A runner cheating thanks to Riemann:

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Abstract: Pure mathematics should be used very carefully when applying it to many fields that have special considerations and special axioms.
I used a simple example of runners waiting for the start of a race. I concluded thanks to Riemann's definition of integrals that a runner can cheat in order to win.
The demonstration in this paper is very simple but the analogy of the proposed example with many fields can make the researcher be careful when using the definition of Riemann for the integrals.
Keywords: Riemann's definition, integrals, applied mathematics, analogy.

1. Introduction:

Let's consider that a runner wants to run a race.
The chronometer starts if and only if he overrides the starting line A.
The chronometer stops if and only if he reaches the finishing line B.
Let's consider that A and B are two points of the line of real numbers. Consequently, the distance between A and B is the distance where the chronometer isn't stopped, and thus the distance is the length l of the interval]A,B].
We conclude that l= B-A and consequently the length l_1 of]A,B[is l_1= B-A-ε and the length l_2 of [A,B] is l_2= B-A+ε
In many fields, ε exists otherwise the intervals will be considered the same during the study.
with ε as follows:
$\forall A, B \in \mathbb{R}$ with B>A we have: $\exists M \in \mathbb{R} \setminus \mathbb{Q}$ with M= Max {x/x ϵ [A,B[}
Hence: $\exists \varepsilon \in \mathbb{R} \setminus \mathbb{Q}$ with ε>0 and M+ ε=B.
M exists even if we can't determine it exactly . It is the real number that is sticking to B at the left in the real number line.
And thus we have:

$$B - \frac{C}{n} \leqslant B - \epsilon < B \qquad \forall n \in \mathbb{N} \setminus 0 \quad \text{and} \quad \forall C \ \epsilon \ ]0,+\infty[$$

So we can consider that $\varepsilon = 0^+$.
Remark:
In many fields where applied mathematics are required, a part of the line of the real numbers (an interval) has a length L>0 otherwise it doesn't exist.
The infinitesimal length of a singleton is l=ε>0 and it exists since it is the infinite sum of singletons lengths that make the length L of an interval.

2. Investigation:

Now let's make a definition to the integral by using Riemann's Definition: $\int_A^B f(x).dx$.

The correct subdivision of the interval [A,B] should be made with intervals which are: $[a_{i-1},a_i[$ since their length is exactly (a_i- a_{i-1}) and since they are separated (they make separate surfaces).
This is very important because the intervals $[a_{i-1},a_i[$ sum of infinite infinitesimal common surfaces can make a surface that can't be neglected.

Also: $\bigcup_{i=1}^{n} [a_{i-1},a_i[\Leftrightarrow [A,B[$

by considering: $a_i > a_{i-1}$ $\forall n \in \mathbb{N} \setminus 0$ and a_0=A and a_n=B
and $\forall i \in \mathbb{N} \setminus 0$: $A_i = A + i\frac{B-A}{n}$ and $a_i - a_{i-1} = h = \frac{B-A}{n}$

we define also: $x_i = A + \frac{(i-1+\gamma)\times(B-A)}{n} = a_{i-1} + \gamma \times h$ with: γ ϵ [0,1[.

Finally we have: $S_f = \sum_{i=1}^{n} (f(x_i)\times(a_i - a_{i-1})) = \frac{B-A}{n}\times\sum_{i=1}^{n}(f(A+\frac{(i-1+\gamma)\times(B-A)}{n}))$

We also have: [A,B] <=> $\underset{i=1}{\overset{n}{U}}$ $[a_{i-1},a_i[$ $U\{B\}$

By considering: M= Max {x/x ϵ $[a_{n-1},B[$}

We have {B}<=>]M,B]

Hence: [A,B] <=> $\underset{i=1}{\overset{n}{U}}$ $[a_{i-1},a_i[$ *U]M,B]*

And we have B-M= ε so we conclude that:

$S_f + (B-M)\times f(B) = S_f + \varepsilon\times f(B) = F_A^B$

Important:

We considered that: $\frac{B-A}{n} \geqslant \epsilon > 0$ from the beginning. With F_A^B the correct **Riemann sum** to $\int_A^B f(x).dx$ which respects: $\lim_{n\to+\infty} F_A^B = \int_A^B f(x).dx$

which means that: $\int_A^B f(x).dx - \varepsilon\times f(B) = \lim_{n\to+\infty} \frac{B-A}{n}\times\sum_{i=1}^{+\infty}(f(A+(i-1+\gamma))\times\frac{B-A}{n})$

Now if we use a subdivision with the intervals: $]a_{i-1},a_i]$ and γ ϵ]0,1] and ε is the same for all the real numbers since the line of the real numbers is homogeneous.

S_f stays the same but: $\int_A^B f(x).dx - \varepsilon\times f(A) = \lim_{n\to+\infty} \frac{B-A}{n}\times\sum_{i=1}^{+\infty}(f(A+(i-1+\gamma))\times\frac{B-A}{n})$

We conclude that: $\varepsilon\times(f(A)-f(B)) = 0 \Rightarrow \varepsilon = 0$.

Consequently:

When using riemann's definition of integrals:

M= Max {x/x ϵ $[a_{n-1},B[$}= Sup {x/x ϵ $[a_{n-1},B[$}=B

And M'= Min {x/x ϵ $]A,a_1]$}= Inf {x/x ϵ $]A,a_1]$}=A

3. Conclusion:

Now let's return back to our example:

We conclude that if our runners will wait infinite time for the race to start, then a runner has enough time to cheat with a distance ε to get closer to the finishing line without that any referee notices until he wins.

The analogy exists between this example and many fields.