G_F(B) \subset X \times A$. For any $b \in B$ and $y \in Y$, we have $(F(b))(y) = (G_F(b))(b, y) \subset (X \times A)(b, y) = A(y)$. Then $(F(b))(y) \subset A(y)$ for all $y \in Y$. Thus, $F(b) \subset A$ for any $b \in B$. Hence, F is neutrosophic upper semi-continuous. □

Theorem 3.17. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction. Then the following are equivalent:*

(1) *F is neutrosophic lower semi-continuous,*
(2) *For any $x \in X$ and any net $(x_i)_{i\in I}$ converging to x in X and each neutrosophic open set B in Y with $x \in F^-(B)$, the net $(x_i)_{i\in I}$ is eventually in $F^-(B)$.*

Proof. (1) ⇒ (2): Let (x_i) be a net converging to x in X and B be any neutrosophic open set in Y with $x \in F^-(B)$. Since F is neutrosophic lower semi-continuous, there exists an open set $A \subset X$ containing x such that $A \subset F^-(B)$. Since $x_i \to x$, there exists an index $i_0 \in I$ such that $x_i \in A$ for every $i \geq i_0$. We have $x_i \in A \subset F^-(B)$ for all $i \geq i_0$. Hence, $(x_i)_{i\in I}$ is eventually in $F^-(B)$.
(2) ⇒ (1): Suppose that F is not neutrosophic lower semi-continuous. There exists a point x and a neutrosophic open set A with $x \in F^-(A)$ such that $B \not\subseteq F^-(A)$ for any open set $B \subset X$ containing x. Let $x_i \in B$ and $x_i \notin F^-(A)$ for each open set $B \subset X$ containing x. Then the neighborhood net (x_i) converges to x but $(x_i)_{i\in I}$ is not eventually in $F^-(A)$. This is a contradiction. □

Theorem 3.18. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction. Then the following are equivalent:*

(1) *F is neutrosophic upper semi-continuous,*
(2) *For any $x \in X$ and any net (x_i) converging to x in X and any neutrosophic open set B in Y with $x \in F^+(B)$, the net (x_i) is eventually in $F^+(B)$.*

Proof. The proof is similar to that of Theorem 3.17. □

Theorem 3.19. *The set of all points of X at which a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$ is not neutrosophic upper semi-continuous is identical with the union of the frontier of the upper inverse image of neutrosophic open sets containing $F(x)$.*

Proof. Suppose F is not neutrosophic upper semi-continuous at $x \in X$. Then there exists a neutrosophic open set A in Y containing $F(x)$ such that $A \cap (X \backslash F^+(B)) \neq \emptyset$ for every open set A containing x. We have $x \in \mathrm{Cl}(X \backslash F^+(B)) = X \backslash \mathrm{Int}(F^+(B))$ and $x \in F^+(B)$. Thus, $x \in Fr(F^+(B))$. Conversely, let B be a neutrosophic open set in Y containing $F(x)$ with $x \in Fr(F^+(B))$. Suppose that F is neutrosophic upper semi-continuous at x. There exists an open set A containing x such that $A \subset F^+(B)$. We have $x \in \mathrm{Int}(F^+(B))$. This is a contradiction. Thus, F is not neutrosophic upper semi-continuous at x. □

Theorem 3.20. *The set of all points of X at which a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$ is not neutrosophic lower semi-continuous is identical with the union of the frontier of the lower inverse image of neutrosophic closed sets which are quasi-coincident with $F(x)$.*

Proof. It is similar to that of Theorem 3.19. □

Definition 3.21. *A neutrosophic set λ of a neutrosophic topological space Y is said to be neutrosophic compact relative to Y if every cover $\{\lambda_\alpha\}_{\alpha \in \Delta}$ of λ by neutrosophic open sets of Y has a finite subcover $\{\lambda_i\}_{i=1}^n$ of λ.*

Definition 3.22. *A neutrosophic set λ of a neutrosophic topological space Y is said to be neutrosophic Lindelof relative to Y if every cover $\{\lambda_\alpha\}_{\alpha \in \Delta}$ of λ by neutrosophic open sets of Y has a countable subcover $\{\lambda_n\}_{n \in N}$ of λ.*

Definition 3.23. *A neutrosophic topological space Y is said to be neutrosophic compact if χ_Y (characteristic function of Y) is neutrosophic compact relative to Y.*

Definition 3.24. *A neutrosophic topological space Y is said to be neutrosophic Lindelof if χ_Y (characteristic function of Y) is neutrosophic Lindelof relative to Y.*

Definition 3.25. *A neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ is said to be punctually neutrosophic compact (resp. punctually neutrosophic Lindelof) if for each $x \in X, F(x)$ is neutrosophic compact (resp. neutrosophic Lindelof).*

Theorem 3.26. *Let the neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ be a neutrosophic upper semicontinuous and F is punctually neutrosophic compact. If A is compact relative to X, then $F(A)$ is neutrosophic compact relative to Y.*

Proof. Let $\{\lambda_\alpha | \alpha \in \Delta\}$ be any cover of $F(Z)$ by neutrosophic copen sets of Y. We claim that $F(A)$ is neutrosophic compact relative to Y. For each $x \in A$, there exists a finite subset $\Delta(x)$ of Δ such that $F(x) \leq \cup\{\lambda_\alpha | \alpha \in \Delta(x)\}$. Put $\lambda(x) = \cup\{\lambda_\alpha | \alpha \in \Delta(x)\}$. Then $F(x) \leq \lambda(x) \in NO(Y)$ and there exists $U(x) \in O(X, x)$ such that $F(U(x)) \leq \lambda(x)$. Since $\{U(x) | x \in A\}$ is an open cover of A there exists a finite number of A, say, $x_1, x_2, .., x_n$ such that $A \subseteq \cup\{U(x_i) | i = 1, 2, .., n\}$. Therefore we obtain $F(A) \leq F(\bigcup\limits_{i=1}^{n} U(x_i)) \leq \bigcup\limits_{i=1}^{n} F(U(x_i)) \leq \bigcup\limits_{i=1}^{n} \lambda(x_i) \leq \bigcup\limits_{i=1}^{n} (\bigcup\limits_{\alpha \in \Delta(x_i)} \lambda_\alpha)$. This shows that $F(A)$ is neutrosophic compact relative to Y. □

Theorem 3.27. *Let the neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ be a neutrosophic upper semicontinuous and F is punctually neutrosophic Lindelof. If A is Lindelof relative to X, then $F(A)$ is neutrosophic Lindelof relative to Y.*

Proof. The proof is similar to that of Theorem 3.26 □

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Department of Mathematics, Lady Doak College, Madurai, Tamil Nadu, India.
E-mail address: dhavaseelan.r@gmail.com

College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark.
E-mail address: jafaripersia@gmail.com

Department of Mathematics, Rajah Serfoji Govt. College, Thanjavur-613005, Tamilnadu, India.
E-mail address: nrajesh_topology@yahoo.co.in

Mathematics & Science Department, University of New Maxico, 705 Gurley Ave, Gallup, NM 87301, USA.
E-mail address: fsmarandache@gmail.com