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PREOPEN SETS IN IDEAL GENERALIZED TOPOLOGICAL SPACES

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Dedicated to the late Professor Ákos Császár

Abstract. The aim of this paper is to introduce and characterize the concepts of preopen sets and their related notions in ideal generalized topological spaces.

1. Introduction

In [4], Császár introduced the important and useful notions of generalized neighborhood systems and generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In the same paper he investigated characterizations of generalized continuous functions ($=(g,g')$-continuous functions). A subfamily g of the power set $P(X)$ of a nonempty set X is called a generalized topology [4] on X if $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \underset{i\in I}{\cup} G_i \in g$. We call the pair (X, g) a generalized topological space (briefly GTS) on X. The members of g are called g-open sets [4] and the complement of a g-open set is called a g-closed set. The generalized closure of a set S of X, denoted by $g\,\mathrm{cl}(A)$, is the intersection of all g-closed sets containing A and the generalized interior of A, denoted by $g\,\mathrm{Int}(A)$, is the union of g-open sets included in A. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [18] and Vaidyanathasamy [23]. Hamlett and Janković (see [12], [13], [16] and [17]) used topological ideals to generalize many notions and properties in general topology. The research in this direction continued by many researchers such as M. E. Abd El-Monsef, A. Al-Omari, F.

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G. Arenas, M. Caldas, J. Dontchev, M. Ganster, D. N. Georgiou, T. R. Hamlett, E. Hatir, S. D. Iliadis, S. Jafari, D. Jankovic, E. F. Lashien, M. Maheswari, , H. Maki, A. C. Megaritis, F. I. Michael, A. A. Nasef, T. Noiri, B. K. Papadopoulos, M. Parimala, G. A. Prinos, M. L. Puertas, M. Rajamani, N. Rajesh, D. Rose, A. Selvakumar, Jun-Iti Umehara and many others (see [1], [2], [3], [6], [7], [8], [10], [11], [14], [15], [17], [19], [20], [22]). An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

2. Preliminaries

Definition 2.1. A subset A of a GTS (X, g) is said to be g-preopen [5] if $A \subset g\,\mathrm{Int}(g\,\mathrm{cl}(A))$.

Definition 2.2. Let g and g' be generalized topologies on X and Y, respectively. Then a function $f\colon X \to Y$ is said to be (g, g')-continuous [4] if $f^{-1}(G) \in g$ for every $G \in g'$.

Definition 2.3 ([9]). Given a GTS (X, g) with an ideal $\mathcal{I}$ on X (for short, IGTS) and if $\mathcal{P}(X)$ is the set of all subsets of X, the generalized local function of A with respect to g and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^*_g(g, \mathcal{I}) = \{x \in X\colon A \cap \cap U \notin \mathcal{I}$ for every g-open set containing $x\}$. When there is no ambiguity, we will write A^*_g for $A^*_g(g, \mathcal{I})$.

Remark 2.4 ([9]). Let $(X, g, \mathcal{I})$ be an IGTS and A a subset of X. Then we have the following:

(1) $A^*_g(g, \{\emptyset\}) = g\,\mathrm{cl}(A)$.
(2) $A^*_g(g, \mathcal{P}(X)) = \emptyset$.
(3) If $A \in \mathcal{I}$, then $A^*_g = \emptyset$.
(4) Neither $A \subset A^*_g$ nor $A^*_g \subset A$.

Theorem 2.5 ([9]). *Let $(X, g, \mathcal{I})$ be an IGTS and A, B subsets of X. Then we have the following:*

(1) If $A \subset B$, then $A^*_g \subset B^*_g$.
(2) $A^*_g = g\,\mathrm{cl}(A^*_g) \subset g\,\mathrm{cl}(A)$ and A^*_g is a g-closed set in (X, g).
(3) $(A^*_g)^*_g \subset A^*_g$.
(4) $(A \cup B)^*_g = A^*_g \cup B^*_g$.
(5) $A^*_g \setminus B^*_g = (A \setminus B)^*_g \setminus B^*_g \subset (A \setminus B)^*_g$
(6) If $C \in \mathcal{I}$, then $(A \setminus C)^*_g \subset A^*_g = (A \cup C)^*_g$.

Definition 2.6 ([9]). Let $(X, g, \mathcal{I})$ be an IGTS. The set operator $g\,\mathrm{cl}^*$ is called a g-$*$-closure and is denoted as $g\,\mathrm{cl}^*(A) = A \cup A_g^*$ for $A \subset X$.

We will denote by $g^*(\mathcal{I}, g)$ the generalized topology generated by $g\,\mathrm{cl}^*$, that is $g^*(\mathcal{I}, g) = \{U \subset X : g\,\mathrm{cl}^*(X \setminus U) = X \setminus U\}$. Clearly, $g^*(\mathcal{I}, g)$ is finer than g. The elements of $g^*(\mathcal{I}, g)$ are called g-$*$-open sets and the complement of g-$*$-open set is a called g-$*$-closed set. Also, $g\,\mathrm{Int}^*(A)$ denotes the interior of A in $g^*(\mathcal{I})$.

Proposition 2.7 ([9]). *The set operator $g\,\mathrm{cl}^*$ satisfies the following:*

(1) $A \subset g\,\mathrm{cl}^*(A)$.
(2) $g\,\mathrm{cl}^*(\emptyset) = \emptyset$ and $g\,\mathrm{cl}^*(X) = X$.
(3) If $A \subset B$, then $g\,\mathrm{cl}^*(A) \subset g\,\mathrm{cl}^*(B)$.
(4) $g\,\mathrm{cl}^*(A) \cup g\,\mathrm{cl}^*(B) \subset g\,\mathrm{cl}^*(A \cup B)$.
(5) If $\mathcal{I} = \{\emptyset\}$, then $g\,\mathrm{cl}^*(A) = g\,\mathrm{cl}(A)$ for $A \subset X$.

Definition 2.8. A subset A of an IGTS $(X, g, \mathcal{I})$ is said to be g-$\mathcal{I}$-open [9] if $A \subset g\,\mathrm{Int}(A_g^*)$.

Definition 2.9. A function $f\colon (X, g, \mathcal{I}) \to (Y, g')$ is called (g, g')-$\mathcal{I}$-continuous [9] if $f^{-1}(V)$ is g-$\mathcal{I}$-open in X for every g'-open set V of Y.

3. Generalized pre-$\mathcal{I}$-open sets

Definition 3.1. A subset A of an IGTS $(X, g, \mathcal{I})$ is said to be

(1) generalized pre-$\mathcal{I}$-open (briefly, g-pre-$\mathcal{I}$-open) if $A \subset g\,\mathrm{Int}(g\,\mathrm{cl}^*(A))$.

(2) generalized preopen (briefly g-preopen) if $A \subset gInt(gCl(A))$.

The family of all g-pre-$\mathcal{I}$-open sets of $(X, g, \mathcal{I})$ is denoted by $P\mathcal{I}O(X, g)$. Also, The family of all g-pre-$\mathcal{I}$-open sets of $(X, g, \mathcal{I})$ containing x is denoted by $GP\mathcal{I}O(X, x)$.

Proposition 3.2. *For an IGTS $(X, g, \mathcal{I})$, we have the following*

(i) Every g-open sets is g-pre-$\mathcal{I}$-open
(ii) Every g-$\mathcal{I}$-open set is g-pre-$\mathcal{I}$-open.
(iii) Every g-pre-$\mathcal{I}$-open is g-preopen.

Proof. (i). Clear.

(ii). Let A be a g-$\mathcal{I}$-open set. Then $A \subset g\,\mathrm{Int}(A_g^*) \subset g\,\mathrm{Int}(A \cup A_g^*) = g\,\mathrm{Int}(g\,\mathrm{cl}^*(A))$. Therefore, A is g-$\mathcal{I}$-open in $(X, g, \mathcal{I})$.

(iii). Let $(X, g, \mathcal{I})$ be an IGTS and let $A \in PIO(X, g)$. Then $A \subset \subset g\,\mathrm{Int}(g\,\mathrm{cl}^*(A)) = g\,\mathrm{Int}(A \cup A^*_g) \subset g\,\mathrm{Int}(g\,\mathrm{cl}(A) \cup A) = g\,\mathrm{Int}(g\,\mathrm{cl}(A))$. This shows that A is g-preopen. ∎

The converses of the above Proposition is not true in general as they can be seen from the following example.

Example 3.3. *Let $X = \{a, b, c\}$, $g = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the set $\{b\}$ is g-pre-$\mathcal{I}$-open but not g-open, the set $\{a\}$ is g-pre-$\mathcal{I}$-open but not g-$\mathcal{I}$-open and the set $\{a, c\}$ is g-preopen but not g-pre-$\mathcal{I}$-open.*

Proposition 3.4. *For an IGTS $(X, g, \mathcal{I})$ and $A \subset X$, we have:*
(i) If $\mathcal{I} = \emptyset$, then A is g-pre-$\mathcal{I}$-open if and only if A is g-preopen.
(ii) If $\mathcal{I} = \mathcal{P}(X)$, then A is g-pre-$\mathcal{I}$-open if and only if A is g-open.

Proof. The proof follows from the fact that
(i) if $\mathcal{I} = \emptyset$, then $A^*_g = g\,\mathrm{cl}(A)$.
(ii) if $\mathcal{I} = \mathcal{P}(X)$, then $A^*_g = \emptyset$ for every subset A of X.

∎

Proposition 3.5. *Let A be a subset of an IGTS $(X, g, \mathcal{I})$ and A be an g-pre-$\mathcal{I}$-open set. Then we have the following:*
(1) $g\,\mathrm{cl}(g\,\mathrm{Int}(g\,\mathrm{cl}^*(A))) = g\,\mathrm{cl}(A)$.
(2) $g\,\mathrm{cl}^*(g\,\mathrm{Int}(g\,\mathrm{cl}^*(A))) = g\,\mathrm{cl}^*(A)$.

Proof. The proof is obvious. ∎

Remark 3.6. The intersection of two g-pre-$\mathcal{I}$-open sets need not be g-pre-$\mathcal{I}$-open as it can be seen from the following example.

Example 3.7. *Let $X = \{a, b, c\}$, $g = \{\emptyset, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the sets $\{a, b\}$ and $\{a, c\}$ are g-pre-$\mathcal{I}$-open sets of $(X, g, \mathcal{I})$ but their intersection $\{a\}$ is not a g-pre-$\mathcal{I}$-open set of $(X, g, \mathcal{I})$.*

Theorem 3.8. *If $\{A_\alpha\}_{\alpha \in \Omega}$ is a family of g-pre-$\mathcal{I}$-open sets in $(X, g, \mathcal{I})$, then $\bigcup_{\alpha \in \Omega} A_\alpha$ is g-pre-$\mathcal{I}$-open in $(X, g, \mathcal{I})$.*

Proof. Since $\{A_\alpha : \alpha \in \Omega\} \subset PIO(X, g)$, then $A_\alpha \subset g\,\mathrm{Int}(g\,\mathrm{cl}^*(A_\alpha))$ for every $\alpha \in \Omega$. Thus, $\bigcup_{\alpha \in \Omega} A_\alpha \subset \bigcup_{\alpha \in \Omega} g\,\mathrm{Int}(g\,\mathrm{cl}^*(A_\alpha)) \subset g\,\mathrm{Int}(g\,\mathrm{cl}^*(\bigcup_{\alpha \in \Omega} A_\alpha))$. Therefore, we obtain $\bigcup_{\alpha \in \Omega} A_\alpha \subset g\,\mathrm{Int}(g\,\mathrm{cl}^*(\bigcup_{\alpha \in \Omega} A_\alpha))$. Hence any union of g-pre-$\mathcal{I}$-open sets is g-pre-$\mathcal{I}$-open. ∎

Definition 3.9. In an IGTS $(X, g, \mathcal{I})$, $A \subset X$ is said to be g-pre-$\mathcal{I}$-closed if $X \setminus A$ is g-pre-$\mathcal{I}$-open in X.

Theorem 3.10. *If A is an g-pre-$\mathcal{I}$-closed set in an IGTS $(X, g, \mathcal{I})$ if and only if $g\,\mathrm{cl}(g\,\mathrm{Int}^*(A)) \subset A$.*

Proof. The proof follows from the definitions. ∎

Theorem 3.11. *If a subset A of an IGTS $(X, g, \mathcal{I})$ is g-pre-$\mathcal{I}$-closed, then $g\,\mathrm{cl}^*(g\,\mathrm{Int}(A)) \subset A$.*

Proof. Since A is g-pre-$\mathcal{I}$-closed, $X \setminus A$ is g-pre-$\mathcal{I}$-open in $(X, g, \mathcal{I})$. Then $X \setminus A \subset g\,\mathrm{Int}(g\,\mathrm{cl}^*(X \setminus A)) \subset g\,\mathrm{Int}^*(g\,\mathrm{cl}(X \setminus A)) = X \setminus (g\,\mathrm{cl}^*(gInt(A))$. Therefore, we obtain $g\,\mathrm{cl}^*(g\,\mathrm{Int}(A)) \subset A$. ∎

Theorem 3.12. *Arbitrary intersection of g-pre-$\mathcal{I}$-closed sets is always g-pre-$\mathcal{I}$-closed.*

Proof. The proof follows from Theorems 3.8 and 3.11. ∎

Definition 3.13. Let $(X, g, \mathcal{I})$ be an IGTS, S a subset of X and x be a point of X. Then

(i) x is called a g-pre-$\mathcal{I}$-interior point of S if there exists $V \in P\mathcal{I}O(X, g)$ such that $x \in V \subset S$.
(ii) the set of all g-pre-$\mathcal{I}$-interior points of S is called the g-pre-$\mathcal{I}$-interior of S and is denoted by $gp\mathcal{I}\,\mathrm{Int}(S)$.

Theorem 3.14. *Let A and B be subsets of $(X, g, \mathcal{I})$. Then the following properties hold:*

(i) $gp\mathcal{I}\,\mathrm{Int}(A) = \cup\{T : T \subset A \text{ and } T \in P\mathcal{I}O(X, g)\}$.
(ii) $gp\mathcal{I}\,\mathrm{Int}(A)$ *is the largest g-pre-$\mathcal{I}$-open subset of X contained in A.*
(iii) A is g-pre-$\mathcal{I}$-open if and only if $A = gp\mathcal{I}\,\mathrm{Int}(A)$.
(iv) $gp\mathcal{I}\,\mathrm{Int}(gp\mathcal{I}\,\mathrm{Int}(A)) = gp\mathcal{I}\,\mathrm{Int}(A)$.
(v) If $A \subset B$, then $gp\mathcal{I}\,\mathrm{Int}(A) \subset gp\mathcal{I}\,\mathrm{Int}(B)$.
(vi) $gp\mathcal{I}\,\mathrm{Int}(A) \cup gp\mathcal{I}\,\mathrm{Int}(B) \subset gp\mathcal{I}\,\mathrm{Int}(A \cup B)$.
(vii) $gp\mathcal{I}\,\mathrm{Int}(A \cap B) \subset gp\mathcal{I}\,\mathrm{Int}(A) \cap gp\mathcal{I}\,\mathrm{Int}(B)$.

Proof. (i) Let $x \in \cup\{T : T \subset A$ and $T \in P\mathcal{I}O(X, g)\}$. Then, there exists $T \in GP\mathcal{I}O(X, x)$ such that $x \in T \subset A$ and hence $x \in gp\mathcal{I}\,\mathrm{Int}(A)$. This shows that $\cup\{T : T \subset A$ and $T \in P\mathcal{I}O(X, g)\} \subset gp\mathcal{I}\,\mathrm{Int}(A)$. For the reverse inclusion, let $x \in gp\mathcal{I}\,\mathrm{Int}(A)$. Then there exists $T \in GP\mathcal{I}O(X, x)$ such that $x \in T \subset A$. we obtain $x \in \cup\{T : T \subset A$ and $T \in P\mathcal{I}O(X, g)\}$. This shows that $gp\mathcal{I}\,\mathrm{Int}(A) \subset \cup\{T\colon T \subset A$ and $T \in P\mathcal{I}O(X, g)\}$. Therefore, we obtain $gp\mathcal{I}\,\mathrm{Int}(A) = \cup\{T\colon T \subset A$ and $T \in P\mathcal{I}O(X, g)\}$.

The proofs of (ii)–(v) are obvious.

(vi) Clearly, $gp\mathcal{I}\,\mathrm{Int}(A) \subset gp\mathcal{I}\,\mathrm{Int}(A \cup B)$ and $gp\mathcal{I}\,\mathrm{Int}(B) \subset gp\mathcal{I}\,\mathrm{Int}(A \cup B)$. Then by (v) we obtain $gp\mathcal{I}\,\mathrm{Int}(A) \cup gp\mathcal{I}\,\mathrm{Int}(B) \subset gp\mathcal{I}\,\mathrm{Int}(A \cup B)$.

(vii) Since $A \cap B \subset A$ and $A \cap B \subset B$, by (v), we have $gp\mathcal{I}\,\mathrm{Int}(A \cap B) \subset$ $\subset gp\mathcal{I}\,\mathrm{Int}(A)$ and $gp\mathcal{I}\,\mathrm{Int}(A \cap B) \subset gp\mathcal{I}\,\mathrm{Int}(B)$. By (v) $gp\mathcal{I}\,\mathrm{Int}(A \cap B) \subset$ $\subset gp\mathcal{I}\,\mathrm{Int}(A) \cap gp\mathcal{I}\,\mathrm{Int}(B)$. ∎

Lemma 3.15. *If A is any subset of an IGTS $(X, g, \mathcal{I})$, then*

$$gp\mathcal{I}\,\mathrm{Int}(A) = A \cap g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A)).$$

Proof. Since

$$A \cap g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A)) \subset g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A)) =$$

$$= g\,\mathrm{Int}(g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A))) = g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A) \cap g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A))) \subset$$

$$\subset g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A \cap g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A))),$$

$A \cap g\,\mathrm{Int}(g\,\mathrm{cl}^*(A))$ is a g-pre-$\mathcal{I}$-open set contained in A and so

$$A \cap g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A)) \subset gp\mathcal{I}\,\mathrm{Int}(A).$$

Since $gp\mathcal{I}\,\mathrm{Int}(A)$ is g-pre-$\mathcal{I}$-open,

$$gp\mathcal{I}\,\mathrm{Int}(A) \subset g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(gp\mathcal{I}\,\mathrm{Int}(A)))) \subset g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A))$$

and so

$$gp\mathcal{I}\,\mathrm{Int}(A) \subset A \cap g\,\mathrm{Int}(g\overset{*}{\mathrm{cl}}(A)).$$

Hence $gp\mathcal{I}\,\mathrm{Int}(A) = A \cap g\,\mathrm{Int}(g\,\mathrm{cl}^*(A))$. ∎

Definition 3.16. Let $(X, g, \mathcal{I})$ be an IGTS, S a subset of X and x be a point of X. Then

(i) x is called a g-pre-$\mathcal{I}$-cluster point of S if $V \cap S \neq \emptyset$ for every $V \in$ $\in GP\mathcal{I}O(X, x)$.

(ii) the set of all g-pre-$\mathcal{I}$-cluster points of S is called the g-pre-$\mathcal{I}$-closure of S and is denoted by $gp\mathcal{I}\,\mathrm{cl}(S)$.

Theorem 3.17. *Let A and B be subsets of $(X, g, \mathcal{I})$. Then the following properties hold:*

(i) $gp\mathcal{I}\,\mathrm{cl}(A) = \cap\{F\colon A \subset F$ *and* $F \in P\mathcal{I}C(X, g)\}$.

(ii) *$gp\mathcal{I}$ $\mathrm{cl}(A)$ is the smallest g-pre-$\mathcal{I}$-closed subset of X containing A.*
(iii) *A is g-pre-$\mathcal{I}$-closed if and only if $A = gp\mathcal{I}\ \mathrm{cl}(A)$.*
(iv) *$gp\mathcal{I}\ \mathrm{cl}(gp\mathcal{I}\ \mathrm{cl}(A) = gp\mathcal{I}\ \mathrm{cl}(A)$.*
(v) *If $A \subset B$, then $gp\mathcal{I}\ \mathrm{cl}(A) \subset gp\mathcal{I}\ \mathrm{cl}(B)$.*
(vi) *$gp\mathcal{I}\ \mathrm{cl}(A \cup B) \supset gp\mathcal{I}\ \mathrm{cl}(A) \cup gp\mathcal{I}\ \mathrm{cl}(B)$.*
(vii) *$gp\mathcal{I}\ \mathrm{cl}(A \cap B) \subset gp\mathcal{I}\ \mathrm{cl}(A) \cap gp\mathcal{I}\ \mathrm{cl}(B)$.*

Proof. (i) Suppose that $x \notin gp\mathcal{I}\ \mathrm{cl}(A)$. Then there exists $V \in GP\mathcal{I}O(X, x)$ such that $V \cap A = \emptyset$. Since $X \setminus V$ is a g-pre-$\mathcal{I}$-closed set containing A and $x \notin X \setminus V$, we obtain $x \notin \cap\{F\colon A \subset F$ and $F \in P\mathcal{I}C(X, g)\}$. Conversely, suppose that $x \notin \cap\{F \mid A \subset F$ and $F \in PIC(X, g)\}$. Then there exists $F \in P\mathcal{I}C(X, g)$ such that $A \subset F$ and $x \notin F$. Since $X \setminus F$ is a g-pre-$\mathcal{I}$-open set containing x, we obtain $(X \setminus F) \cap A = \emptyset$. This shows that $x \notin gp\mathcal{I}\ \mathrm{cl}(A)$. Therefore, we obtain $gp\mathcal{I}\ \mathrm{cl}(A) = \cap\{F\colon A \subset F$ and $F \in P\mathcal{I}C(X, g)$.

The other proofs are obvious. ∎

Theorem 3.18. *Let $(X, g, \mathcal{I})$ be an IGTS and $A \subset X$. A point $x \in gp\mathcal{I}\ \mathrm{cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in GP\mathcal{I}O(X, x)$.*

Proof. This is obvious by Definition 3.16. ∎

Theorem 3.19. *Let $(X, g, \mathcal{I})$ be an IGTS and $A \subset X$. Then the following propeties hold:*
(i) *$gp\mathcal{I}\ \mathrm{Int}(X \setminus A) = X \setminus gp\mathcal{I}\ \mathrm{cl}(A)$;*
(ii) *$gp\mathcal{I}\ \mathrm{cl}(X \setminus A) = X \setminus gp\mathcal{I}\ \mathrm{Int}(A)$.*

Proof. (i) Let $x \notin gp\mathcal{I}\ \mathrm{cl}(A)$. There exists $V \in GP\mathcal{I}O(X, x)$ such that $V \cap A = \emptyset$; hence $x \in V \subset X \setminus A$ and we obtain $x \in gp\mathcal{I}\ \mathrm{Int}(X \setminus A)$. This shows that $X \setminus gp\mathcal{I}\ \mathrm{cl}(A) \subset gp\mathcal{I}\ \mathrm{Int}(X \setminus A)$. Let $x \in gp\mathcal{I}\ \mathrm{Int}(X \setminus A)$. Then, there exists $U \in GP\mathcal{I}O(X, x)$ such that $x \in U \subset X \setminus A$. Therefore, $x \in U$ and $U \cap A = \emptyset$. We obtain $x \notin gp\mathcal{I}\ \mathrm{cl}(A)$; hence $x \in X \setminus gp\mathcal{I}\ \mathrm{cl}(A)$. Therefore, we obtain $gp\mathcal{I}\ \mathrm{Int}(X \setminus A) = X \setminus gp\mathcal{I}\ \mathrm{cl}(A)$.
(ii) It follows from (i). ∎

Definition 3.20. A subset B_x of an IGTS $(X, g, \mathcal{I})$ is called a g-pre-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists a g-pre-$\mathcal{I}$-open set U such that $x \in U \subset B_x$.

Theorem 3.21. *A subset of an IGTS $(X, g, \mathcal{I})$ is g-pre-$\mathcal{I}$-open if and only if it is a g-pre-$\mathcal{I}$-neighbourhood of each of its points.*

Proof. Let G be a g-pre-$\mathcal{I}$-open set of X. Then by definition, it is clear that G is a g-pre-$\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is g-pre-$\mathcal{I}$-open. Conversely, suppose G is a g-pre-$\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in P\mathcal{I}O(X, g)$ such that $x \in S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each S_x is g-pre-$\mathcal{I}$-open and arbitrary union of g-pre-$\mathcal{I}$-open sets is g-pre-$\mathcal{I}$-open, G is g-pre-$\mathcal{I}$-open in $(X, g, \mathcal{I})$. ∎

4. (g, g')-pre-$\mathcal{I}$-continuous functions

Definition 4.1. A function $f : (X, g, \mathcal{I}) \to (Y, g')$ is said to be (g, g')-pre-$\mathcal{I}$-continuous (resp. (g, g')-precontinuous [9]) if the inverse image of every g'-open set of Y is g-pre-$\mathcal{I}$-open (resp. g-preopen) in $(X, g, \mathcal{I})$.

Proposition 4.2. *For a function $f : (X, g, \mathcal{I}) \to (Y, g')$, we have the following*

(i) *Every (g, g')-continuous function is (g, g')-pre-$\mathcal{I}$-continuous.*
(ii) *Every (g, g')-$\mathcal{I}$-continuous function is (g, g')-pre-$\mathcal{I}$-continuous.*
(iii) *Every (g, g')-pre-$\mathcal{I}$-continuous function is (g, g')-precontinuous.*

Proof. The proof follows from Proposition 3.2. ∎

However, the converses in Proposition 4.2 are not necessarily true as we illustrate by means of the examples below.

Example 4.3. *Let*

$$X = \{a, b, c\}, g = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}, \quad g' = \{\emptyset, \{b\}\} \quad \text{and} \quad \mathcal{I} = \{\emptyset, \{a\}\}.$$

Then the identity function $f : (X, g, \mathcal{I}) \to (Y, g')$ is (g, g')-pre-$\mathcal{I}$-continuous but it is not (g, g')-continuous.

Example 4.4. *Let*

$$X = \{a, b, c\}, \quad g = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}, \quad g' = \{\emptyset, \{a, c\}\} \text{ and } \mathcal{I} = \{\emptyset, \{a\}\}.$$

Then the identity function $f : (X, g, \mathcal{I}) \to (Y, g')$ is (g, g')-precontinuous but it is not (g, g')-pre-$\mathcal{I}$-continuous.

Example 4.5. *Let $X = \{a, b, c\}$, $g = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}$, $g' = \{\emptyset, \{a\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the identity function $f : (X, g, \mathcal{I}) \to (Y, g')$ is (g, g')-pre-$\mathcal{I}$-continuous but it is not (g, g')-$\mathcal{I}$-continuous.*

THEOREM 4.6. *For a function $f\colon (X, g, \mathcal{I}) \to (Y, g')$, the following statements are equivalent:*

(i) f is (g, g')-pre-$\mathcal{I}$-continuous;
(ii) For each point x in X and each g'-open set F of Y such that $f(x) \in F$, there is a g-pre-$\mathcal{I}$-open set A in X such that $x \in A$, $f(A) \subset F$;
(iii) The inverse image of each g'-closed set in Y is g-pre-$\mathcal{I}$-closed in X;
(iv) For each subset A of X, $f(gp\mathcal{I}\operatorname{cl}(A)) \subset g'\operatorname{cl}(f(A))$;
(v) For each subset B of Y, $gp\mathcal{I}\operatorname{cl}(f^{-1}(B)) \subset f^{-1}(g'\operatorname{cl}(B))$;
(vi) For each subset C of Y, $f^{-1}(g'\operatorname{Int}(C)) \subset gp\mathcal{I}\operatorname{Int}(f^{-1}(C))$.

PROOF. (i) $\Rightarrow$ (ii): Let $x \in X$ and F be a g'-open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is g-pre-$\mathcal{I}$-open in X. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$.

(ii) $\Rightarrow$ (i): Let F be g'-open in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (ii), there is a g-pre-$\mathcal{I}$-open set U_x in X such that $x \in U_x$ and $f(U_x) \subset F$. Then $x \in U_x \subset f^{-1}(F)$. Hence $f^{-1}(F)$ is g-pre-$\mathcal{I}$-open in X.

(i)$\Leftrightarrow$(iii): This follows due to the fact that for any subset B of Y, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.

(iii) $\Rightarrow$ (iv): Let A be a subset of X. Since $A \subset f^{-1}(f(A))$, we have $A \subset f^{-1}(g'\operatorname{cl}(f(A)))$. Now, $g'\operatorname{cl}(f(A))$ is g'-closed in Y and hence $gp\mathcal{I}\operatorname{cl}(A) \subset f^{-1}(g'\operatorname{cl}(f(A)))$, for $gp\mathcal{I}\operatorname{cl}(A)$ is the smallest g-pre-$\mathcal{I}$-closed set containing A. Then $f(gp\mathcal{I}\operatorname{cl}(A)) \subset g'\operatorname{cl}(f(A))$.

(iv) $\Rightarrow$ (v): Let B be any subset of Y. Now,

$$f(gp\mathcal{I}\operatorname{cl}(f^{-1}(B))) \subset g'\operatorname{cl}(f(f^{-1}(B))) \subset g'\operatorname{cl}(B).$$

Consequently, $gp\mathcal{I}\operatorname{cl}(f^{-1}(B)) \subset f^{-1}(g'\operatorname{cl}(B))$.

(v) $\Rightarrow$ (vi): Let C be any subset of Y. Then by (v)

$$gp\mathcal{I}\operatorname{cl}(f^{-1}(Y \setminus C) \subset g^{-1}(g'\operatorname{cl}(Y \setminus C)).$$

Therefore,

$$\begin{aligned} X \setminus gp\mathcal{I}\operatorname{Int}(f^{-1}(C)) &= pg\mathcal{I}\operatorname{cl}(X \setminus f^{-1}(C)) = \\ &= gp\mathcal{I}\operatorname{cl}(f^{-1}(Y \setminus C)) \subset f^{-1}(g'\operatorname{cl}(Y \setminus C)) \\ &= f^{-1}(Y \setminus g'\operatorname{Int}(C)) = \\ &= X \setminus f^{-1}(g'\operatorname{Int}(C)). \end{aligned}$$

This shows that $f^{-1}(g'\operatorname{Int}(C)) \subset gp\mathcal{I}\operatorname{Int}(f^{-1}(C))$.

(vi) $\Rightarrow$ (i): Let B be a g'-open set in Y. Then $g'\operatorname{Int}(B) = B$ and

$$f^{-1}(B) \subset f^{-1}(g'\operatorname{Int}(B)) \subset gp\mathcal{I}\operatorname{Int}(f^{-1}(B)).$$

Hence we have $f^{-1}(B) = gp\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is g-pre-$\mathcal{I}$-open in X. ∎

DEFINITION 4.7. The graph $G(f)$ of a function $f\colon (X, g, \mathcal{I}) \to (Y, g')$ is said to be (g, g')-pre-$\mathcal{I}$-closed in $X \times Y$ if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in GP\mathcal{I}O(X, x)$ and a g'-open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

LEMMA 4.8. *The graph of a function $f\colon (X, g, \mathcal{I}) \to (Y, g')$ is (g, g')-pre-$\mathcal{I}$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in GP\mathcal{I}O(X, x)$ and a g'-open set V of Y containing y such that $f(U) \cap V = \emptyset$.*

PROOF. The proof is an immediate consequence of Definition 4.7. ∎

THEOREM 4.9. *If $f\colon (X, g, \mathcal{I}) \to (Y, g')$ is a (g, g')-pre-$\mathcal{I}$-continuous function and (Y, g') is g'-T_2, then $G(f)$ is (g, g')-pre-$\mathcal{I}$-closed.*

PROOF. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since Y is g'-T_2, there exist g'-open sets V and W of Y such that $f(x) \in W$, $y \in V$ and $V \cap W = \emptyset$. Since f is (g, g')-pre-$\mathcal{I}$-continuous, there exists $U \in GP\mathcal{I}O(X, x)$ such that $f(U) \subset W$. Therefore, $f(U) \cap V = \emptyset$. Hence, by Lemma 4.8, $G(f)$ is (g, g')-pre-$\mathcal{I}$-closed. ∎

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