Strip Configuration of the Poincaré Sphere

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Abstract

In this paper we find and discuss the Strip Configuration of the Poincaré Sphere.

Key Words: compact manifold, topology.

1 Introduction

The Poincaré homology sphere, first introduced by Henri Poincaré, is an example of a closed 3-manifold with homology groups homologous to a 3-sphere but which is not homeomorphic to it. As a matter of fact it has a finite fundamental group of order 120 known as as the binary icosahedral group. There are many ways to construct the Poincaré homology sphere. Among all, the simplest construction is by identifying opposite faces of a dodecahedron using the minimal clockwise twist to line up the faces.

In [1] we have defined the idea of a solid strip configurations which is a way for describing 3-dimensional compact manifolds alternative to Δ-complexes and CW complexes.

In this paper we want to find and discuss the Solid Strip Configuration of the Poincaré homology sphere.

2 CW Complex Definition

We want to find the solid strip configuration of the Poincaré homology sphere. In order to do it we have to define the Poincaré homology sphere CW complex first.

As mentioned in the introduction the CW complex of the Poincaré homology sphere is composed by a single 3-cell which is a dodecahedron where opposite faces of the dodecahedron are identified using the minimal clockwise twist to line up them. By doing so some edges an vertices of the original dodecahedron get identified themselves. We have worked out identified edges (labelled by letters from A to J) and vertices (labelled by numbers from 1 to 5) and the result is shown in Fig. 1a where an orientation for each edge is also given.

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By doing so we get a final CW complex with:

$$\begin{array}{rl} 5 & Vertices \\ 10 & Edges \\ 6 & Faces \\ 1 & 3-cell \end{array} \tag{1}$$

Which gives an Euler characteristics equal to $\chi = 0$, as expected since all the homology groups of the Poincaré sphere are trivial.

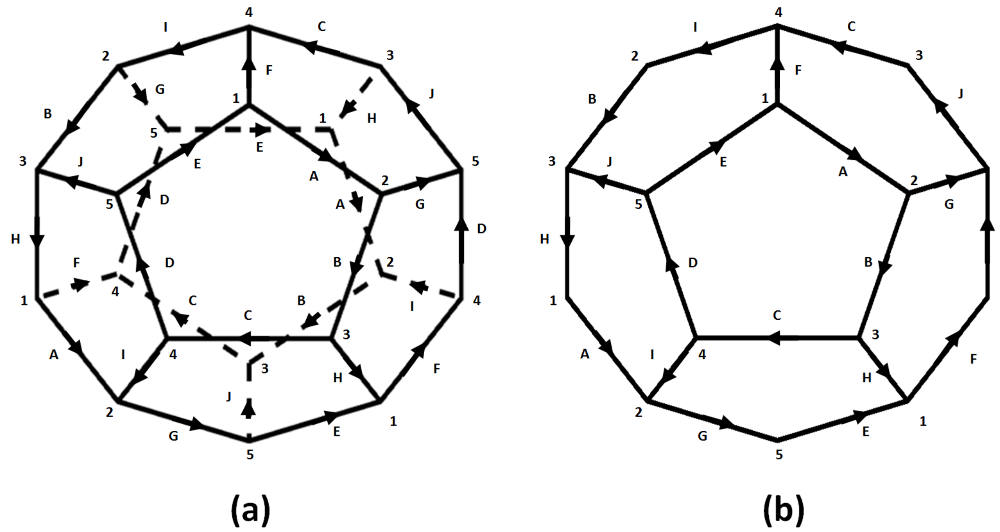

Figure 1: Poincaré Sphere CW complex

3 Finding the Solid Strip Configuration

As described in [1], if the rank of the homology group $\mathbf{H}_2$ of a space X is k, then there are $k+1$ classes of closed surfaces defined in the CW complex each of which contain one or more surfaces that compose the strip configuration of X. Since the Poincaré homology sphere has trivial group $\mathbf{H}_2$, then in this case there is only one of such class.

We can find the above mentioned surfaces from the chain complex of X. For 3-manifolds, the way to do it is to find all the cycles starting from the 3-cells space and using the boundary maps of the chain to see in which class they end up. Each class can be analysed to see how many homeomorphic classes of surfaces there are and in which way the cross (see [1]). Eventually, when all the surfaces and the way they cross, are identified they can be turned in solid strips (i.e. thick surfaces) and used to build the solid strip configuration.

In this case the analysis is very simple since there is only a 3-cell and therefore the only class mentioned above contains only one surfaces σ which is, obviously the cycles defined by the only 3-cell present and by the map from the 3-cell space to the 2-cell space.

The surface σ can be extracted easily from the CW complex and its 2-dimensional CW complex is shown in Fig. 1b. The surface σ is orientable (as expected since the group $\mathbf{H}_2$ of the Poincaré sphere has no torsion $\mathbb{Z}_2$, see [1]) and it is not a manifold since at each edge of σ three edges of three separate faces (i.e. 2-cells) are identified.

The relevant solid strip of σ can be found by turning it to a thick surface. since σ is orientable, this can be done embedding σ in $\mathbb{R}^4$ and shifting it by a small δL along the 5^{th} dimension. The space swept by σ during the shift is a 3-dimensional manifold ξ with boundaries, which is the solid strip we where looking for.

The Poincaré homology sphere is the manifold associated to ξ (i.e. $\Omega(\xi)$ see [1]) and it is the simplest manifold in which ξ can be embedded.

References

[1] V. Nardozza. *Solid Strips Configurations.* http://vixra.org/abs/1912.0161 (2019)