$\widetilde{G}\alpha$-CLOSED SETS IN TERMS OF GRILLS

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Abstract

In this paper, we define the $\widetilde{g}\alpha(\theta)$-convergence and $\widetilde{g}\alpha(\theta)$-adherence using the concept of grills and study some of their properties.

Keywords. Grill, $\widetilde{g}\alpha(\theta)$-convergence and $\widetilde{g}\alpha(\theta)$-adherence of a grill, $\widetilde{g}\alpha$-closed space.

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1. Introduction and Preliminaries

Recently, R. Devi et al. [2] introduced and studied the concept of $\widetilde{g}\alpha$-closed sets in topological spaces. The idea of grill was introduced by G. Choquent [1] in 1947 and since then it has been observed in connection with many mathematical investigation such as the theories of proximity spaces, compactification etc, that grills as a tool (like filters) are extremely useful and convenient for many situations. In 2006, M.N. Mukherjee and B. Roy [4] studied the notion of p-closed sets in topological spaces in terms of grills.

In this paper, we introduce the notions of $\widetilde{g}\alpha(\theta)$-adherence and $\widetilde{g}\alpha(\theta)$-convergence of a grill and develop the concept to some extent so that the result derived here may support our subsequent deliberations.

Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure and the interior of A will be denoted by $cl(A)$ and $int(A)$, respectively.

Definition 1.1. [1] A grill $\mathcal{G}$ on a topological space X is defined to be a collection of non empty subsets of X such that

(i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$ and

(ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 1.2. A subset A of a space (X, τ) is called a

1. semi-open set [3] if $A \subseteq cl(int(A))$ and a semi-closed set [4] if $int(cl(A)) \subseteq A$ and

2. α-open set [5] if $A \subseteq int(cl(int(A)))$ and an α-closed set [6] if $cl(int(cl(A))) \subseteq A$.

The semi-closure (resp. α-closure) of a subset A of a space (X, τ) is the intersection of all semi-closed (resp. α-closed) sets that contain A and is denoted by $scl(A)$ (resp. $\alpha cl(A)$).

Definition 1.3. A subset A of a space (X, τ) is called a

1. $\widehat{g}$-closed set [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ); the complement of $\widehat{g}$-closed set is $\widehat{g}$-open,

2. ${}^*g$-closed set [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\widehat{g}$-open in (X, τ); the complement of ${}^*g$-closed set is ${}^*g$-open.

3. ${}^\sharp gs$-closed set [9] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is ${}^*g$-open in (X, τ); the complement of ${}^\sharp gs$-closed set is ${}^\sharp gs$-open.

4. $\widetilde{g}\alpha$-closed set [2] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ${}^\sharp gs$-open in (X, τ); the complement of $\widetilde{g}\alpha$-closed set is $\widetilde{g}\alpha$-open.

The set of all $\widetilde{g}\alpha$-open sets of X will be denoted by $\widetilde{G}\alpha O(X)$ and the set of all those members of $\widetilde{G}\alpha O(X)$, which contain a given point x of X will be designated by $\widetilde{G}\alpha O(x)$. The intersection of all $\widetilde{g}\alpha$-closed sets in X, which are contained in a given set $A(\subseteq X)$ is called the $\widetilde{g}\alpha$-closure of A, to be denoted by $\mathrm{cl}_{\widetilde{g}\alpha}(A)$. It is known that for $x \in X$ and $A \subseteq X$, $x \in \widetilde{g}\alpha$-cl$(A)$ if and only if $U \cap A \neq \phi$, for all $U \in \widetilde{G}\alpha O(x)$. Again for any set A in X, $\widetilde{g}\alpha(\theta)$-cl$(A)$, denoted by $\widetilde{g}\alpha(\theta)$-cl$(A)$, is defined as $\widetilde{g}\alpha(\theta)\text{-cl}(A) = \left\{x \in X : \widetilde{g}\alpha\text{-cl}(U) \cap A \neq \phi \text{ for all } U \in \widetilde{G}\alpha O(x)\right\}$.

2. Grills: $\widetilde{G}\alpha(\theta)$-convergence and $\widetilde{G}\alpha(\theta)$-adherence

Definition 2.1. A grill $\mathcal{G}$ on a topological space X is said to

(i) $\widetilde{g}\alpha(\theta)$-adhere at $x \in X$ if for each $U \in \widetilde{G}\alpha O(x)$ and each $G \in \mathcal{G}$, $\mathrm{cl}_{\widetilde{g}\alpha}(U) \cap G \neq \phi$,

(ii) $\widetilde{g}\alpha(\theta)$-converge to a point $x \in X$ if for each $U \in \widetilde{G}\alpha O(x)$, there is some $G \in \mathcal{G}$ such that $G \subseteq \mathrm{cl}_{\widetilde{g}\alpha}(U)$ (in this case we shall also say that $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-convergent to x).

Remark 2.2. It at once follows that a grill $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-convergent to a point $x \in X$ if and only if $\mathcal{G}$ contains the collection $\left\{\mathrm{cl}_{\widetilde{g}\alpha}(U) : U \in \widetilde{G}\alpha O(x)\right\}$.

Definition 2.3. A filter $\mathcal{F}$ on a topological space X is said to $\widetilde{g}\alpha(\theta)$-adhere at $x \in X$ ($\widetilde{g}\alpha(\theta)$-converge to $x \in X$) if for each $F \in \mathcal{F}$ and each $U \in \widetilde{G}\alpha O(x)$, $F \cap \mathrm{cl}_{\widetilde{g}\alpha}(U) \neq \phi$ (resp. to each $U \in \widetilde{G}\alpha O(x)$, there corresponds $F \in \mathcal{F}$ such that $F \subseteq \mathrm{cl}_{\widetilde{g}\alpha}(U)$).

Definition 2.4. [6] If $\mathcal{G}$ is a grill (or a filter) on a space X, then the section of $\mathcal{G}$, denoted by sec$\mathcal{G}$ is given by $\mathrm{sec}\mathcal{G} = \left\{A \subseteq X : A \cap G \neq \phi, \text{ for all } G \in \mathcal{G}\right\}$.

Lemma 2.5. [6]

(a) For any grill (filter) $\mathcal{G}$ on a space X, sec$\mathcal{G}$ is a filter(resp. grill) on X.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are respectively a filter and a grill on a space X with $\mathcal{F} \subseteq \mathcal{G}$, then there is an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{G}$.

Theorem 2.6. If a grill $\mathcal{G}$ on a topological space X, $\widetilde{g}\alpha(\theta)$-adheres at some point $x \in X$, then $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-convergent to x.

Proof. Let a grill $\mathcal{G}$ on X, $\widetilde{g}\alpha(\theta)$-adhere at $x \in X$. Then for each $U \in \widetilde{G}\alpha O(x)$ and each $G \in \mathcal{G}$, $\text{cl}_{\widetilde{g}\alpha}(U) \cap G \neq \phi$ so that $\text{cl}_{\widetilde{g}\alpha}(U) \in \text{sec}\mathcal{G}$, for each $U \in \widetilde{G}\alpha O(x)$ and hence $X - \text{cl}_{\widetilde{g}\alpha}(U) \notin \mathcal{G}$. Then $\text{cl}_{\widetilde{g}\alpha}(U) \in \mathcal{G}$ (as $\mathcal{G}$ is a grill and $X \in \mathcal{G}$), for each $U \in \widetilde{G}\alpha O(x)$. Hence $\mathcal{G}$ must $\widetilde{g}\alpha(\theta)$-converge to x.

The reverse need not be true by the following Example.

Example 2.7. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. It is easy to verify that (X, τ) is a topological space such that $\widetilde{G}\alpha O(X) = \tau$. Let $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-convergent but not $\widetilde{g}\alpha(\theta)$-adherent.

Notation 2.8. Let X be a topological space. Then for any $x \in X$, we have the following notation:

$$\begin{aligned} \mathcal{G}(\widetilde{g}\alpha(\theta), x) &= \Big\{A \subseteq X : x \in \widetilde{g}\alpha(\theta)\text{-cl}(A)\Big\} \\ sec\mathcal{G}(\widetilde{g}\alpha(\theta), x) &= \Big\{A \subseteq X : A \cap G \neq \phi, \text{ for all } G \in \mathcal{G}(\widetilde{g}\alpha(\theta), x)\Big\} \end{aligned}$$

In the next two theorems, we characterize the $\widetilde{g}\alpha(\theta)$-adherence and $\widetilde{g}\alpha(\theta)$-convergence of grills in terms of the above notations.

Theorem 2.9. A grill $\mathcal{G}$ on a space X, $\widetilde{g}\alpha(\theta)$-adheres to a point $x \in X$ if and only if $\mathcal{G} \subseteq \mathcal{G}(\widetilde{g}\alpha(\theta), x)$.

Proof. A grill $\mathcal{G}$ on a space X, $\widetilde{g}\alpha(\theta)$-adheres at $x \in X$.

$$\begin{aligned} &\Rightarrow \ cl_{\widetilde{g}\alpha}(U) \cap G \neq \phi, \text{ for all } U \in \widetilde{G}\alpha O(x) \text{ and all } G \in \mathcal{G} \\ &\Rightarrow \ x \in \widetilde{g}\alpha(\theta)\text{-}cl(G), \text{ for all } G \in \mathcal{G} \\ &\Rightarrow \ G \in \mathcal{G}(\widetilde{g}\alpha(\theta), x), \text{ for all } G \in \mathcal{G} \\ &\Rightarrow \ \mathcal{G} \subseteq \mathcal{G}(\widetilde{g}\alpha(\theta), x). \end{aligned}$$

Conversely, let $\mathcal{G} \subseteq \mathcal{G}(\widetilde{g}\alpha(\theta), x)$. Then for all $G \in \mathcal{G}$, $x \in \widetilde{g}\alpha(\theta)\text{-cl}(G)$, so that for all $U \in \widetilde{G}\alpha O(x)$ and for all $G \in \mathcal{G}$, $\mathrm{cl}_{\widetilde{g}\alpha}(U) \cap G \neq \phi$. Hence $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-adheres at x.

Theorem 2.10. A grill $\mathcal{G}$ on a topological space X is $\widetilde{g}\alpha(\theta)$-convergent to a point x of X if and only if $\sec\mathcal{G}(\widetilde{g}\alpha(\theta), x) \subseteq \mathcal{G}$.

Proof. Let $\mathcal{G}$ be a grill on X, $\widetilde{g}\alpha(\theta)$-converging to $x \in X$. Then for each $U \in \widetilde{G}\alpha O(x)$ there exists $G \in \mathcal{G}$ such that $G \subseteq \mathrm{cl}_{\widetilde{g}\alpha}(U)$ and hence

$$cl_{\widetilde{g}\alpha}(U) \in \mathcal{G} \text{ for each } U \in \widetilde{G}\alpha O(x) \tag{1}$$

Now, $B \in \sec\mathcal{G}(\widetilde{g}\alpha(\theta), x) \Rightarrow X - B \notin \mathcal{G}(\widetilde{g}\alpha(\theta), x) \Rightarrow x \notin \widetilde{g}\alpha(\theta)\text{-cl}(X - B) \Rightarrow$ there exists $U \in \widetilde{G}\alpha O(x)$ such that $\mathrm{cl}_{\widetilde{g}\alpha}(U) \cap (X - B) = \phi \Rightarrow \mathrm{cl}_{\widetilde{g}\alpha}(U) \subseteq B$, where $U \in \widetilde{G}\alpha O(x) \Rightarrow B \in \mathcal{G}$ (by (1)).

Conversely, let if possible, $\mathcal{G}$ not to $\widetilde{g}\alpha(\theta)$-converge to x. Then for some $U \in \widetilde{G}\alpha O(x)$, $\mathrm{cl}_{\widetilde{g}\alpha}(U) \notin \mathcal{G}$ and hence $\mathrm{cl}_{\widetilde{g}\alpha}(U) \notin \sec\mathcal{G}(\widetilde{g}\alpha(\theta), x)$. Thus for some $A \in \mathcal{G}(\widetilde{g}\alpha(\theta), x)$,

$$A \cap cl_{\widetilde{g}\alpha}(U) = \phi \tag{2}$$

But $A \in \mathcal{G}(\widetilde{g}\alpha(\theta), x) \Rightarrow x \in \widetilde{g}\alpha(\theta)\text{-cl}(A) \Rightarrow cl_{\widetilde{g}\alpha}(U) \cap A \neq \phi$, contradicting (2).

Definition 2.11. A non empty subset A of a topological space X is called $\widetilde{g}\alpha$-closed relative to X if for every cover $\mathcal{U}$ of A by $\widetilde{g}\alpha$-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup\Big\{\mathrm{cl}_{\widetilde{g}\alpha}(U) : U \in \mathcal{U}_0\Big\}$. If, in addition, $A = X$, then X is called a $\widetilde{g}\alpha$-closed space.

Theorem 2.12. A subset A of a topological space X is $\widetilde{g}\alpha$-closed relative to X if and only if every grill $\mathcal{G}$ on X with $A \in \mathcal{G}$, $\widetilde{g}\alpha(\theta)$-converges to a point in A.

Proof. Let A be $\widetilde{g}\alpha$-closed relative to X and $\mathcal{G}$ a grill on X satisfying $A \in \mathcal{G}$ such that $\mathcal{G}$ does not $\widetilde{g}\alpha(\theta)$-converges to any $a \in A$. Then to each $a \in A$, there corresponds some $U_a \in \widetilde{G}\alpha O(a)$ such that $\mathrm{cl}_{\widetilde{g}\alpha}(U_a) \notin \mathcal{G}$. Now $\{U_a : a \in A\}$ is a cover of A by $\widetilde{g}\alpha$-open sets of X. Then $A \subseteq \cup_{i=1}^{n}\mathrm{cl}_{\widetilde{g}\alpha}(U_{a_i}) = U$ (say), for some positive integer n. Since $\mathcal{G}$ is a grill, $U \notin \mathcal{G}$ and hence $A \notin \mathcal{G}$, which is a contradiction.

Conversely, let A be not $\widetilde{g}\alpha$-closed relative to X. Then for some cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of A by $\widetilde{g}\alpha$-open sets of X, $\mathcal{F} = \Big\{A - \cup_{\alpha\in\Lambda_0}\mathrm{cl}_{\widetilde{g}\alpha}(U_\alpha) : \Lambda_0$ is a finite subset of $\Lambda\Big\}$ is a filterbase on X. Then the family $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{F}^*$ on X. Then $\mathcal{F}^*$ is a grill on X with $A \in \mathcal{F}^*$. Now for each $x \in A$, there must exist $\beta \in \Lambda$ such that $x \in U_\beta$, as $\mathcal{U}$ is a cover of A. Then for any $G \in \mathcal{F}^*$, $G \cap (A - \mathrm{cl}_{\widetilde{g}\alpha}(U_\beta)) \neq \phi$, so that G does not contained in $\mathrm{cl}_{\widetilde{g}\alpha}(U_\beta)$, for all $G \in \mathcal{G}$. Hence $\mathcal{F}^*$ cannot $\widetilde{g}\alpha(\theta)$-converge to any point of A. The contradiction proves the desired result.

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