# Operations on Neutrosophic Vague Graphs

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**Abstract:** In this manuscript, the operations on neutrosophic vague graphs are introduced. Moreover, Cartesian product, cross product, lexicographic product, strong product and composition of neutrosophic vague graph are investigated and the proposed concepts are illustrated with examples.

**Keywords:** Neutrosophic vague graph, operations of neutrosophic vague graph, Cartesian product.
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## 1. Introduction

Vague sets are regarded as a special case of context-dependent fuzzy sets. Initially, vague set theory was first investigated by Gau and Buehrer [25] which is an extension of fuzzy set theory. In order to handle the indeterminate and inconsistent information, the neutrosophic set is introduced by Florentin Smarandache and has been studied extensively (see [5] - [24]). Neutrosophic set and related notions have shown applications in many different fields. In the definition of neutrosophic set, the indeterminacy value is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are defined completely independent, if the sum of these values lies between 0 and 3. Neutrosophic vague set is introduced in [6]. Al-Quran and Hassan in [2] introduced the concept of neutrosophic vague soft expert set as a combination of neutrosophic vague set and soft expert set in order to improve the reasonability of decision making in reality. Neutrosophic vague graphs are investigated in [16]. Motivated by papers [6, 16], we introduce the concept of operations on neutrosophic vague graphs. The major contributions of this work are as follows:

- Operations on neutrosophic vague graphs are established.
- Cartesian product, cross product, lexicographic product, strong product and composition of neutrosophic vague graph are explained with examples.

## 2. Preliminaries

**Definition 2.1.** *[25] A vague set $A$ on a non empty set $X$ is a pair $(T_A, F_A)$, where $T_A : X \to [0,1]$ and $F_A : X \to [0,1]$are true membership and false membership functions, respectively, such that*

$$0 \leq T_A(x) + F_A(y) \leq 1 \text{ for any } x \in X.$$

*Let $X$ and $Y$ be two non-empty sets. A vague relation $R$ of $X$ to $Y$ is a vague set $R$ on $X \times Y$ that is $R = (T_R, F_R)$, where $T_R : X \times Y \to [0,1], F_R : X \times Y \to [0,1]$ and satisfy the condition:*

$$0 \leq T_R(x, y) + F_R(x, y) \leq 1 \text{ for any } x \in X.$$

**Definition 2.2.** *[7] Let $G^* = (V, E)$ be a graph. A pair $G = (J, K)$ is called a vague graph on $G^*$, where $J = (T_J, F_J)$ is a vague set on $V$ and $K = (T_K, F_K)$ is a vague set on $E \subseteq V \times V$ such that for each $xy \in E$,*

$$T_K(xy) \leq \min(T_J(x), T_J(y)) \text{ and } F_K(xy) \geq \max(T_J(x), F_J(y)).$$

**Definition 2.3.** *[17] A Neutrosophic set $A$ is contained in another neutrosophic set $B$, (i.e) $A \subseteq B$ if $\forall x \in X, T_A(x) \leq T_B(x), I_A(x) \geq I_B(x)$and $F_A(x) \geq F_B(x)$.*

**Definition 2.4.** *[11, 17] Let $X$ be a space of points (objects), with a generic elements in $X$ denoted by $x$. A single valued neutrosophic set $A$ in $X$ is characterised by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership-function $F_A(x)$,*

*For each point $x$ in $X$, $T_A(x), F_A(x), I_A(x) \in [0,1]$. Also*

$$A = \{x, T_A(x), F_A(x), I_A(x)\} \text{ and } 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

**Definition 2.5.** *[1, 22] A neutrosophic graph is defined as a pair $G^* = (V, E)$ where*

*(i) $V = \{v_1, v_2, .., v_n\}$ such that $T_1 = V \to [0,1]$, $I_1 = V \to [0,1]$ and $F_1 = V \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and*

$$0 \leq T_1(v) + I_1(v) + F_1(v) \leq 3$$

*(ii)* $E \subseteq V \times V$ *where* $T_2 = E \to [0,1]$, $I_2 = E \to [0,1]$ *and* $F_2 = E \to [0,1]$ *are such that*

$$T_2(uv) \leq \min\{T_1(u), T_1(v)\},$$
$$I_2(uv) \leq \min\{I_1(u), I_1(v)\},$$
$$F_2(uv) \leq \max\{F_1(u), F_1(v)\},$$
$$\text{and } 0 \leq T_2(uv) + I_2(uv) + F_2(uv) \leq 3, \ \forall uv \in E$$

**Definition 2.6.** *[6] A neutrosophic vague set* $A_{NV}$ *(NVS in short) on the universe of discourse* $X$ *written as*

$$A_{NV} = \{\langle x, \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), \hat{F}_{A_{NV}}(x)\rangle, x \in X\}$$

*whose truth-membership, indeterminacy membership and falsity-membership function is defined as*

$$\hat{T}_{A_{NV}}(x) = [T^-(x), T^+(x)], \hat{I}_{A_{NV}}(x) = [I^-(x), I^+(x)] \text{ and } \hat{F}_{A_{NV}}(x) = [F^-(x), F^+(x)],$$

*where* $T^+(x) = 1 - F^-(x), F^+(x) = 1 - T^-(x)$, *and* $0 \leq T^-(x) + I^-(x) + F^-(x) \leq 2.$

**Definition 2.7.** *[6] The complement of NVS* $A_{NV}$ *is denoted by* $A^c_{NV}$ *and it is defined by*

$$\hat{T}^c_{A_{NV}}(x) = [1 - T^+(x), 1 - T^-(x)],$$
$$\hat{I}^c_{A_{NV}}(x) = [1 - I^+(x), 1 - I^-(x)],$$
$$\hat{F}^c_{A_{NV}}(x) = [1 - F^+(x), 1 - F^-(x)],$$

**Definition 2.8.** *[6] Let* $A_{NV}$ *and* $B_{NV}$ *be two NVSs of the universe* $U$*. If for all* $u_i \in U$,

$$\hat{T}_{A_{NV}}(u_i) \leq \hat{T}_{B_{NV}}(u_i), \hat{I}_{A_{NV}}(u_i) \geq \hat{I}_{B_{NV}}(u_i), \hat{F}_{A_{NV}}(u_i) \geq \hat{F}_{B_{NV}}(u_i)$$

*then the NVS,* $A_{NV}$ *are included in* $B_{NV}$*, denoted by* $A_{NV} \subseteq B_{NV}$ *where* $1 \leq i \leq n$.

**Definition 2.9.** *[6] The union of two NVSs* $A_{NV}$ *and* $B_{NV}$ *is a NVSs,* $C_{NV}$*, written as* $C_{NV} = A_{NV} \cup B_{NV}$*, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of* $A_{NV}$ *and* $B_{NV}$ *by*

$$\hat{T}_{C_{NV}}(x) = [\max(T^-_{A_{NV}}(x), T^-_{B_{NV}}(x)), \max(T^+_{A_{NV}}(x), T^+_{B_{NV}}(x))]$$
$$\hat{I}_{C_{NV}}(x) = [\min(I^-_{A_{NV}}(x), I^-_{B_{NV}}(x)), \min(I^+_{A_{NV}}(x), I^+_{B_{NV}}(x))]$$
$$\hat{F}_{C_{NV}}(x) = [\min(F^-_{A_{NV}}(x), F^-_{B_{NV}}(x)), \min(F^+_{A_{NV}}(x), F^+_{B_{NV}}(x))]$$

**Definition 2.10.** *[6] The intersection of two NVSs,* $A_{NV}$ *and* $B_{NV}$ *is a NVSs* $C_{NV}$*, written as* $C_{NV} = A_{NV} \cap B_{NV}$*, whose truth-membership function, indeterminacy-membership function and false-membership function are related to those of* $A_{NV}$ *and* $B_{NV}$ *by*

$$\hat{T}_{C_{NV}}(x) = [\min(T^-_{A_{NV}}(x), T^-_{B_{NV}}(x)), \min(T^+_{A_{NV}}(x), T^+_{B_{NV}}(x))]$$

$$\hat{I}_{C_{NV}}(x) = [\max(I^-_{A_{NV}}(x), I^-_{B_{NV}}(x)), \max(I^+_{A_{NV}}(x), I^+_{B_{NV}}(x))]$$

$$\hat{F}_{C_{NV}}(x) = [\max(F^-_{A_{NV}}(x), F^-_{B_{NV}}(x)), \max(F^+_{A_{NV}}(x), F^+_{B_{NV}}(x))]$$

**Definition 2.11.** *[16] Let* $G^* = (R, S)$ *be a graph. A pair* $G = (A, B)$ *is called a neutrosophic vague graph (NVG) on* $G^*$ *or a neutrosophic vague graph where* $A = (\hat{T}_A, \hat{I}_A, \hat{F}_A)$ *is a neutrosophic vague set on* $R$ *and* $B = (\hat{T}_B, \hat{I}_B, \hat{F}_B)$ *is a neutrosophic vague set* $S \subseteq R \times R$ *where*

(1)$R = \{v_1, v_2, ..., v_n\}$ *such that* $T^-_A : R \to [0,1], I^-_A : R \to [0,1], F^-_A : R \to [0,1]$ *which satisfies the condition* $F^-_A = [1 - T^+_A]$

$T^+_A : R \to [0,1], I^+_A : R \to [0,1], F^+_A : R \to [0,1]$ *which satisfies the condition* $F^+_A = [1 - T^-_A]$

*denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element* $v_i \in R$*, and*

$$0 \leq T^-_A(v_i) + I^-_A(v_i) + F^-_A(v_i) \leq 2$$

$$0 \leq T^+_A(v_i) + I^+_A(v_i) + F^+_A(v_i) \leq 2.$$

*(2)* $S \subseteq R \times R$ *where*

$$T^-_B : R \times R \to [0,1], I^-_B : R \times R \to [0,1], F^-_B : R \times R \to [0,1]$$

$$T^+_B : R \times R \to [0,1], I^+_B : R \times R \to [0,1], F^+_B : R \times R \to [0,1]$$

*denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element* $v_i, v_j \in S$*, respectively and such that,*

$$0 \leq T^-_B(v_iv_j) + I^-_B(v_iv_j) + F^-_B(v_iv_j) \leq 2$$

$$0 \leq T^+_B(v_iv_j) + I^+_B(v_iv_j) + F^+_B(v_iv_j) \leq 2,$$

*such that*

$$T_B^-(v_iv_j) \leq \min\{T_A^-(v_i), T_A^-(v_j)\}$$

$$I_B^-(v_iv_j) \leq \min\{I_A^-(v_i), I_A^-(v_j)\}$$

$$F_B^-(v_iv_j) \leq \max\{F_A^-(v_i), F_A^-(v_j)\},$$

*and similarly*

$$T_B^+(v_iv_j) \leq \min\{T_A^+(v_i), T_A^+(v_j)\}$$

$$I_B^+(v_iv_j) \leq \min\{I_A^+(v_i), I_A^+(v_j)\}$$

$$F_B^+(v_iv_j) \leq \max\{F_A^+(v_i), F_A^+(v_j)\}.$$

**Example 2.1.** *Consider a neutrosophic vague graph* $G = (R, S)$ *such that* $A = \{a, b, c\}$ *and* $B = \{ab, bc, ca\}$ *are defined by*

$\hat{a} = T[0.5, 0.6], I[0.4, 0.3], F[0.4, 0.5],$ $\hat{b} = T[0.4, 0.6], I[0.7, 0.3], F[0.4, 0.6],$

$\hat{c} = T[0.4, 0.4], I[0.5, 0.3], F[0.6, 0.6]$

$a^- = (0.5, 0.4, 0.4), b^- = (0.4, 0.7, 0.4), c^- = (0.4, 0.5, 0.6)$

$a^+ = (0.6, 0.3, 0.5), b^+ = (0.6, 0.3, 0.6), c^+ = (0.4, 0.3, 0.6)$

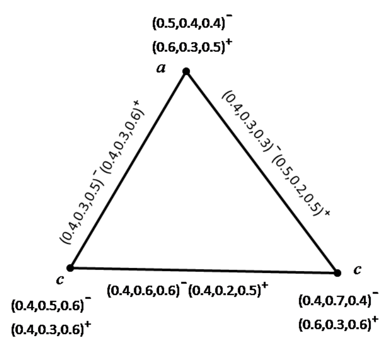


*Figure* 1 *NEUTROSOPHIC VAGUE GRAPH*

## 3. Operations on Neutrosophic Vague Graphs

**Definition 3.1.** *The Cartesian product of two NVGs $G_1$ and $G_2$ is denoted by the pair $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ and defined as*

$$T^-_{A_1 \times A_2}(kl) = T^-_{A_1}(k) \wedge T^-_{A_2}(l)$$

$$I^-_{A_1 \times A_2}(kl) = I^-_{A_1}(k) \wedge I^-_{A_2}(l)$$

$$F^-_{A_1 \times A_2}(kl) = F^-_{A_1}(k) \vee F^-_{A_2}(l)$$

$$T^+_{A_1 \times A_2}(kl) = T^+_{A_1}(k) \wedge T^+_{A_2}(l)$$

$$I^+_{A_1 \times A_2}(kl) = I^+_{A_1}(k) \wedge I^+_{A_2}(l)$$

$$F^+_{A_1 \times A_2}(kl) = F^+_{A_1}(k) \vee F^+_{A_2}(l)$$

*for all $(k, l) \in R_1 \times R_2$.*

*The membership value of the edges in $G_1 \times G_2$ can be calculated as,*

$$(1) T^-_{B_1 \times B_2}(kl_1)(kl_2) = T^-_{A_1}(k) \wedge T^-_{B_2}(l_1 l_2)$$

$$T^+_{B_1 \times B_2}(kl_1)(kl_2) = T^+_{A_1}(k) \vee T^+_{B_2}(l_1 l_2),$$

$$(2) I^-_{B_1 \times B_2}(kl_1)(kl_2) = I^-_{A_1}(k) \wedge I^-_{B_2}(l_1 l_2)$$

$$I^+_{B_1 \times B_2}(kl_1)(kl_2) = I^+_{A_1}(k) \vee I^+_{B_2}(l_1 l_2),$$

$$(3) F^-_{B_1 \times B_2}(kl_1)(kl_2) = F^-_{A_1}(k) \vee F^-_{B_2}(l_1 l_2)$$

$$F^+_{B_1 \times B_2}(kl_1)(kl_2) = F^+_{A_1}(k) \wedge F^+_{B_2}(l_1 l_2),$$

*for all $k \in R_1, l_1 l_2 \in S_2$.*

$$(4) T^-_{B_1 \times B_2}(k_1 l)(k_2 l) = T^-_{A_2}(l) \wedge T^-_{B_2}(k_1 k_2)$$

$$T^+_{B_1 \times B_2}(k_1 l)(k_2 l) = T^+_{A_2}(l) \vee T^+_{B_2}(k_1 k_2),$$

$$(5) I^-_{B_1 \times B_2}(k_1 l)(k_2 l) = I^-_{A_2}(l) \wedge I^-_{B_2}(k_1 k_2)$$

$$I^{+}_{B_1\times B_2}(k_1l)(k_2l)=I^{+}_{A_2}(l)\vee I^{+}_{B_2}(k_1k_2),$$

$$(6)F^{-}_{B_1\times B_2}(k_1l)(k_2l)=F^{-}_{A_2}(l)\vee F^{-}_{B_2}(k_1k_2)$$

$$F^{+}_{B_1\times B_2}(k_1l)(k_2l)=F^{+}_{A_2}(l)\wedge F^{+}_{B_2}(k_1k_2),$$

*for all* $k_1k_2\in S_1, l\in R_2$.

**Example 3.1.** *Consider* $G_1=(R_1,S_1)$ *and* $G_2=(R_2,S_2)$ *are two NVGs of* $G=(R,S)$ *, as represented in Figure 2, now we get* $G_1\times G_2$ *as follows Figure 3.*

$\hat{k}_1=T[0.5,0.6],I[0.6,0.4],F[0.4,0.5],\hat{k}_2=T[0.4,0.6],I[0.7,0.3],F[0.4,0.6],$

$\hat{k}_3=T[0.6,0.4],I[0.3,0.7],F[0.6,0.4],\hat{k}_4=T[0.4,0.4],I[0.4,0.6],F[0.6,0.6]$

$\hat{l}_1=T[0.4,0.4],I[0.5,0.3],F[0.6,0.6],\hat{l}_2=T[0.5,0.6],I[0.4,0.3],F[0.4,0.5],$

$\hat{l}_3=T[0.4,0.6],I[0.7,0.3],F[0.4,0.6]$

$k_1^-=(0.5,0.6,0.4),k_2^-=(0.4,0.7,0.4),k_3^-=(0.6,0.3,0.6),k_4^-=(0.4,0.4,0.6)$

$k_1^+=(0.6,0.4,0.5),k_2^+=(0.6,0.3,0.6),k_3^+=(0.4,0.7,0.4),k_4^-=(0.4,0.6,0.6)$

$l_1^-=(0.4,0.5,0.6),l_2^-=(05,0.4,0.4),l_3^-=(0.4,0.7,0.4)$

$l_1^+=(0.4,0.3,0.6),l_2^+=(0.6,0.3,0.5),l_3^+=(0.6,0.3,0.6)$

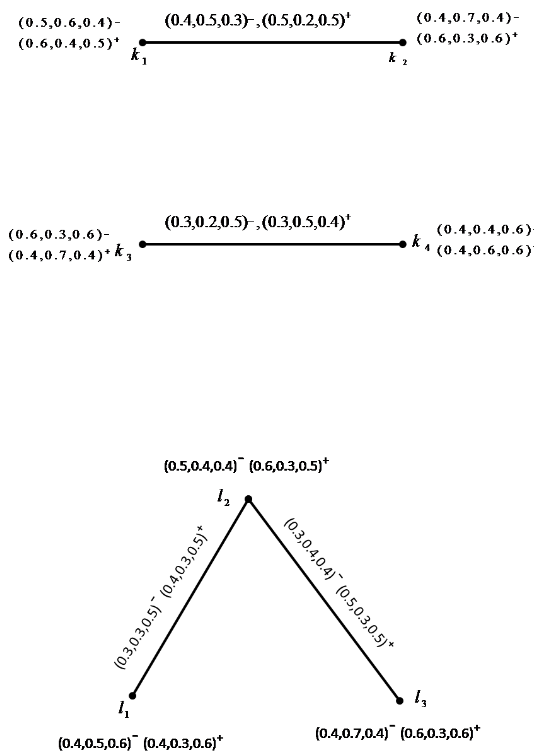


*Figure* 2
NEUTROSOPHIC VAGUE GRAPH

**Theorem 3.2.** *The Cartesian product $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ of two NVG $G_1$ and $G_2$ also an NVG of $G_1 \times G_2$.*

*Proof.* We consider,

Case 1: for $k \in R_1, l_1l_2 \in S_2$,

$$\begin{aligned}\hat{T}_{(B_1\times B_2)}((kl_1)(kl_2)) &= \hat{T}_{A_1}(k) \wedge \hat{T}_{B_2}(l_1l_2)\\ &\le \hat{T}_{A_1}(k) \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]\\ &= [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_2)]\\ &= \hat{T}_{(A_1\times A_2)}(k,l_1) \wedge \hat{T}_{(A_1\times A_2)}(k,l_2)\end{aligned}$$

$$\hat{I}_{(B_1\times B_2)}((kl_1)(kl_2)) = \hat{I}_{A_1}(k) \wedge \hat{I}_{B_2}(l_1l_2)$$

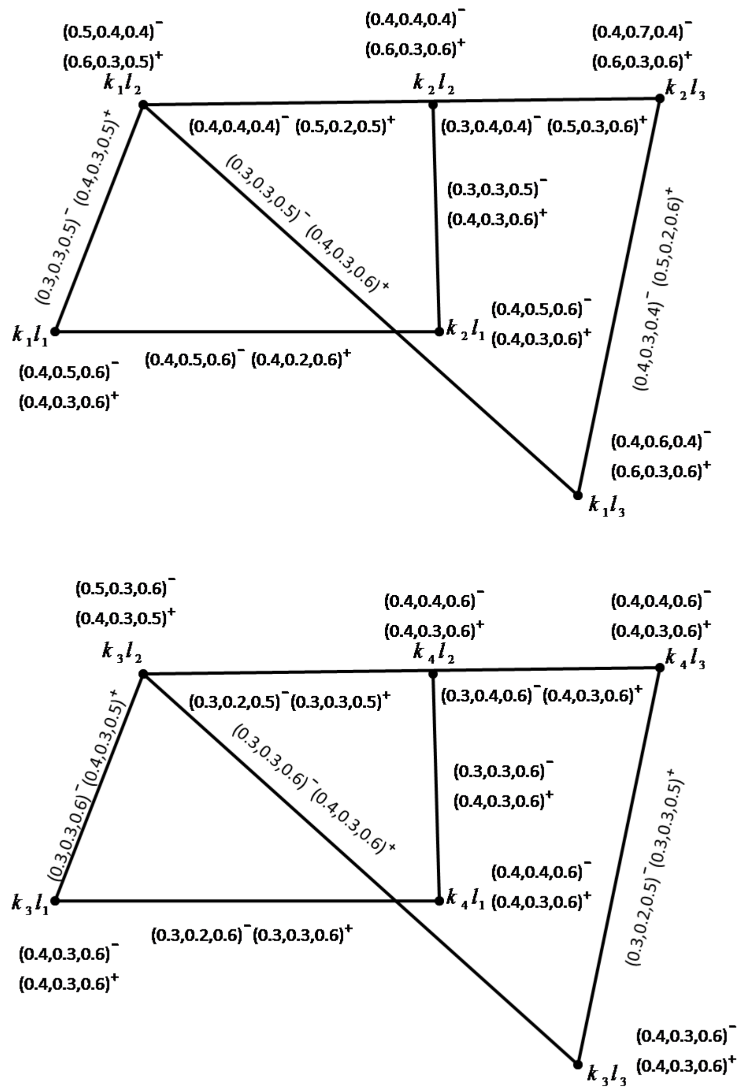


*Figure* 3
CARTESIAN PRODUCT OF NEUTROSOPHIC VAGUE GRAPH

$$\leq \hat{I}_{A_1}(k) \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]$$

$$= [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_2)]$$

$$= \hat{I}_{(A_1 \times A_2)}(k, l_1) \wedge \hat{I}_{(A_1 \times A_2)}(k, l_2)$$

$$\hat{F}_{(B_1 \times B_2)}((kl_1)(kl_2)) = \hat{F}_{A_1}(k) \vee \hat{F}_{B_2}(l_1 l_2)$$

$$\leq \hat{F}_{A_1}(k) \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)]$$

$$= [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_2)]$$

$$= \hat{F}_{(A_1 \times A_2)}(k, l_1) \vee \hat{F}_{(A_1 \times A_2)}(k, l_2)$$

for all $kl_1, kl_2 \in G_1 \times G_2$.

Case 2: for $k \in R_2, l_1l_2 \in S_1$.

$$\begin{aligned}\hat{T}_{(B_1 \times B_2)}((l_1k)(l_2k)) &= \hat{T}_{A_2}(k) \wedge \hat{T}_{B_1}(l_1l_2)\\ &\leq \hat{T}_{A_2}(k) \wedge [\hat{T}_{A_1}(l_1) \wedge \hat{T}_{A_1}(l_2)]\\ &= [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_1)] \wedge [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_2)]\\ &= \hat{T}_{(A_1 \times A_2)}(l_1, k) \wedge \hat{T}_{(A_1 \times A_2)}(l_2, k)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1 \times B_2)}((l_1k)(l_2k)) &= \hat{I}_{A_2}(k) \wedge \hat{I}_{B_1}(l_1l_2)\\ &\leq \hat{I}_{A_2}(k) \wedge [\hat{I}_{A_1}(l_1) \wedge \hat{I}_{A_1}(l_2)]\\ &= [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_1)] \wedge [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_2)]\\ &= \hat{I}_{(A_1 \times A_2)}(l_1, k) \wedge \hat{I}_{(A_1 \times A_2)}(l_2, k)\end{aligned}$$

$$\begin{aligned}\hat{F}_{(B_1 \times B_2)}((l_1k)(l_2k)) &= \hat{F}_{A_2}(k) \vee \hat{F}_{B_1}(l_1l_2)\\ &\leq \hat{F}_{A_2}(k) \vee [\hat{F}_{A_1}(l_1) \vee \hat{F}_{A_1}(l_2)]\\ &= [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_1)] \vee [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_2)]\\ &= \hat{F}_{(A_1 \times A_2)}(l_1, k) \vee \hat{F}_{(A_1 \times A_2)}(l_2, k)\end{aligned}$$

for all $l_1k, l_2k \in G_1 \times G_2$ and hence the proof. □

**Definition 3.3.** *The Cross product of two NVGs $G_1$ and $G_2$ is denoted by the pair $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ and is defined as*

$$\begin{aligned}(i)T^-_{A_1 \times A_2}(kl) &= T^-_{A_1}(k) \wedge T^-_{A_2}(l)\\ I^-_{A_1 \times A_2}(kl) &= I^-_{A_1}(k) \wedge I^-_{A_2}(l)\\ F^-_{A_1 \times A_2}(kl) &= F^-_{A_1}(k) \vee F^-_{A_2}(l)\end{aligned}$$

$$T^{+}_{A_1\times A_2}(kl) = T^{+}_{A_1}(k) \wedge T^{+}_{A_2}(l)$$

$$I^{+}_{A_1\times A_2}(kl) = I^{+}_{A_1}(k) \wedge I^{+}_{A_2}(l)$$

$$F^{+}_{A_1\times A_2}(kl) = F^{+}_{A_1}(k) \vee F^{+}_{A_2}(l)$$

*for all* $k, l \in R_1 \times R_2$.

$$(ii) T^{-}_{(B_1\times B_2)}(k_1l_1)(k_2l_2) = T^{-}_{B_1}(k_1k_2) \wedge T^{-}_{B_2}(l_1l_2)$$

$$I^{-}_{(B_1\times B_2)}(k_1l_1)(k_2l_2) = I^{-}_{B_1}(k_1k_2) \wedge I^{-}_{B_2}(l_1l_2)$$

$$F^{-}_{(B_1\times B_2)}(k_1l_1)(k_2l_2) = F^{-}_{B_1}(k_1k_2) \vee F^{-}_{B_2}(l_1l_2)$$

$$(iii) T^{+}_{(B_1\times B_2)}(k_1l_1)(k_2l_2) = T^{+}_{B_1}(k_1k_2) \wedge T^{+}_{B_2}(l_1l_2)$$

$$I^{+}_{(B_1\times B_2)}(k_1l_1)(k_2l_2) = I^{+}_{B_1}(k_1k_2) \wedge I^{+}_{B_2}(l_1l_2)$$

$$F^{+}_{(B_1\times B_2)}(k_1l_1)(k_2l_2) = F^{+}_{B_1}(k_1k_2) \vee F^{+}_{B_2}(l_1l_2)$$

*for all* $k_1k_2 \in S_1, l_1l_2 \in S_2$.

**Example 3.2.** *Consider* $G_1 = (R_1, S_1)$ *and* $G_2 = (R_2, S_2)$ *as two NVG of* $G = (R, S)$ *respectively, (see Figure 2). We obtain the cross product of* $G_1 \times G_2$ *as follows (see Figure 4).*

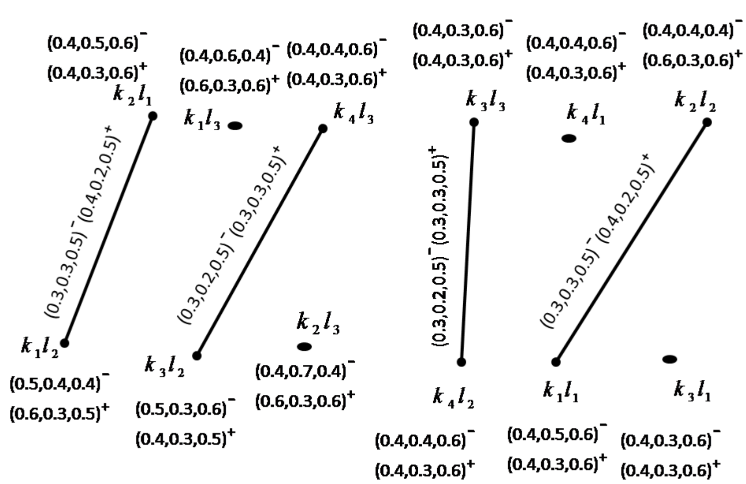


*Figure* 4
CROSS PRODUCT OF NEUTROSOPHIC VAGUE GRAPH

**Theorem 3.4.** *The cross product $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ of two NVG of NVG $G_1$ and $G_2$ is an NVG of $G_1 \times G_2$.*

*Proof.* For all $k_1 l_1, k_2 l_2 \in G_1 \times G_2$

$$\begin{aligned}\hat{T}_{(B_1 \times B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{T}_{B_1}(k_1 k_2) \wedge \hat{T}_{B_2}(l_1 l_2)\\ &\leq [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_1}(k_2)] \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]\\ &= [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k_2) \wedge \hat{T}_{A_2}(l_2)]\\ &= \hat{T}_{(A_1 \times A_2)}(k_1 l_1) \wedge \hat{T}_{(A_1 \times A_2)}(k_2, l_2)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1 \times B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{I}_{B_1}(k_1 k_2) \wedge \hat{I}_{B_2}(l_1 l_2)\\ &\leq [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_1}(k_2)] \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]\\ &= [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k_2) \wedge \hat{I}_{A_2}(l_2)]\\ &= \hat{I}_{(A_1 \times A_2)}(k_1 l_1) \wedge \hat{I}_{(A_1 \times A_2)}(k_2, l_2)\end{aligned}$$

$$\begin{aligned}\hat{F}_{(B_1 \times B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{F}_{B_1}(k_1 k_2) \vee \hat{F}_{B_2}(l_1 l_2)\\ &\leq [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_1}(k_2)] \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)]\\ &= [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k_2) \vee \hat{F}_{A_2}(l_2)]\\ &= \hat{F}_{(A_1 \times A_2)}(k_1 l_1) \vee \hat{F}_{(A_1 \times A_2)}(k_2, l_2)\end{aligned}$$

This completes the proof. □

**Definition 3.5.** *The lexicographic product of two NVGs $G_1$ and $G_2$ is denoted by the pair $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ and defined as*

$$\begin{aligned}(i) T^-_{(A_1 \bullet A_2)}(kl) &= T^-_{A_1}(k) \wedge T^-_{A_2}(l)\\ I^-_{(A_1 \bullet A_2)}(kl) &= I^-_{A_1}(k) \wedge I^-_{A_2}(l)\\ F^-_{(A_1 \bullet A_2)}(kl) &= F^-_{A_1}(k) \vee F^-_{A_2}(l)\\ T^+_{(A_1 \bullet A_2)}(kl) &= T^+_{A_1}(k) \wedge T^+_{A_2}(l)\\ I^+_{(A_1 \bullet A_2)}(kl) &= I^+_{A_1}(k) \wedge I^+_{A_2}(l)\end{aligned}$$

$$F^{+}_{(A_1 \bullet A_2)}(kl) = F^{+}_{A_1}(k) \vee F^{+}_{A_2}(l),$$

*for all* $kl \in R_1 \times R_2$

$$(ii) T^{-}_{(B_1 \bullet B_2)}(kl_1)(kl_2) = T^{-}_{A_1}(k) \wedge T^{-}_{B_2}(l_1 l_2)$$

$$I^{-}_{(B_1 \bullet B_2)}(kl_1)(kl_2) = I^{-}_{A_1}(k) \wedge I^{-}_{B_2}(l_1 l_2)$$

$$F^{-}_{(B_1 \bullet B_2)}(kl_1)(kl_2) = F^{-}_{A_1}(k) \vee F^{-}_{B_2}(l_1 l_2)$$

$$T^{+}_{(B_1 \bullet B_2)}(kl_1)(kl_2) = T^{+}_{A_1}(k) \wedge T^{+}_{B_2}(l_1 l_2)$$

$$I^{+}_{(B_1 \bullet B_2)}(kl_1)(kl_2) = I^{+}_{A_1}(k) \wedge I^{+}_{B_2}(l_1 l_2)$$

$$F^{+}_{(B_1 \bullet B_2)}(kl_1)(kl_2) = F^{+}_{A_1}(k) \vee F^{+}_{B_2}(l_1 l_2),$$

*for all* $k \in R_1, l_1 l_2 \in S_2$.

$$(iii) T^{-}_{(B_1 \bullet B_2)}(k_1 l_1)(k_2 l_2) = T^{-}_{B_1}(k_1 k_2) \wedge T^{-}_{B_2}(l_1 l_2)$$

$$I^{-}_{(B_1 \bullet B_2)}(k_1 l_1)(k_2 l_2) = I^{-}_{B_1}(k_1 k_2) \wedge I^{-}_{B_2}(l_1 l_2)$$

$$F^{-}_{(B_1 \bullet B_2)}(k_1 l_1)(k_2 l_2) = F^{-}_{B_1}(k_1 k_2) \vee F^{-}_{B_2}(l_1 l_2)$$

$$T^{+}_{(B_1 \bullet B_2)}(k_1 l_1)(k_2 l_2) = T^{+}_{B_1}(k_1 k_2) \wedge T^{+}_{B_2}(l_1 l_2)$$

$$I^{+}_{(B_1 \bullet B_2)}(k_1 l_1)(k_2 l_2) = I^{+}_{B_1}(k_1 k_2) \wedge I^{+}_{B_2}(l_1 l_2)$$

$$F^{+}_{(B_1 \bullet B_2)}(k_1 l_1)(k_2 l_2) = F^{+}_{B_1}(k_1 k_2) \vee F^{+}_{B_2}(l_1 l_2),$$

*for all* $k_1 k_2 \in S_1, l_1 l_2 \in S_2$.

**Example 3.3.** *The lexicographic product of NVG* $G_1 = (S_1, T_1)$ *and* $G_2 = (S_2, T_2)$ *shown in Figure 2 is defined as* $G_1 \bullet G_2 = (S_1 \bullet S_2, T_1 \bullet T_2)$ *and is presented in Figure 5.*

**Theorem 3.6.** *The lexicographic product* $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ *of two NVG of NVG* $G_1$ *and* $G_2$ *is an NVG of* $G_1 \bullet G_2$.

*Proof.* We have two cases.

Case 1: For $k \in R_1, l_1 l_2 \in S_2$,

$$\hat{T}_{(B_1 \bullet B_2)}((kl_1)(kl_2)) = \hat{T}_{A_1}(k) \wedge \hat{T}_{B_2}(l_1 l_2)$$

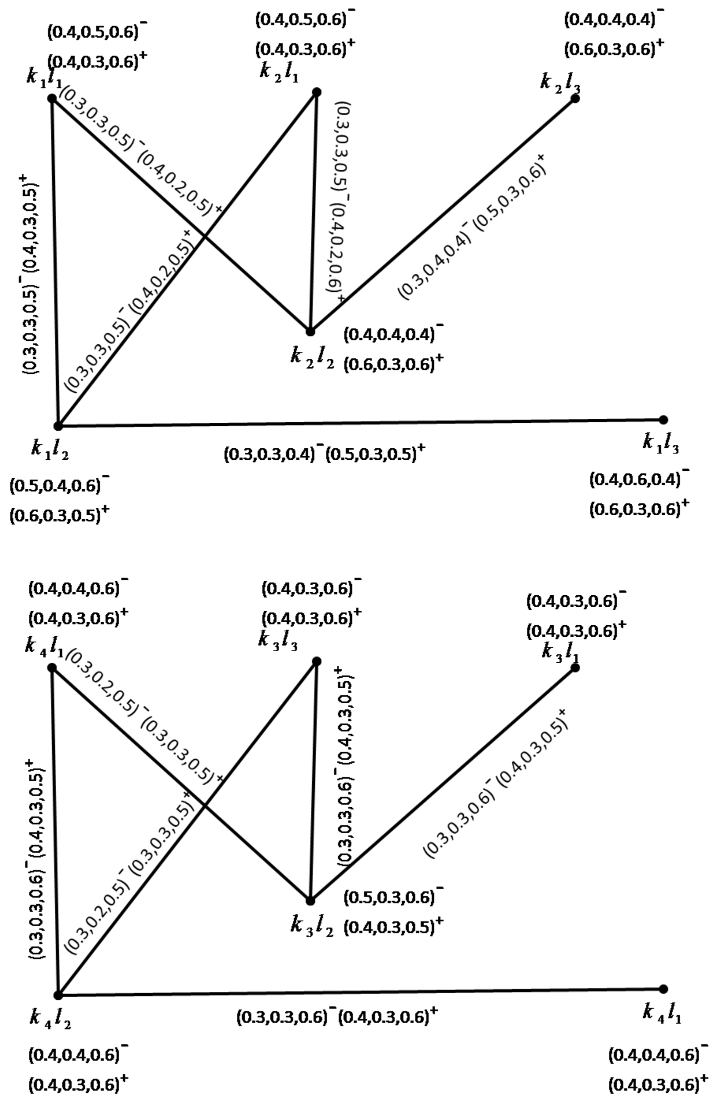


*Figure* 5
LEXICOGRAPHIC PRODUCT OF NEUTROSOPHIC VAGUE GRAPH

$$\leq \hat{T}_{A_1}(k) \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]$$

$$= [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_2)]$$

$$= \hat{T}_{(A_1 \bullet A_2)}(k, l_1) \wedge \hat{T}_{(A_1 \bullet A_2)}(k, l_2)$$

$$\hat{I}_{(B_1 \bullet B_2)}((kl_1)(kl_2)) = \hat{I}_{A_1}(k) \wedge \hat{I}_{B_2}(l_1 l_2)$$

$$\leq \hat{I}_{A_1}(k) \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]$$

$$= [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_2)]$$

$$= \hat{I}_{(A_1 \bullet A_2)}(k, l_1) \wedge \hat{I}_{(A_1 \bullet A_2)}(k, l_2)$$

$$\begin{aligned}
\hat{F}_{(B_1 \bullet B_2)}((kl_1)(kl_2)) &= \hat{F}_{A_1}(k) \vee \hat{F}_{B_2}(l_1 l_2) \\
&\leq \hat{F}_{A_1}(k) \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)] \\
&= [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_2)] \\
&= \hat{F}_{(A_1 \bullet A_2)}(k, l_1) \vee \hat{F}_{(A_1 \bullet A_2)}(k, l_2)
\end{aligned}$$

for all $kl_1, kl_2 \in S_1 \times S_2$.

Case 2: For all $k_1 l_1 \in S_1, k_2 l_2 \in S_2$,

$$\begin{aligned}
\hat{T}_{(B_1 \bullet B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{T}_{B_1}(k_1 k_2) \wedge \hat{T}_{B_2}(l_1 l_2) \\
&\leq [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_1}(k_2)] \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)] \\
&= [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k_2) \wedge \hat{T}_{A_2}(l_2)] \\
&= \hat{T}_{(A_1 \bullet A_2)}(k_1 l_1) \wedge \hat{T}_{(A_1 \bullet A_2)}(k_2, l_2)
\end{aligned}$$

$$\begin{aligned}
\hat{I}_{(B_1 \bullet B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{I}_{B_1}(k_1 k_2) \wedge \hat{I}_{B_2}(l_1 l_2) \\
&\leq [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_1}(k_2)] \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)] \\
&= [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k_2) \wedge \hat{I}_{A_2}(l_2)] \\
&= \hat{I}_{(A_1 \bullet A_2)}(k_1 l_1) \wedge \hat{I}_{(A_1 \bullet A_2)}(k_2, l_2)
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{(B_1 \bullet B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{F}_{B_1}(k_1 k_2) \vee \hat{F}_{B_2}(l_1 l_2) \\
&\leq [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_1}(k_2)] \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)] \\
&= [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k_2) \vee \hat{F}_{A_2}(l_2)] \\
&= \hat{F}_{(A_1 \bullet A_2)}(k_1 l_1) \vee \hat{F}_{(A_1 \bullet A_2)}(k_2, l_2)
\end{aligned}$$

for all $k_1, l_1 \in k_2, l_2 \in R_1 \bullet R_2$. □

**Definition 3.7.** *The strong product of two NVG $G_1$ and $G_2$ is denoted by the pair $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ and defined as*

$$(i) T^-_{(A_1 \boxtimes A_2)}(kl) = T^-_{A_1}(k) \wedge T^-_{A_2}(l)$$
$$I^-_{(A_1 \boxtimes A_2)}(kl) = I^-_{A_1}(k) \wedge I^-_{A_2}(l)$$
$$F^-_{(A_1 \boxtimes A_2)}(kl) = F^-_{A_1}(k) \vee F^-_{A_2}(l)$$
$$T^+_{(A_1 \boxtimes A_2)}(kl) = T^+_{A_1}(k) \wedge T^+_{A_2}(l)$$
$$I^+_{(A_1 \boxtimes A_2)}(kl) = I^+_{A_1}(k) \wedge I^+_{A_2}(l)$$
$$F^+_{(A_1 \boxtimes A_2)}(kl) = F^+_{A_1}(k) \vee F^+_{A_2}(l)$$

*for all $kl \in R_1 \boxtimes R_2$*

$$(ii) T^-_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = T^-_{A_1}(k) \wedge T^-_{B_2}(l_1 l_2)$$
$$I^-_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = I^-_{A_1}(k) \wedge I^-_{B_2}(l_1 l_2)$$
$$F^-_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = F^-_{A_1}(k) \vee F^-_{B_2}(l_1 l_2)$$
$$T^+_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = T^+_{A_1}(k) \wedge T^+_{B_2}(l_1 l_2)$$
$$I^+_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = I^+_{A_1}(k) \wedge I^+_{B_2}(l_1 l_2)$$
$$F^+_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = F^+_{A_1}(k) \vee F^+_{B_2}(l_1 l_2),$$

*for all $k \in R_1, l_1 l_2 \in S_2$.*

$$(iii) T^-_{B_1 \boxtimes B_2}(k_1 l)(k_2 l) = T^-_{A_2}(l) \wedge T^-_{B_2}(k_1 k_2)$$
$$I^-_{B_1 \boxtimes B_2}(k_1 l)(k_2 l) = I^-_{A_2}(l) \wedge I^-_{B_2}(k_1 k_2)$$
$$F^-_{B_1 \boxtimes B_2}(k_1 l)(k_2 l) = F^-_{A_2}(l) \vee F^-_{B_2}(k_1 k_2)$$
$$T^+_{B_1 \boxtimes B_2}(k_1 l)(k_2 l) = T^+_{A_2}(l) \wedge T^+_{B_2}(k_1 k_2)$$
$$I^+_{B_1 \boxtimes B_2}(k_1 l)(k_2 l) = I^+_{A_2}(l) \wedge I^+_{B_2}(k_1 k_2)$$
$$F^+_{B_1 \boxtimes B_2}(k_1 l)(k_2 l) = F^+_{A_2}(l) \vee F^+_{B_2}(k_1 k_2),$$

*for all* $k_1k_2 \in S_1, l \in R_2$.

$$(iv) T^-_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = T^-_{B_1}(k_1k_2) \wedge T^-_{B_2}(l_1l_2)$$

$$I^-_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = I^-_{B_1}(k_1k_2) \wedge I^-_{B_2}(l_1l_2)$$

$$F^-_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = F^-_{B_1}(k_1k_2) \vee F^-_{B_2}(l_1l_2)$$

$$T^+_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = T^+_{B_1}(k_1k_2) \wedge T^+_{B_2}(l_1l_2)$$

$$I^+_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = I^+_{B_1}(k_1k_2) \wedge I^+_{B_2}(l_1l_2)$$

$$F^+_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = F^+_{B_1}(k_1k_2) \vee F^N_{B_2}(l_1l_2)$$

*for all* $k_1k_2 \in S_1, l_1l_2 \in S_2$.

**Example 3.4.** *The strong product of NVG* $G_1 = (R_1, S_1)$ *and* $G_2 = (R_2, S_2)$ *shown in Figure 2 is defined as* $G_1 \boxtimes G_2 = (S_1 \boxtimes S_2, T_1 \boxtimes T_2)$ *and is presented in Figure 6.*

**Theorem 3.8.** *The strong product* $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ *of two NVG of NVG* $G_1$ *and* $G_2$ *is a NVG of* $G_1 \boxtimes G_2$.

*Proof.* There are three cases:

Case 1: for $k \in R_1, l_1l_2 \in S_2$,

$$\begin{aligned}\hat{T}_{(B_1 \boxtimes B_2)}((kl_1)(kl_2)) &= \hat{T}_{A_1}(k) \wedge \hat{T}_{B_2}(l_1l_2)\\ &\leq \hat{T}_{A_1}(k) \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]\\ &= [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_2)]\\ &= \hat{T}_{(A_1 \boxtimes A_2)}(k, l_1) \wedge \hat{T}_{(A_1 \boxtimes A_2)}(k, l_2)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1 \boxtimes B_2)}((kl_1)(kl_2)) &= \hat{I}_{A_1}(k) \wedge \hat{I}_{B_2}(l_1l_2)\\ &\leq \hat{I}_{A_1}(k) \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]\\ &= [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_2)]\\ &= \hat{I}_{(A_1 \boxtimes A_2)}(k, l_1) \wedge \hat{I}_{(A_1 \boxtimes A_2)}(k, l_2)\end{aligned}$$

$$\hat{F}_{(B_1 \boxtimes B_2)}((kl_1)(kl_2)) = \hat{F}_{A_1}(k) \vee \hat{F}_{B_2}(l_1l_2)$$

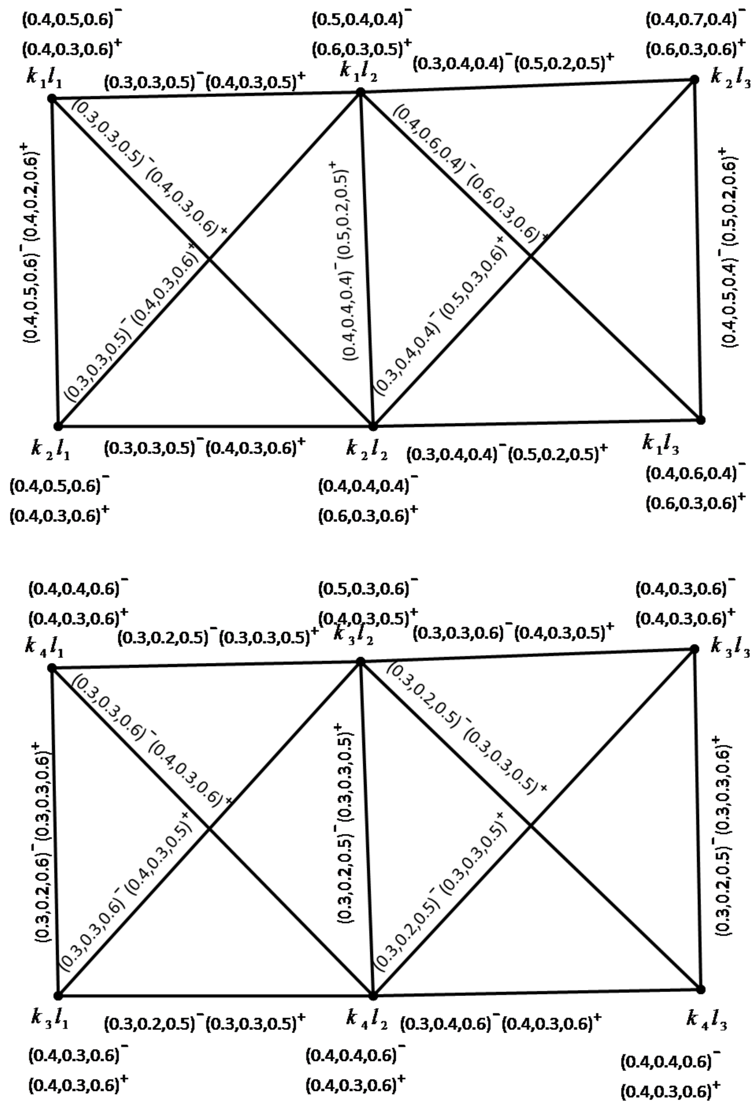


*Figure* 6
STRONG PRODUCT NEUTROSOPHIC VAGUE GRAPH

$$\leq \hat{F}_{A_1}(k) \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)]$$

$$= [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_2)]$$

$$= \hat{F}_{(A_1 \boxtimes A_2)}(k, l_1) \vee \hat{F}_{(A_1 \boxtimes A_2)}(k, l_2)$$

for all $kl_1, kl_2 \in R_1 \boxtimes R_2$.

Case 2: for $k \in R_2, l_1l_2 \in S_1$,

$$\begin{aligned}\hat{T}_{(B_1 \boxtimes B_2)}((l_1k)(l_2k)) &= \hat{T}_{A_2}(k) \wedge \hat{T}_{B_1}(l_1l_2)\\ &\leq \hat{T}_{A_2}(k) \wedge [\hat{T}_{A_1}(l_1) \wedge \hat{T}_{A_1}(l_2)]\\ &= [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_1)] \wedge [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_2)]\\ &= \hat{T}_{(A_1 \boxtimes A_2)}(l_1, k) \wedge \hat{T}_{(A_1 \boxtimes A_2)}(l_2, k)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1 \boxtimes B_2)}((l_1k)(l_2k)) &= \hat{I}_{A_2}(k) \wedge \hat{I}_{B_1}(l_1l_2)\\ &\leq \hat{I}_{A_2}(k) \wedge [\hat{I}_{A_1}(l_1) \wedge \hat{I}_{A_1}(l_2)]\\ &= [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_1)] \wedge [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_2)]\\ &= \hat{I}_{(A_1 \boxtimes A_2)}(l_1, k) \wedge \hat{I}_{(A_1 \boxtimes A_2)}(l_2, k)\end{aligned}$$

$$\begin{aligned}\hat{F}_{(B_1 \boxtimes B_2)}((l_1k)(l_2k)) &= \hat{F}_{A_2}(k) \vee \hat{F}_{B_1}(l_1l_2)\\ &\leq \hat{F}_{A_2}(k) \vee [\hat{F}_{A_1}(l_1) \vee \hat{F}_{A_1}(l_2)]\\ &= [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_1)] \vee [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_2)]\\ &= \hat{F}_{(A_1 \boxtimes A_2)}(l_1, k) \vee \hat{F}_{(A_1 \boxtimes A_2)}(l_2, k)\end{aligned}$$

for all $l_1k, l_2k \in R_1 \boxtimes R_2$.

Case 3: for $k_1, k_2 \in S_1, l_1l_2 \in S_2$

$$\begin{aligned}\hat{T}_{(B_1 \boxtimes B_2)}((k_1l_1)(k_2l_2)) &= \hat{T}_{B_1}(k_1k_2) \wedge \hat{T}_{B_2}(l_1l_2)\\ &\leq [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_1}(k_2)] \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]\\ &= [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k_2) \wedge \hat{T}_{A_2}(l_2)]\\ &= \hat{T}_{(A_1 \boxtimes A_2)}(k_1l_1) \wedge \hat{T}_{(A_1 \boxtimes A_2)}(k_2, l_2)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1 \boxtimes B_2)}((k_1l_1)(k_2l_2)) &= \hat{I}_{B_1}(k_1k_2) \wedge \hat{I}_{B_2}(l_1l_2)\\ &\leq [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_1}(k_2)] \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]\end{aligned}$$

$$= [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k_2) \wedge \hat{I}_{A_2}(l_2)]$$

$$= \hat{I}_{(A_1 \boxtimes A_2)}(k_1 l_1) \wedge \hat{I}_{(A_1 \boxtimes A_2)}(k_2, l_2)$$

$$\begin{aligned}
\hat{F}_{(B_1 \boxtimes B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{F}_{B_1}(k_1 k_2) \vee \hat{F}_{B_2}(l_1 l_2) \\
&\leq [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_1}(k_2)] \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)] \\
&= [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k_2) \vee \hat{F}_{A_2}(l_2)] \\
&= \hat{F}_{(A_1 \boxtimes A_2)}(k_1 l_1) \vee \hat{F}_{(A_1 \boxtimes A_2)}(k_2, l_2)
\end{aligned}$$

for all $l_1 k_1, l_2 k_1 \in R_1 \boxtimes R_2$. Hence the proof. □

**Definition 3.9.** *The composition of two NVG $G_1$ and $G_2$ is denoted by the pair $G_1 \circ G_2 = (R_1 \boxtimes R_2, S_1 \circ S_2)$ and defined as*

$$\begin{aligned}
(i) T^-_{(A_1 \circ A_2)}(kl) &= T^-_{A_1}(k) \wedge T^-_{A_2}(l) \\
I^-_{(A_1 \circ A_2)}(kl) &= I^-_{A_1}(k) \wedge I^-_{A_2}(l) \\
F^-_{(A_1 \circ A_2)}(kl) &= F^-_{A_1}(k) \vee F^-_{A_2}(l) \\
T^+_{(A_1 \circ A_2)}(kl) &= T^+_{A_1}(k) \wedge T^+_{A_2}(l) \\
I^+_{(A_1 \circ A_2)}(kl) &= I^+_{A_1}(k) \wedge I^+_{A_2}(l) \\
F^+_{(A_1 \circ A_2)}(kl) &= F^+_{A_1}(k) \vee F^+_{A_2}(l)
\end{aligned}$$

*for all $kl \in R_1 \circ R_2$.*

$$\begin{aligned}
(ii) T^-_{(B_1 \circ B_2)}(kl_1)(kl_2) &= T^-_{A_1}(k) \wedge T^-_{B_2}(l_1 l_2) \\
I^-_{(B_1 \circ B_2)}(kl_1)(kl_2) &= I^-_{A_1}(k) \wedge I^-_{B_2}(l_1 l_2) \\
F^-_{(B_1 \circ B_2)}(kl_1)(kl_2) &= F^-_{A_1}(k) \vee F^-_{B_2}(l_1 l_2) \\
T^+_{(B_1 \circ B_2)}(kl_1)(kl_2) &= T^+_{A_1}(k) \wedge T^+_{B_2}(l_1 l_2) \\
I^+_{(B_1 \circ B_2)}(kl_1)(kl_2) &= I^+_{A_1}(k) \wedge I^+_{B_2}(l_1 l_2) \\
F^+_{(B_1 \circ B_2)}(kl_1)(kl_2) &= F^+_{A_1}(k) \vee F^+_{B_2}(l_1 l_2),
\end{aligned}$$

*for all* $k \in R_1, l_1l_2 \in S_2$.

$$(iii) T^-_{B_1 \circ B_2}(k_1l)(k_2l) = T^-_{A_2}(l) \wedge T^-_{B_2}(k_1k_2)$$
$$I^-_{B_1 \circ B_2}(k_1,l)(k_2,l) = I^-_{A_2}(l) \wedge I^-_{B_2}(k_1k_2)$$
$$F^-_{B_1 \circ B_2}(k_1,l)(k_2,l) = F^-_{A_2}(l) \vee F^-_{B_2}(k_1k_2)$$
$$T^+_{B_1 \circ B_2}(k_1,l)(k_2,l) = T^+_{A_2}(l) \wedge T^+_{B_2}(k_1k_2)$$
$$I^+_{B_1 \circ B_2}(k_1,l)(k_2,l) = I^+_{A_2}(l) \wedge I^+_{B_2}(k_1k_2)$$
$$F^+_{B_1 \circ B_2}(k_1,l)(k_2,l) = F^+_{A_2}(l) \vee F^+_{B_2}(k_1k_2),$$

*for all* $k_1k_2 \in S_1, l \in R_2$.

$$(iv) T^-_{(B_1 \circ B_2)}(k_1l_1)(k_2l_2) = T^-_{B_1}(k_1k_2) \wedge T^-_{A_2}(l_1) \wedge T^-_{A_2}(l_2)$$
$$I^-_{(B_1 \circ B_2)}(k_1l_1)(k_2l_2) = I^-_{B_1}(k_1k_2) \wedge I^-_{A_2}(l_1) \wedge I^-_{A_2}(l_2)$$
$$F^-_{(B_1 \circ B_2)}(k_1l_1)(k_2l_2) = F^-_{B_1}(k_1k_2) \vee F^-_{A_2}(l_1) \vee F^-_{A_2}(l_2)$$
$$T^+_{(B_1 \circ B_2)}(k_1l_1)(k_2l_2) = T^-_{B_1}(k_1k_2) \wedge T^+_{A_2}(l_1) \wedge T^+_{A_2}(l_2)$$
$$I^+_{(B_1 \circ B_2)}(k_1l_1)(k_2l_2) = I^+_{B_1}(k_1k_2) \wedge I^+_{A_2}(l_1) \wedge I^+_{A_2}(l_2)$$
$$F^+_{(B_1 \circ B_2)}(k_1l_1)(k_2l_2) = F^+_{B_1}(k_1k_2) \vee F^+_{A_2}(l_1) \vee F^+_{A_2}(l_2)$$

*for all* $k_1k_2 \in S_1, l_1l_2 \in S_2$.

**Example 3.5.** *The composition of NVG* $G_1 = (R_1, S_1)$ *and* $G_2 = (R_2, S_2)$ *shown in Figure 2 is defined as* $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ *and is presented in Figure 7.*

**Theorem 3.10.** *Composition* $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ *of two NVG of NVG* $G_1$ *and* $G_2$ *is an NVG of* $G_1 \circ G_2$.

*Proof.* There are three cases: Case:1 For $k \in R_1, l_1l_2 \in S_2$,

$$\begin{aligned}\hat{T}_{(B_1 \circ B_2)}((kl_1)(kl_2))7 &= \hat{T}_{A_1}(k) \wedge \hat{T}_{B_2}(l_1l_2)\\ &\leq \hat{T}_{A_1}(k) \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]\\ &= [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_2)]\end{aligned}$$

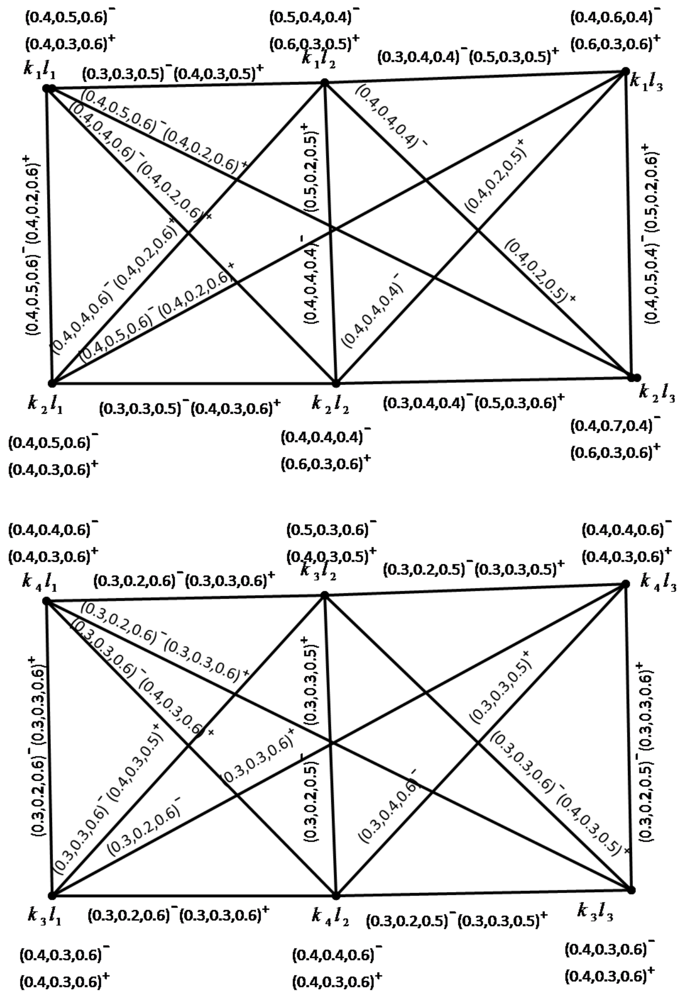


*Figure* 7
COMPOSITION OF NEUTROSOPHIC VAGUE GRAPH

$$= \hat{T}_{(A_1 \circ A_2)}(k, l_1) \wedge \hat{T}_{(A_1 \circ A_2)}(k, l_2)$$

$$\begin{aligned}
\hat{I}_{(B_1 \circ B_2)}((kl_1)(kl_2)) &= \hat{I}_{A_1}(k) \wedge \hat{I}_{B_2}(l_1 l_2) \\
&\leq \hat{I}_{A_1}(k) \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)] \\
&= [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_2)] \\
&= \hat{I}_{(A_1 \circ A_2)}(k, l_1) \wedge \hat{I}_{(A_1 \circ A_2)}(k, l_2)
\end{aligned}$$

$$\hat{F}_{(B_1 \circ B_2)}((kl_1)(kl_2)) = \hat{F}_{A_1}(k) \vee \hat{F}_{B_2}(l_1 l_2)$$

$$\le \hat{F}_{A_1}(k) \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)]$$

$$= [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_2)]$$

$$= \hat{F}_{(A_1 \circ A_2)}(k, l_1) \vee \hat{F}_{(A_1 \circ A_2)}(k, l_2)$$

for all $kl_1, kl_2 \in R_1 \circ R_2$.

Case 2: for $k \in R_2, l_1 l_2 \in S_1$,

$$\begin{aligned}
\hat{T}_{(B_1 \circ B_2)}((l_1k)(l_2k)) &= \hat{T}_{A_2}(k) \wedge \hat{T}_{B_1}(l_1 l_2) \\
&\le \hat{T}_{A_2}(k) \wedge [\hat{T}_{A_1}(l_1) \wedge \hat{T}_{A_1}(l_2)] \\
&= [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_1)] \wedge [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_2)] \\
&= \hat{T}_{(A_1 \circ A_2)}(l_1, k) \wedge \hat{T}_{(A_1 \circ A_2)}(l_2, k)
\end{aligned}$$

$$\begin{aligned}
\hat{I}_{(B_1 \circ B_2)}((l_1k)(l_2k)) &= \hat{I}_{A_2}(k) \wedge \hat{I}_{B_1}(l_1 l_2) \\
&\le \hat{I}_{A_2}(k) \wedge [\hat{I}_{A_1}(l_1) \wedge \hat{I}_{A_1}(l_2)] \\
&= [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_1)] \wedge [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_2)] \\
&= \hat{I}_{(A_1 \circ A_2)}(l_1, k) \wedge \hat{I}_{(A_1 \circ A_2)}(l_2, k)
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{(B_1 \circ B_2)}((l_1k)(l_2k)) &= \hat{F}_{A_2}(k) \vee \hat{F}_{B_1}(l_1 l_2) \\
&\le \hat{F}_{A_2}(k) \vee [\hat{F}_{A_1}(l_1) \vee \hat{F}_{A_1}(l_2)] \\
&= [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_1)] \vee [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_2)] \\
&= \hat{F}_{(A_1 \circ A_2)}(l_1, k) \vee \hat{F}_{(A_1 \circ A_2)}(l_2, k)
\end{aligned}$$

for all $l_1k, l_2k \in R_1 \circ R_2$.

Case 3: For $k_1k_2 \in S_1, l_1, l_2 \in R_2$ such that $l_1 \neq l_2$,

$$\begin{aligned}
\hat{T}_{(B_1 \circ B_2)}((k_1l_1)(k_2l_2)) &= \hat{T}_{B_1}(k_1, k_2) \wedge \hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2) \\
&\le [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_1}(k_2)] \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)] \\
&= [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k_2) \wedge \hat{T}_{A_2}(l_2)]
\end{aligned}$$

$$= \hat{T}_{(A_1 \circ A_2)}(k_1 l_1) \wedge \hat{T}_{(A_1 \circ A_2)}(k_2 l_2)$$

$$\begin{aligned}
\hat{I}_{(B_1 \circ B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{I}_{B_1}(k_1, k_2) \wedge \hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2) \\
&\leq [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_1}(k_2)] \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)] \\
&= [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k_2) \wedge \hat{I}_{A_2}(l_2)] \\
&= \hat{I}_{(A_1 \circ A_2)}(k_1 l_1) \wedge \hat{I}_{(A_1 \circ A_2)}(k_2 l_2)
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{(B_1 \circ B_2)}((k_1 l_1)(k_2 l_2)) &= \hat{F}_{B_1}(k_1, k_2) \vee \hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2) \\
&\leq [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_1}(k_2)] \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)] \\
&= [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k_2) \vee \hat{F}_{A_2}(l_2)] \\
&= \hat{F}_{(A_1 \circ A_2)}(k_1 l_1) \vee \hat{F}_{(A_1 \circ A_2)}(k_2 l_2)
\end{aligned}$$

for all $k_1 l_1, k_2 l_2 \in R_1 \circ R_2$. □

## Conclusion

This paper deals with the operations on neutrosophic vague graphs. Moreover, Cartesian product, cross product, lexicographic product, strong product and composition of neutrosophic vague graph are investigated and the proposed concepts are illustrated with examples. Further we are able to extend by investigating the regular and isomorphic properties of the proposed graph.

**Conflict of Interest:** The authors declare that they have no conflict of interest.

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