Intuitionistic fuzzy ideals on approximation systems

H. Jude Immaculate,
Department of Mathematics,
Karunya Institute of Technology and Sciences ,
Coimbatore, India.

Saeid Jafari,
College of Vestsjaelland South,
Herrestraede 11,4200 Slagelse,
Denmark.

I.Arockiarani
Department of Mathematics,
Nirmala College for Women
Coimbatore, India.

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Abstract

In this paper, we initiate the concept of intuitionistic fuzzy ideals on rough sets. Using a new relation we discuss some of the algebraic nature of intuitionistic fuzzy ideals of a ring.

1 Introduction

Rough set theory, proposed by Pawlak [25] is a new mathematical tool that supports uncertainty reasoning. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields. The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximation spaces are induced. Using this approximation knowledge hidden information may be revealed and expressed in the form of decision rules. A key notion in Pawlak rough set model is an equivalence relation. Atanassov [2] presented intuitionistic fuzzy sets in 1986 which is very effective to deal with vagueness. As a generalization of fuzzy set the concept of intuitionistic fuzzy set has played an important role in analysis of uncertainty of data. Various notions of intuitionistic fuzzy rough set were explored to extend rough set theory in the intuitionistic fuzzy environment. This paper concerns a relationship between rough sets, intuitionistic fuzzy sets and ring theory. We consider a ring as a universal set and assume the knowledge about objects is restricted by an intuitionistic fuzzy ideal. In fact, we apply the notion of intuitionistic fuzzy ideal of a ring for definitions of lower and upper approximations in a ring. Some of its characterizations are discussed.

2 Preliminaries

Definition 2.1. [2]:*An intuitionistic fuzzy set (IFS in short) A in X is an object having the form* $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$ *where the function* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *denote the degree of membership (namely* $\mu_A(x)$*) and the degree of non membership (namely* $\nu_A(x)$*) of each element* $x \in X$ *to the set A, respectively and* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *for each* $x \in X$*. Denote by IFS(X) the set of all intuitionistic fuzzy set in X.*

Definition 2.2. [2]:
Let A and B be IFS's of the form $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)/x \in X\rangle\}$*. Then*

1. $A \subseteq B$ *if and only if* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for all* $x \in X$.

2. $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$.

3. $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)/x \in X\rangle\}$.*(Complement of A)*

4. $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)/x \in X\rangle\}$.

5. $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)/x \in X\rangle\}$.

For the sake of simplicity we use the notion $A = \langle x, \mu_A, \nu_A\rangle$ *instead of* $A = \{\langle x, \mu_A(x), \nu_A(x)/x \in X\rangle\}$. *The intuitionistic fuzzy set* $0 \sim= \{\langle x, 0\sim, 1\sim\rangle /x \in X\}$ *and* $1 \sim= \{\langle x, 1\sim, 0\sim\rangle /x \in X\}$ *are respectively the empty set and the whole set of* X.

Definition 2.3. [4]:
Let $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be any two IFS of R. Then their sum* $A + B$ *is defined by*

$$A + B = (\mu_A + \mu_B, \nu_A + \nu_B))$$

where $(\mu_A + \mu_B)(x) = \bigvee_{x=y+z} [\mu_A(y) \wedge \mu_B(z)]$ *and* $(\nu_A + \nu_B)(x) = \bigwedge_{x=y+z} [\nu_A(y) \vee \nu_B(z)]$ *for all* $x \in R$

Definition 2.4. [4]:
An IFS $A = (\mu_A, \nu_A)$ *is called an intuitionistic fuzzy ideal of R if for all* $x, y, i \in R$

(IF1) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ *and* $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;

(IF2) $\mu_A(xy) \geq \mu_A(y)$ *and* $\nu_A(xy) \leq \nu_A(y)$.

Definition 2.5. [14]:
For an approximation space (U, θ) *by a rough approximation in* (U, θ) *we mean a mapping* $(U, \theta, -) : P(U) \to P(U) \times P(U)$ *defined for every* $X \in P(U)$ *by* $(U, \theta, X) = ((\underline{U}, \theta, X), (\overline{U}, \theta, X))$ *where* $(\underline{U}, \theta, X) = \{x \in U | [x]_\theta \subseteq X\}$, $(\overline{U}, \theta, X) = \{x \in U | [x]_\theta \cap X \neq \phi\}$ *where* $(\underline{U}, \theta, X)$ *is called a lower rough approximation of X in* (U, θ) *, whereas* $(\overline{U}, \theta, X)$ *is called an upper approximation of X in* (U, θ)*. Given an approximation space* (U, θ)*, a pair* $(A, B) \in P(U) \times P(U)$ *is called a rough set in* (U, θ) *if and only if* $(A, B) = (U, \theta, X)$ *for some* $X \in P(U)$ *. Let* (U, θ) *be an approximation space and X a non-empty subset of U.*

(i) If $(\underline{U}, \theta, X) = (\overline{U}, \theta, X)$ *, then X is called definable.*

(ii) If $(\underline{U}, \theta, X) = \phi$ *then X is called empty interior.*

(iii) If $(\overline{U}, \theta, X) = U$*, then X is called empty interior.*

The lower approximation of X in (U, θ) *is the greatest definable set in U contained in X. The upper approximation of X in is the least definable set in U containing X. Therefore we have*

$$(\underline{U}, \theta, X) = \bigcup \{S | S \subseteq X, S \text{ is definable}\}$$
$$(\overline{U}, \theta, X) = \bigcap \{S | X \subseteq S, S \text{ is definable}\}$$

A rough set of X is the family of all subsets of U having the same upper approximation of X.

3 Intuitionistic fuzzy ideals and congruence relations

Theorem 3.1. *For an intuitionistic fuzzy ideal A of a ring R we have the following*

(i) $\mu_A(0) \geq \mu_A(x)$ *and* $\nu_A(0) \leq \nu_A(x)$,

(ii) $\mu_A(-x) = \mu_A(x)$ *and* $\nu_A(-x) = \nu_A(x)$ *for all* $x \in R$.

Proof. (i) For any $x \in R$ we have
$\mu_A(0) = \mu_A(x - x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$,
$\nu_A(0) = \nu_A(x - x) \leq \nu_A(x) \vee \nu_A(x) = \nu_A(x)$.

(ii) By using (i) we get
$\mu_A(-x) = \mu_A(0 - x) \geq \mu_A(0) \wedge \mu_A(x) = \mu_A(x)$,
$\nu_A(-x) = \nu_A(0 - x) \leq \nu_A(0) \vee \nu_A(x) = \nu_A(x)$.
Since x is arbitrary we conclude that $\mu_A(-x) = \mu_A(x)$ and $\nu_A(-x) = \nu_A(x)$. □

Theorem 3.2. *If an intuitioistic fuzzy set $A = (\mu_A, \nu_A)$ in R stisfies (IFI) then*

(i) $\mu_A(x - y) = \mu_A(0) \Rightarrow \mu_A(x) = \mu_A(y)$,

(ii) $\nu_A(x - y) = \nu_A(0) \Rightarrow \nu_A(x) = \nu_A(y)$ *for all* $x, y \in R$.

Proof. Let $x, y \in R$ such that $\mu_A(x - y) = \mu_A(0)$. Then $\mu_A(x) = \mu_A(x - y + y) \geq \mu_A(x - y) \wedge \mu_A(y) = \mu_A(0) \wedge \mu_A(y) = \mu_A(y)$. Similarly $\nu_A(y) = \nu_A(x - x + y) = \nu_A(x - (x - y)) \leq \nu_A(x) \vee \nu_A(x - y) = \mu_A(x)$. So $\mu_A(x) = \mu_A(y)$.

(ii) $\nu_A(x - y) = \nu_A(0)$ for all $x, y \in R$ then $\nu_A(x) = \nu_A(x - y + y) \leq \nu_A(x) \vee \nu_A(y) \leq \nu_A(0) \vee \nu_A(y) = \nu_A(y)$.Similarly $\nu_A(x) \leq \nu_A(y)$ and so $\nu_A(x) = \nu_A(y)$. □

Definition 3.3. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set on R and let $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. Then the set $A_{\alpha,\beta} = \{x \in R | \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is called a (α, β)-level subset of A. The set of all $(\alpha, \beta) \in Im(\mu_A) \times Im(\nu_A)$ such that $\alpha + \beta \leq 1$ is called the image of $A = (\mu_A, \nu_A)$ denoted by $Im(A)$.*

Definition 3.4. *Let A be an intuitionistic fuzzy ideal of R for each $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ the set*

$$U(A_{(\alpha,\beta)}) = \{(a, b) \in R \times R | \mu_A(a - b) \geq \alpha, \nu_A(a - b) \leq \beta\}.$$

is called a (α, β)-level relation on A. An equivalence relation θ on a ring R is called a congruence relation if $(a, b) \in \theta \Rightarrow (a + x, b + x) \in \theta, (x + a, x + b) \in \theta$ for all $x \in R$.

Theorem 3.5. *Let A be an intuitionistic fuzzy ideal of R and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ then $U(A_{(\alpha,\beta)})$ is a congruence relation on R.*

Proof. For any element $a \in R, \mu_A(a - a) = \mu_A(0) \geq \alpha$ and $\nu_A(a - a) = \nu_A(0) \leq \beta$ and so $(a, a) \in U(A_{(\alpha,\beta)})$ then $\mu_A(a - b) \geq \alpha$ and $\nu_A(a - b) \leq \beta$ implies $(a, b) \in U(A_{(\alpha,\beta)})$. Since A is an ideal of R $\mu_A(b - a) = \mu_A(-(a - b)) = \mu_A(a - b) \geq \alpha$. And $\nu_A(b - a) = \nu_A(-(a - b)) = \nu_A(a - b) \leq \beta$. Which yields $(b, a) \in U(A_{(\alpha,\beta)})$. If $(a, b) \in U(A_{(\alpha,\beta)})$ and $(b, c) \in U(A_{(\alpha,\beta)})$ then since A is an intuitionistic fuzzy ideal of R

$$\mu_A(a - c) = \mu_A((a - b) + (b - c)) \geq min\{\mu_A((a - b), \mu_A(b - c))\} \geq \min\{\alpha, \alpha\} = \alpha,$$
$$\nu_A(a - c) = \nu_A((a - b) + (b - c)) \leq max\{\nu_A((a - b), \nu_A(b - c))\} \leq \max\{\beta, \beta\} = \beta.$$

And hence $(a, c) \in U(A_{(\alpha,\beta)})$. Therefore $U(A_{(\alpha,\beta)})$ is an equivalence relation on R. Now let $(a, b) \in U(A_{(\alpha,\beta)})$ and x be an element of R. Then since $\mu_A(a - b) \geq \alpha, \nu_A(a - b) \leq \beta, \mu_A((a + x) - (b + x)) = \mu_A((a + x) + (-x - b)) = \mu_A(a + (x - x) - b) = \mu_A(a + 0 - b) = \mu_A(a - b) \geq \alpha, \nu_A((a + x) - (b + x)) = \nu_A((a + x) + (-x - b)) = \nu_A(a + (x - x) - b) = \nu_A(a + 0 - b) = \nu_A(a - b) \leq \beta$ and so $(a + x, b + x) \in U(A_(\alpha, \beta))$. Since $(R, +)$ is an abelian group, we have $(x + a, x + b) \in U(A_{(\alpha,\beta)})$. Therefore $U(A_{(\alpha,\beta)})$ is a congruence relation.
We denote $[x]_{A_{(\alpha,\beta)}}$ the equivalence class of $U(A_{(\alpha,\beta)})$ containing x of R. □

Lemma 3.6. *Let A be an intuitionistic fuzzy ideal of R. If $a, b \in R$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ then*

(i) $[a]_{A_{(\alpha,\beta)}} + [b]_{A_{(\alpha,\beta)}} = [a + b]_{A_{(\alpha,\beta)}}$.

(ii) $[-a]_{A_{(\alpha,\beta)}} = -([a]_{A_{(\alpha,\beta)}})$.

Proof. (i) Suppose $x \in [a]_{A_{(\alpha,\beta)}} + [b]_{A_{(\alpha,\beta)}}$. Then there exists an $y \in [a]_{A_{(\alpha,\beta)}}$ and $[z] \in [b]_{A_{(\alpha,\beta)}}$ such that $x = y + z$. Since $(a, y) \in U(A_{(\alpha,\beta)})$ and $(b, z) \in U(A_{(\alpha,\beta)})$ we have $(a + b, y + z) \in U(A_{(\alpha,\beta)})$ or $(a + b, x) \in U(A_{(\alpha,\beta)})$ and so $x \in [a + b]_{A_{(\alpha,\beta)}}$.
Conversely let $x \in [a + b]_{A_{(\alpha,\beta)}}$ then $(x, a + b) \in U(A_{(\alpha,\beta)})$. Hence $(x - b, a) \in U(A_{(\alpha,\beta)})$and so $x - b \in U(A_{(\alpha,\beta)})$ or $x \in [a]_{A_{(\alpha,\beta)}} + [b] \Rightarrow x \in [a]_{A_{(\alpha,\beta)}} + [b]_{A_{(\alpha,\beta)}}$.

(ii) We have $x \in [-a]]_{A_{(\alpha,\beta)}}$
$\Leftrightarrow (x,-a) \in U(A_{(\alpha,\beta)})$
$\Leftrightarrow (0,-a-x) \in U(A_{(\alpha,\beta)})$
$\Leftrightarrow (a,-x) \in U(A_{(\alpha,\beta)})$
$\Leftrightarrow -x \in [a]_{A_{(\alpha,\beta)}}$
$\Leftrightarrow x \in -([a]_{A_{(\alpha,\beta)}}$.

□

Lemma 3.7. *Let A and B be two intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ then $U((A \cap B)_{(\alpha,\beta)}) = U(A_{(\alpha,\beta)}) \cap U(B_{(\alpha,\beta)})$.*

Lemma 3.8. *Let A be an intuitionistic fuzzy ideal of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. For any $a \in R$ we have $a + [0]_{A_{(\alpha,\beta)}} = [a]_{A_{(\alpha,\beta)}}$.*

Proof. Assume that $a \in R$ then we have

$$\begin{aligned} x \in a + [0]_{A_{(\alpha,\beta)}} &\Leftrightarrow x - a \in [0]_{A_{(\alpha,\beta)}} \\ &\Leftrightarrow (x-a, 0) \in U(A_{(\alpha,\beta)}) \\ &\Leftrightarrow (x-a) \in U(A_{(\alpha,\beta)}) \\ &\Leftrightarrow x \in [a]_{A_{(\alpha,\beta)}} \end{aligned}$$

□

Lemma 3.9. *Let A and B be two intuitionistic fuzzy ideal of R such that $B \subseteq A$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$.Then $[x]_{B_{(\alpha,\beta)}} \subseteq [x]_{A_{(\alpha,\beta)}}$ for every $x \in R$.*

Proof. We have $y \in [x]_{A_{(\alpha,\beta)}} \Rightarrow (x,y) \in U(B_{(\alpha,\beta)})$
$\Rightarrow \mu_B(x-y) \geq \alpha$ and $\nu_B(x-y) \leq \beta$
$\Rightarrow \mu_A(x-y) \geq \alpha$ and $\nu_A(x-y) \leq \beta$
$\Rightarrow (x,y) \in U(A_{(\alpha,\beta)})$
$\Rightarrow y \in [x]_{A_{(\alpha,\beta)}}$. □

Definition 3.10. *Let A and B be two intuitionistic fuzzy ideals of a ring R. Then the composition of the congruence relation $U(A_{(\alpha,\beta)})$ and $U(B_{(\alpha,\beta)})$ is defined by*

$$U(A_{(\alpha,\beta)}) \circ U(B_{(\alpha,\beta)}) = \{(a,b) \in R \times R | \exists y \in R \ \textit{ such that } \ (a,c) \in U(A_{(\alpha,\beta)}), (c,b) \in U(B_{(\alpha,\beta)})\}.$$

We have $U(A_{(\alpha,\beta)}) \circ U(B_{(\alpha,\beta)})$ is also a congruence relation. We denote the congruence relation by $U((A \circ B)_{(\alpha,\beta)})$

Lemma 3.11. *Let A and B be two intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then $U((A \circ B)_{(\alpha,\beta)}) \subseteq U((A+B)_{(\alpha,\beta)})$.*

Proof. □

Assume that (a,b) be an arbitrary element of $U((A \circ B)_{(\alpha,\beta)})$. Then there exist an element $c \in R$ such that $(a,c) \in U(A_{(\alpha,\beta)}))$ and $(c,b) \in U(B_{(\alpha,\beta)}))$. Therefore we have $\mu_A(a-c) \geq \alpha, \nu_A(a-c) \leq \beta, \mu_B(c-b) \geq \alpha, \nu_B(c-b) \leq \beta$. Then $\mu_A + \mu_B(a-b) = \bigvee_{u+v=a-b} (\mu_A(u) \wedge \mu_B(v))$
$= \mu_A(a-c) \wedge \mu_B(c-b) \geq \alpha \wedge \alpha = \alpha$, $(\nu_A + \nu_B)(a-b) = \bigwedge_{u+v=a-b} (\nu_A(u) \vee \nu_B(v))$
$= \nu_A(a-c) \vee \nu_B(c-b) \geq \beta \vee \beta = \beta$ and so $U((A+B)_{(\alpha,\beta)})$.

Lemma 3.12. *Let A and B be two intuitionistic fuzzy ideals of a ring R with finite images and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then $U((A \circ B)_{(\alpha,\beta)}) = U((A+B)_{(\alpha,\beta)})$.*

Proof. □

Assume that $(a,b) \in U((A \circ B)_{(\alpha,\beta)})$. *Then* $(\mu_A + \mu_B)(a-b) \geq \alpha, (\nu_A + \nu_B)(a-b) \leq \beta$. *Thus we have* $\bigvee_{a-b=x+y} (\mu_A(x) \wedge \mu_B(y)) \geq \alpha$, $\bigwedge_{a-b=x+y} (\nu_A(u) \vee \nu_B(v)) \leq \beta$. *Since* $Im\mu_A$ *and* $Im\mu_B$ *are finite* $\mu_A(x_0) \wedge \mu_B(y_0) \geq \alpha$ *for some* $x_0, y_0 \in R$ *such that* $a - b = x_0 + y_0$. *Thus* $\mu_A(x_0) \geq \alpha$ *and* $\mu_A(y_0) \geq \alpha \Rightarrow$ $\mu_A(x_0 - 0) \geq \alpha$ *and* $\mu_A(a - b - x_0) \geq \alpha$*......(1)*
and $\nu_A(x_0) \vee \nu_B(y_0) \leq \beta$ *for some* $x_0, y_0 \in R$. *Thus* $\nu_A(x_0) \leq \beta$ *and* $\nu_B(y_0) \leq \beta \Rightarrow \nu_A(x_0 - 0) \leq \beta$ *and* $\nu_A(a - b - x_0) \leq \beta$*......(2)*
. From (i) and (ii) $(x_0, 0) \in U(A_{(\alpha,\beta)}))$ *and* $(a - b, x_0) \in U(B_{(\alpha,\beta)})$. *Therefore* $(a - b, 0) \in U((A \circ B)_{(\alpha,\beta)}$. *Since* $U((A \circ B)_{(\alpha,\beta)}$ *is a congruence relation we get* $(a,b) \in U((A \circ B)_{(\alpha,\beta)}$. *Thus* $U((A \circ B)_{(\alpha,\beta)} = U((A + B)_{(\alpha,\beta)}$, *if* $Im\mu_A$ *and* $Im\mu_B$ *are finite.*

4 Approximation based on intuitionistic fuzzy ideals

Let A be an intuitionistic fuzzy ideal of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Since $U(A_{(\alpha,\beta)})$ is an equivalence(congruence) relation on R we use $(R, A_{(\alpha,\beta)})$ instead of the approximation space (U, θ) where $U = R$ and θ is the above equivalence relation.

Definition 4.1. *Let A be an intuitionistic fuzzy ideal of a ring R and* $U(A_{(\alpha,\beta)})$ *be an* (α,β)*-level congruence relation of A on R. Let X be a non-empty subset of R. Then the sets*

$$\underline{U}(A_{(\alpha,\beta)}, X) = \{x \in R | [x]_{A_{(\alpha,\beta)}} \subseteq X\},$$
$$\overline{U}(A_{(\alpha,\beta)}, X) = \{x \in R | [x]_{A_{(\alpha,\beta)}} \cap X \neq \phi\}.$$

are respectively the lower and upper approximation of the set X with respect to $U(A_{(\alpha,\beta)})$.

Proposition 4.2. *For every approximation space* $(R, A_{(\alpha,\beta)})$ *and every subset* A, B *of R, we have:*

(i) $\underline{U}(A_{(\alpha,\beta)}, B) \subseteq B \subseteq \overline{U}(A_{(\alpha,\beta)}, B)$;

(ii) $\underline{U}(A_{(\alpha,\beta)}, \phi) = \phi = \overline{U}(A_{(\alpha,\beta)}, \phi)$;

(iii) $\underline{U}(A_{(\alpha,\beta)}, R) = R = \overline{U}(A_{(\alpha,\beta)}, B)$;

(iv) If $B \subset C$, *then* $\underline{U}(A_{(\alpha,\beta)}, B) \subseteq \underline{U}(A_{(\alpha,\beta)}, C)$;$\overline{U}(A_{(\alpha,\beta)}, B) \subseteq \overline{U}(A_{(\alpha,\beta)}, C)$

(v) $\underline{U}(A_{(\alpha,\beta)}, \underline{U}(A_{(\alpha,\beta)}, B)) = \underline{U}(A_{(\alpha,\beta)}, B)$;

(vi) $\overline{U}(A_{(\alpha,\beta)}, \overline{U}(A_{(\alpha,\beta)}, B)) = \overline{U}(A_{(\alpha,\beta)}, B)$;

(vii) $\overline{U}(A_{(\alpha,\beta)}, \underline{U}(A_{(\alpha,\beta)}, B)) = \underline{U}(A_{(\alpha,\beta)}, B)$;

(viii) $\underline{U}(A_{(\alpha,\beta)}, \overline{U}(A_{(\alpha,\beta)}, B)) = \overline{U}(A_{(\alpha,\beta)}, B)$;

(ix) $\underline{U}(A_{(\alpha,\beta)}, B) = \overline{U}(A_{(\alpha,\beta)}, B^c)^c$;

(x) $\overline{U}(A_{(\alpha,\beta)}, B) = \underline{U}(A_{(\alpha,\beta)}, B^c)^c$;

(xi) $\underline{U}(A_{(\alpha,\beta)}, B \cap C) = \underline{U}(A_{(\alpha,\beta)}, B) \cap \underline{U}(A_{(\alpha,\beta)}, C)$;

(xii) $\overline{U}(A_{(\alpha,\beta)}, B \cap C) \subseteq \overline{U}(A_{(\alpha,\beta)}, B) \cap \overline{U}(A_{(\alpha,\beta)}, C)$;

(xiii) $\underline{U}(A_{(\alpha,\beta)}, B \cup C) \supseteq \underline{U}(A_{(\alpha,\beta)}, B) \cup \underline{U}(A_{(\alpha,\beta)}, C)$;

(xiv) $\overline{U}(A_{(\alpha,\beta)}, B \cup C) = \overline{U}(A_{(\alpha,\beta)}, B) \cup \overline{U}(A_{(\alpha,\beta)}, C)$;

(xv) $\underline{U}(A_{(\alpha,\beta)}, [x]_{A_{(\alpha,\beta)}}) = \overline{U}(A_{(\alpha,\beta)}, [x]_{A_{(\alpha,\beta)}})$ *for all* $x \in R$.

Proof. The proof is obvious. □

The converse of (xii) and (xiii) in proposition 4.2 need not be true seen from the following example.

Example 4.3. *Let $R = \{0, x, y, z\}$ be a set with binary operations as follows:*

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

.	0	x	y	z
0	0	0	0	0
x	0	x	y	z
y	0	x	y	z
z	0	0	0	0

Then clearly R is a ring with $x = -x, y = -y$ and $z = -z$. Now let $\mu_A(0) = \alpha_0, \nu_0 = \beta_0, \mu_A(z) = \alpha_1, \nu_A(z) = \beta_1, \mu_A(x) = \mu_A(y) = \alpha_2, \nu_A(x) = \nu_A(y) = \beta_2$, where $\alpha_1, \beta_1 \in [0,1], i = 0,1,2$ and $\alpha_2 < \alpha_1 < \alpha_0$ and $\beta_0 < \beta_1 < \beta_2$. We have
$A_{(\alpha_0,\beta_0)} = \{(0,0),(x,x),(y,y),(z,z)\}$
$A_{(\alpha_1,\beta_1)} = \{(0,0),(x,x),(y,y),(z,z),(x,y),(y,x),(0,z),(z,0)\}$
$A_{(\alpha_2,\beta_2)} = R \times R$
. Now let $B = \{0, x\}$ and $C = \{0, y, z\}$. Then $\overline{U}(A_{(\alpha_1,\beta_1)}, B) = R; \overline{U}(A_{(\alpha_1,\beta_1)}, C) = R; \overline{U}(A_{(\alpha_1,\beta_1)}, (B \cap C) = \{0, z\}$; and $\underline{U}(A_{(\alpha_1,\beta_1)}, B) = \phi; \underline{U}(A_{(\alpha_1,\beta_1)}, C) = \{0, c\}; \underline{U}(A_{(\alpha_1,\beta_1)}, (B \cup C) = R$. Thus $\overline{U}(A_{(\alpha_1,\beta_1)}, B) \cap \overline{U}(A_{(\alpha_1,\beta_1)}, C) \nsubseteq \overline{U}(A_{(\alpha_1,\beta_1)}, (B \cap C)$; and $\underline{U}(A_{(\alpha_1,\beta_1)}, (B \cup C)) \nsubseteq \underline{U}(A_{(\alpha_1,\beta_1)}, B) \cup \underline{U}(A_{(\alpha_1,\beta_1)}, C)$;

Proposition 4.4.

Let A and B be intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. If X is a non-empty subset of R, then $\overline{U}((A \cap B)_{A_{(\alpha,\beta)}}, X) \subseteq \underline{U}(A_{(\alpha,\beta)}, X) \cap \underline{U}(A_{(\alpha,\beta)}, X)$.

Proof. Let $x \in overline((A \cap B)_{A_{(\alpha,\beta)}}, X)$,
$\Rightarrow [x]_{(A \cap B)_{(\alpha,\beta)}}) \cap X \neq \phi$,
$\Rightarrow a \in [x]_{(A \cap B)_{(\alpha,\beta)}}) \cap X$
$\Rightarrow (a, x) \in U(A \cap B)_{A_{(\alpha,\beta)}}$ and $a \in X$.
$\Rightarrow (\mu_A \cap \mu_B)(a - x) \geq \alpha, (\nu_A \cup \nu_B) \leq \beta$ and $a \in X$.
$\Rightarrow min\{\mu_A(a - x), \mu_B(a - x)\} \geq \alpha, max\{\nu_A(a - x), \nu_B(a - x)\} \leq \beta$ and $a \in X$.
$\Rightarrow \mu_A(a - x) \geq \alpha, \nu_A(a - x) \leq \beta$ and $\mu_B(a - x) \geq \alpha, \nu_B(a - x) \leq \beta$ and $a \in X$,
$\Rightarrow (a, x) \in U(A_{(\alpha,\beta)})$ and $(a, x) \in U(B_{(\alpha,\beta)})$ and $a \in X$.
$\Rightarrow (a, x) \in U(A_{(\alpha,\beta)}), a \in X$ and $(a, x) \in U(B_{(\alpha,\beta)}), a \in X$.
$\Rightarrow a \in [x]_{A_{(\alpha,\beta)}} \cap X$ and $a \in [x]_{B_{(\alpha,\beta)}} \cap X$,
$\Rightarrow x \in \overline{U}(A_{(\alpha,\beta)}, X)$ and $x \in \overline{U}(B_{(\alpha,\beta)}, X)$. The converse of the above proposition need not be true seen from the following example. ☐

Example 4.5. *Let $R = Z_6$(the ring of integers modulo 6). Let $B = Z_6 \to [0,1]$ and $C : Z_6 \to [0,1]$ with $\mu_B(0) = \nu_B(0) = \alpha_0, \mu_B(1) = \mu_B(2) = \mu_B(4) = \mu_B = \alpha_3, \nu_B(1) = \nu_B(2) = \nu_B(4) = \nu_B(5) = \beta_3, \mu_B(3) = \alpha_2, \nu_B(3) = \beta_2$;*
$\mu_C = \alpha_1, \nu_c(0) = \beta_1; \mu_C(1) = \mu_C(3) = \mu_C(5) = \alpha_4, \nu_C(1) = \nu_C(2) = \nu_C(5) = \beta_4, \mu_C(2) = \mu_C(4) = \alpha_2, \nu_C(2) = \nu_C(4) = \beta_2$ where $\alpha_1, \beta_1 \in [0,1], i = 0,1,2,3,4$ and $\alpha_4 < \alpha_3 < \alpha_2 < \alpha_1 < \alpha_0$ and $\beta_0 < \beta_1 < \beta_2 < \beta_3 < \beta_4$. We have $(\mu_B \cap \mu_C)(2) = (\mu_B \cap \mu_C)(4) = \alpha_3; (\mu_B \cap \mu_C)(1) = (\mu_B \cap \mu_C)(3) = (\mu_B \cap \mu_C)(5) = \alpha_4; (\mu_B \cap \mu_C)(0) = \alpha_1$;
$(\nu_B \cup \nu_C)(2) = (\nu_B \cup \nu_C)(4) = \beta_3; (\nu_B \cup \nu_C)(1) = (\nu_B \cup \nu_C)(3) = (\nu_B \cup \nu_C)(5) = \beta_4; (\nu_B \cap \nu_C)(0) = \beta_1$;
Also $B_{(\alpha_0,\beta_0)} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$;
$B_{(\alpha_2,\beta_2)} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),$
$(5,2),(2,5),(4,1)(1,4),(0,3),(3,0)\}$;
$B_{(\alpha_3,\beta_3)} = Z_6 \times Z_6$;
$C_{(\alpha_0,\beta_0)} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$;
$C_{(\alpha_2,\beta_2)} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),$
$(5,3),(3,5),(4,2),(2,4),(1,3),(3,1),$
$(0,2),(2,0),(5,1),(1,5),(0,4),(4,0)\}$;
$C_{(\alpha_4,\beta_4)} = Z_6 \times Z_6$;
$(B \cap C)_{(\alpha_0,\beta_0)} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$;

$(B \cap C)_{(\alpha_3,\beta_3)} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(5,3),(3,5),(4,2),(2,4),(1,3),(3,1),(0,2),(2,0),(5,1),(1,5),(0,4),(4,$
$(B \cap C)_{(\alpha_4,\beta_4)} = Z_6 \times Z_6$.
Now let $X = \{1,2,3\}$*, then* $\overline{U}(B_{(\alpha_2,\beta_2)},X) = Z_6; \overline{U}(C_{(\alpha_2,\beta_2)},X) = Z_6; \overline{U}((B \cap C)_{(\alpha_2,\beta_2)},X) = \{1,2,3\}$*; and* $\overline{U}((B \cap C)_{(\alpha,\beta)},X) = \neq \overline{U}(B_{(\alpha,\beta)},X) \cap \overline{U}(C_{(\alpha,\beta)},X)$

Proposition 4.6. *Let A and B be intuitionistic fuzzy ideals of a ring R and let* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta \le 1$*. If X is a non-empty subset of R, then* $\underline{U}(A_{(\alpha,\beta)},X) \cap \underline{U}(B_{(\alpha,\beta)},X) = \underline{U}((A \cap B)_{(\alpha,\beta)},X)$*.*

Proof. Let $x \in \underline{U}(A_{(\alpha,\beta)},X) \cap \underline{U}(B_{(\alpha,\beta)},X)$,
$\Rightarrow x \in \underline{U}(A_{(\alpha,\beta)},X)$ and $x \in \underline{U}(B_{(\alpha,\beta)},X)$,
$\Rightarrow [x]_{A_{(\alpha,\beta)}} \subseteq X$ and $[x]_{B_{(\alpha,\beta)}} \subseteq X$,
$\Rightarrow [x]_{(A\cap B)_{(\alpha,\beta)}} \subseteq X$,
$\Rightarrow x \in \underline{U}((A \cap B)_{(\alpha,\beta)},X)$ □

Proposition 4.7. *Let A be an intuitionistic fuzzy ideal of a ring R and let* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta \le 1$*. If A is an ideal of R, then A is an upper rough ideal of R.*

Proof. Let $a,b \in \overline{U}(A_{(\alpha,\beta)})$ and $r \in R$, then $[a]_{A_{(\alpha,\beta)}} \cap B \neq \phi$ and $[b]_{A_{(\alpha,\beta)}} \cap B \neq \phi$ so there exists $x \in [a]_{A_{(\alpha,\beta)}} \cap B$ and $y \in [y]_{B_{(\alpha,\beta)}} \cap B$. Since B is an ideal of R we have $x-y \in B$ and $rx \in B$. Thus $x-y \in [a]_{A_{(\alpha,\beta)}} - [b]_{A_{(\alpha,\beta)}} = [a-b]_{A_{(\alpha,\beta)}}$. Hence $[a-b]_{(A\cap B)_{(\alpha,\beta)}} \cap B \neq \phi$ this implies $a-b \in \overline{U}(A_{(\alpha,\beta)},B)$. Since $(x,a) \in U(A)_{(\alpha,\beta)}$, then $\mu_A(x-a) \ge \alpha, \nu_A(x-a) \le \beta$. Now we have

$$\mu_A(rx-ra) = \mu_A(r(x-a)) \ge max\{\mu_A(r),\mu_A(x-a)\} \ge \mu_A(x-a) \ge \alpha$$
$$\nu_A(rx-ra) = \nu_A(r(x-a)) \le min\{\nu_A(r),\nu_A(x-a)\} \le \nu_A(x-a) \le \beta.$$

Hence $(rx,ra) \in U(A_{(\alpha,\beta)})$ or $rx \in [ra]_{A_{(\alpha,\beta)}}$, thus $rx \in [ra]_{A_{(\alpha,\beta)}} \cap B \Rightarrow [ra]_{A_{(\alpha,\beta)}} \cap B \neq \phi$ Therefore $ra \in \overline{U}(A_{(\alpha,\beta)}),B)$. Likewise $ar \in \overline{U}(A_{(\alpha,\beta)}),B)$. Therefore $\overline{U}(A_{(\alpha,\beta)}),B)$ is an ideal of R. □

Lemma 4.8. *Let A be an intuitionistic fuzzy ideal of a ring R and let* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta \le 1$*.If* $\underline{U}(A_{(\alpha,\beta)},B)$ *is a non-empty set, then* $[0]_{A_{(\alpha,\beta)}} \subseteq B$*.*

Proof. Let $\underline{U}(A_{(\alpha,\beta)},B) \neq 0$ then there exists $x \in \underline{U}(A_{(\alpha,\beta)},B)$ or $[x]_{A_{(\alpha,\beta)}} \subseteq B$. So $[x]_{A_{(\alpha,\beta)}} \subseteq B$. So $-([x]_{A_{(\alpha,\beta)}}) \subseteq -B = \{-a|a \in B\} = B$.
$[0]_{A_{(\alpha,\beta)}} = [x+(-x)]_{A_{(\alpha,\beta)}}$;
$= [x]_{A_{(\alpha,\beta)}} + [-x]_{A_{(\alpha,\beta)}}$;
$= [x]_{A_{(\alpha,\beta)}} + (-[x]_{A_{(\alpha,\beta)}}) \subseteq B+B = B$. □

Proposition 4.9. *Let A be an intuitionistic fuzzy ideal of a ring R and let* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta \le 1$*.Let B be an intuitionistic fuzzy ideal of R. If* $\underline{U}(A_{(\alpha,\beta)},B)$ *is a non-empty set then it is equal to B.*

Proof. We know $\underline{U}(A_{(\alpha,\beta)},B) \subseteq B$. Assume tha t a is an arbitrary element of B. Since $[0]_{A_{(\alpha,\beta)}} \subseteq B$. Since A is an ideal of R, we have $a+[0]_{A_{(\alpha,\beta)}} \subseteq a+B \subseteq B$;
$\Rightarrow [a]_{A_{(\alpha,\beta)}} \subseteq B$.
$\Rightarrow a \in \underline{U}(A_{(\alpha,\beta)},B)$. □

Corollary 4.10. *Let A be an intuitionistic fuzzy ideal of a ring R and let* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta \le 1$*.If B is an ideal of R, then* $(\underline{U}(A_{(\alpha,\beta)},B),\overline{U}(A_{(\alpha,\beta)},B)$ *is a rough ideal of R.*

Proposition 4.11. *Let A and B be intuitionistic fuzzy ideals of a ring R and let* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta \le 1$*.If C is a non-empty subset of R, then*

(i) $\overline{U}(B_{(\alpha,\beta)},C) \subseteq \overline{U}(A_{(\alpha,\beta)},C)$;

(ii) $\underline{U}(A_{(\alpha,\beta)},C) \subseteq \underline{U}(B_{(\alpha,\beta)},C)$.

Proof. (i) Let x be an arbitrary element of $\overline{U}(B_{(\alpha,\beta)},C)$ then $[x]_{B_{(\alpha,\beta)}} \cap C \neq \phi$, since $[x]_{B_{(\alpha,\beta)}} \subseteq [x]_{A_{(\alpha,\beta)}}$, we have $[x]_{A_{(\alpha,\beta)}} \cap C \neq \phi$, which implies $x \in \overline{U}(A_{(\alpha,\beta)},C)$.

(ii) Let $x \in \underline{U}(A_{(\alpha,\beta)}, C)$, then $[x]_{A_{(\alpha,\beta)}} \subseteq C \Rightarrow [x]_{B_{(\alpha,\beta)}} \subseteq C$ thus $x \in \overline{U}(B_{(\alpha,\beta)}, C)$.

□

Proposition 4.12. *Let A and B be intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$.Let X be a non-empty subset of R.If $U(B_{(\alpha,\beta)}) \subseteq \overline{U}(A_{(\alpha,\beta)})$ then*

(i) $\overline{U}(B_{(\alpha,\beta)}, X) \subseteq \overline{U}(A_{(\alpha,\beta)}, X)$;

(ii) $\underline{U}(A_{(\alpha,\beta)}, X) \subseteq \underline{U}(B_{(\alpha,\beta)}, X)$.

Proof. (i) Let x be an arbitrary element of $\overline{U}(B_{(\alpha,\beta)}, X)$ then there exists $a \in [x]_{B_{(\alpha,\beta)}} \cap C$. Then $a \in X$ and $(a, x) \in U(B_{(\alpha,\beta)}, X) \subseteq \overline{U}(A_{(\alpha,\beta)})$. Therefore $a \in [x]_{A_{(\alpha,\beta)}} \cap X$and so $x \in \overline{U}(A_{(\alpha,\beta)}, X)$.

(ii) Let x be an arbitrary element of $\underline{U}(A_{(\alpha,\beta)}, X)$, then $[x]_{A_{(\alpha,\beta)}} \subseteq X$. Since $[x]_{B_{(\alpha,\beta)}} \subseteq [x]_{A_{(\alpha,\beta)}}$ we get $[x]_{B_{(\alpha,\beta)}} \subseteq X$ implies $x \in \underline{U}(A_{(\alpha,\beta)}, X)$.

□

Proposition 4.13. *Let A and B be intuitionistic fuzzy ideals of a ring R and let $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$. Let X be a non-empty subset of R then*

(i) $\overline{U}((A \circ B)_{(\alpha,\beta)}, X) \subseteq \overline{U}((A + B)_{(\alpha,\beta)}, X)$;

(ii) $\underline{U}((A + B)_{(\alpha,\beta)}, X) \subseteq \underline{U}((A \circ B)_{(\alpha,\beta)}, X)$.

Proposition 4.14. *Let A and B be an intuitionistic fuzzy ideal of a ring R, with finite images and let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. If X is a non-empty subset of R, then*

(i) $\overline{U}((A \circ B)_{(\alpha,\beta)}, X) \subseteq \overline{U}((A + B)_{(\alpha,\beta)}, X)$;

(ii) $\underline{U}((A + B)_{(\alpha,\beta)}, X) \subseteq \underline{U}((A \circ B)_{(\alpha,\beta)}, X)$.

If A and B are non-empty subsets of R. Let $A.B$ denote the set of all finite sums $\{a_1b_1 + a_2b_2 + ... + a_nb_n, n \in N, a_i \in A, b_i \in B\}$.

Proposition 4.15. *Let A and B be intuitionistic fuzzy ideals of a ring R and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. If A is an ideal of R, then $\overline{U}(A_{(\alpha,\beta)}, C).\overline{U}(B_{(\alpha,\beta)}, C) \subseteq \overline{U}((A \circ B)_{(\alpha,\beta)}, C)$.*

Proof. Suppose that z be any element of $\overline{U}(A_{(\alpha,\beta)}, C).\overline{U}(B_{(\alpha,\beta)}, C)$. Then $z = \sum\limits_{i=1}^{n} a_ib_i$ for some $a_i \in \overline{U}(A_{(\alpha,\beta)}, C)$ and $b_i \in \overline{U}(B_{(\alpha,\beta)}, C)$. Thus $[a_i]_{A_{(\alpha,\beta)}} \cap C \neq \phi$ and $[b_i]_{B_{(\alpha,\beta)}} \cap C \neq \phi$ for $i = 1, 2, 3.........n$. Since C is ideal of R then $\sum\limits_{i=1}^{n} x_iy_i \in C$. Since $(x_i, a_i) \in U(A_{(\alpha,\beta)})$ and $(y_i, b_i) \in U(B_{(\alpha,\beta)})$ we have $\mu_A(x_i - a_i) \geq \alpha, \nu_A(x_i - a_i) \leq \beta$ and $\mu_B(y_i - b_i) \geq \alpha, nu_A(y_i - b_i \leq \beta$. Then
$\mu_A(x_ib_i - a_ib_i) = \mu_A((x_i - a_i)b_i) \geq max\{\mu_A(x_i - a_i), \mu_A(b_i)\} \geq \mu_A(x_i - a_i) \geq \alpha$,
$\nu_A(x_ib_i - a_ib_i) = \nu_A((x_i - a_i)b_i) \leq min\{\nu_A(x_i - a_i), \nu_A(b_i)\} \leq \nu_A(x_i - a_i) \leq \beta$,
$\mu_B(x_iy_i - x_ib_i) = \mu_B((x_i(y_i - b_i)) \geq max\{\mu_B(x_i), \mu_B(y_i - b_i)\} \geq \mu_B(y_i - b_i) \geq \alpha$,
$\nu_B(x_iy_i - x_ib_i) = \nu_B((x_i(y_i - b_i) \leq min\{\nu_B(x_i), \nu_B(y_i - b_i)\} \leq \nu_B(y_i - b_i) \leq \beta$. Hence $(x_ib_i, a_ib_i) \in U(A_{\alpha,\beta})$ and $(x_iy_i, x_ib_i) \in U(B)_{\alpha,\beta}$ and so $(x_iy_i, a_ib_i) \in U(A \circ B)_{\alpha,\beta}$ for all $i = 1, 2, ...n$. Since $U(A \circ B)_{\alpha,\beta}$ is a congruence relation we get
$\left[\sum\limits_{i=1}^{n} x_iy_i, \sum\limits_{i=1}^{n} a_ib_i\right] \in U(A \circ B)_{(\alpha,\beta)}$ and so $\sum\limits_{i=1}^{n} x_iy_i \in \left[\sum\limits_{i=1}^{n} a_ib_i\right]_{(A \circ B)_{(\alpha,\beta)}}$.
Therefore $\left[\sum\limits_{i=1}^{n} a_ib_i\right]_{(A \circ B)_{(\alpha,\beta)}} \cap C \neq \phi \Rightarrow \sum\limits_{i=1}^{n} a_ib_i \in \overline{U}(A \circ B)_{(\alpha,\beta)}$. □

Corollary 4.16. *Let A and B be intuitionistic fuzzy ideal of a ring R and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. If A is an ideal of R, then $\overline{U}(A_{(\alpha,\beta)}, C).\overline{U}(B_{(\alpha,\beta)}, C) \subseteq \overline{U}((A + B)_{(\alpha,\beta)}, C)$.*

5 Conclusion

In this paper we considered the concept of intuitionistic fuzzy ideals on rough sets. The lower and upper approximation of rough sets were defined and a new definition is introduced such that the approximation space satisfies the condition of the ideal. Using this new relation, we have discussed some of the algebraic nature of intuitionistic fuzzy ideals of a ring. In future, the authors may extend this paper to neutrosophic ideals in approximation systems.

References

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, second ed., Springer-Verlag, USA, 1992.

[2] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems 20(1986)87-96.

[3] M. Banerjee, S.K. Pal, Roughness of a fuzzy set, Inform. Sci. 93 (34) (1996) 235246.

[4] B. Banerjee and D. K. Basnet, Intuitionistic fuzzy subrings and ideals, J. of fuzzy Mathematics 11(2003) 139-155.

[5] R. Biswas, S. Nanda, Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math. 42(1994) 251254.

[6] Z. Bonikowaski, Algebraic structures of rough sets, in: W.P. Ziarko (Ed.), Rough Sets Fuzzy Sets, and Knowledge Discovery, Springer-Verlag, Berlin, (1995)242247.

[7] Z. Bonikowaski, E. Bryniarski, U. Wybraniec-Skardowska, Extensions and intentions in the rough set theory, Inform. Sci. 107 (1998) 149167.

[8] K. Chakrabarty, R. Biswas, S. Nanda, Fuzziness in rough sets, Fuzzy Sets Syst. 110 (2000)247251.

[9] S. D. Comer, On connections between information systems, rough sets and algebraic logic, Algebraic Methods in Logic and Computer Science, Vol. 28, Banach Center Publications,(1993) 117124.

[10] B. Davvaz, Rough sets in a fundamental ring, Bull. Iranian Math. Soc. 24 (2) (1998) 4961.

[11] B. Davvaz, Lower and upper approximations in H_v-groups, Ratio Math. 13 (1999) 7186.

[12] B. Davaaz, Fuzzy sets and probabilistic rough sets, Int.J.Sci andTechnol. Univ. KAshan 1(2000)23-28

[13] B. Davvaz, Roughness in rings, Inform. Sci. 164 (2004) 147163.

[14] B. Davvaz, Roughness based on fuzzy ideals, Information Sciences 176(2006)2417-2437.

[15] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. General Syst. 17 (23) (1990) 191209.

[16] T. Iwinski, Algebraic approach to rough sets, Bull. Polish Acad. Sci. Math. 35 (1987) 673683.

[17] J. Jarvinen, On the structure of rough approximations, Fundamental Informatica 53(2002)135153.

[18] N. Kuroki, P.P.Wang, The lower and upper approximations in a fuzzy group, Inform.Sci. 90(1996)203-220.

[19] N. Kuroki, Rough ideals in semigroups, Inform. Sci. 100(1997)139-163.

[20] N. Kuroki, J. N. Mordeson, Structure of rough sets and rough groups, J. Fuzzy Math. 5 (1)(1997) 183191.

[21] W. J. Liu, Fuzzy invariant subgroups and fuzzy ideal, Fuzzy sets and system 8(1982)133-139.

[22] W. J. Liu, Operations on fuzzy ideals, Fuzzy Sets Syst. 11 (1983) 3141.

[23] J. N. Mordeson, Rough set theory applied to (fuzzy) ideal theory, Fuzzy Sets and Systems 121(2001) 315324.

[24] T. K. Mukherjee, M.K. Sen, On fuzzy ideals in rings 1, Fuzzy Sets Syst. 21 (1987) 99104.

[25] Z. Pawlak, Rough sets, Int. J. Inf. Comp. Sci. 11 (1982) 341356.

[26] Z. Pawlak, Rough SetsTheoretical Aspects of Reasoning About Data, Kluwer Academic Publishing, Dordrecht, 1991.

[27] K. Qin, Z. Pei, On the topological properties of fuzzy rough sets, Fuzzy Sets Syst. 151 (3)(2005) 601613.

[28] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512517.

[29] Q. M. Xiao, Z.-L. Zhang, Rough prime ideals and rough fuzzy prime ideals in semigroups, Inform. Sci., 176(2006)725-733.

[30] L. A. Zadeh, Fuzzy sets, Inform. Cont. 8 (1965) 338353.

[31] L. A. Zadeh, The concept of linguistic variable and its applications to approximate reasoning, Part I, Inform. Sci. 8 (1975) 199249; Part II, Inform. Sci. 8 (1975) 301357; Part III, Inform. Sci. 9 (1976) 4380.