ON $\mathcal{I}$-OPEN SETS AND $\mathcal{I}$-CONTINUOUS FUNCTIONS IN IDEAL BITOPOLOGICAL SPACES

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Abstract. The aim of this paper is to introduce and characterize the concepts of $\mathcal{I}$-open sets and their related notions in ideal bitopological spaces.

1. Introduction and Preliminaries

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [19] and Vaidyanathasamy [24]. Hamlett and Janković (see [12], [13], [17] and [18]) used topological ideals to generalize many notions and properties in general topology. The research in this direction continued by many researchers such as M. E. Abd El-Monsef, A. Al-Omari, F. G. Arenas, M. Caldas, J. Dontchev, M. Ganster, D. N. Georgiou, T. R. Hamlett, E. Hatir, S. D. Iliadis, S. Jafari, D. Jankovic, E. F. Lashien, M. Maheswari, , H. Maki, A. C. Megaritis, F. I. Michael, A. A. Nasef, T. Noiri, B. K. Papadopoulos, M. Parimala, G. A. Prinos, M. L. Puertas, M. Rajamani, N. Rajesh, D. Rose, A. Selvakumar, Jun-Iti Umehara and many others (see [1], [2], [5], [7], [8], [9], [10], [11], [14], [15], [18], [23], [21], [22]). An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*$: $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$, called the local function [24] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau_1, \tau_2, \mathcal{I})$ is called an ideal bitopological space. Let A be a subset of a bitopological space (X, τ_1, τ_2). We denote the closure of A and the interior of A with respect to τ_i by τ_i-Cl(A) and τ_i-Int(A), respectively. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-preopen [16] if $A \subset \tau_i$-Int$(\tau_j$-Cl$(A))$, where $i, j = 1, 2$ and $i \neq j$. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be (i, j)-pre-$\mathcal{I}$-open [4] if $S \subset \tau_i$-Int$(\tau_j$-Cl$^*(S))$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-preopen [16] (resp. (i, j)-semi-$\mathcal{I}$-open [3]) if $A \subset \tau_i$-Int$(\tau_j$-Cl$(A))$ (resp. $S \subset \tau_j$-Cl$^*(\tau_i$-Int$(S)))$, where $i, j = 1, 2$

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and $i \neq j$. The complement of an (i,j)-semi-$\mathcal{I}$-open set is called an (i,j)-semi-$\mathcal{I}$-closed set. A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-pre-$\mathcal{I}$-continuous [4] if the inverse image of every σ_i-open set in (Y, σ_1, σ_2) is (i,j)-pre-$\mathcal{I}$-open in $(X, \tau_1, \tau_2, \mathcal{I})$, where $i \neq j$, i, j=1, 2.

2. (i,j)-$\mathcal{I}$-open sets

Definition 2.1. *A subset A of an ideal bitopological space $(X, \tau_i, \tau_2, \mathcal{I})$ is said to be (i,j)-$\mathcal{I}$-open if $A \subset \tau_i$-Int(A^*_j).*
The family of all (i,j)-$\mathcal{I}$-open subsets of $(X, \tau_i, \tau_2, \mathcal{I})$ is denoted by (i,j)-$\mathcal{I}O(X)$.

Remark 2.2. *It is clear that $(1,2)$-$\mathcal{I}$-openness and τ_1-openness are independent notions.*

Example 2.3. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a,b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then τ_1-Int$(\{a,b\}^*_2) = \tau_1$-Int$(\{b\}) = \emptyset \supsetneq \{a,b\}$. Therefore $\{a,b\}$ is a τ_1-open set but not $(1,2)$-$\mathcal{I}$-open.*

Example 2.4. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a,b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a,b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then τ_1-Int$(\{a\}^*_2) = \tau_1$-Int$(X) = X \supset \{a\}$. Therefore, $\{a\}$ is $(1,2)$-$\mathcal{I}$-open set but not τ_1-open.*

Remark 2.5. *Similarly $(1,2)$-$\mathcal{I}$-openness and τ_2-openness are independent notions.*

Example 2.6. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then τ_1-Int$(\{b,c\}^*_2) = \tau_1$-Int$(\{a,b\}) = \{a\} \supsetneq \{b,c\}$. Therefore, $\{b,c\}$ is a τ_2-open set but not $(1,2)$-$\mathcal{I}$-open.*

Example 2.7. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}\}$, $\tau_2 = \{\emptyset, \{b\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then τ_1-Int$(\{a\}^*_2) = \tau_1$-Int$(\{a\}) = \{a\} \supset \{a\}$. Therefore, $\{a\}$ is an $(1,2)$-$\mathcal{I}$-open set but not τ_2-open.*

Proposition 2.8. *Every (i,j)-$\mathcal{I}$-open set is (i,j)-pre-$\mathcal{I}$-open.*

Proof. Let A be an (i,j)-$\mathcal{I}$-open set. Then $A \subset \tau_i$-Int$(A^*_j) \subset \tau_i$-Int$(A \cup A^*_j) = \tau_i$-Int$(\tau_j$-Cl$^*(A))$. Therefore, $A \in (i,j)$-$P\mathcal{I}O(X)$. □

Example 2.9. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the set $\{c\}$ is $(1,2)$-preopen but not $(1,2)$-$\mathcal{I}$-open.*

Remark 2.10. *The intersection of two (i,j)-$\mathcal{I}$-open sets need not be (i,j)-$\mathcal{I}$-open as shown in the following example.*

Example 2.11. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\{a,b\}$, $\{a,c\} \in (1,2)$-$\mathcal{I}O(X)$ but $\{a,b\} \cap \{a,c\} = \{a\} \notin (1,2)$-$\mathcal{I}O(X)$.*

Theorem 2.12. *For an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ and $A \subset X$, we have:*

(1) If $\mathcal{I} = \{\emptyset\}$, then $A_j^*(\mathcal{I}) = \tau_j$-Cl$(A)$ and hence each of (i,j)-$\mathcal{I}$-open set and (i,j)-preopen set are coincide.
(2) If $\mathcal{I} = \mathcal{P}(X)$, then $A_j^*(\mathcal{I}) = \emptyset$ and hence A is (i,j)-$\mathcal{I}$-open if and only if $A = \emptyset$.

Theorem 2.13. *For any (i,j)-$\mathcal{I}$-open set A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, we have $A_j^* = (\tau_i\text{-Int}(A_j^*))_j^*$.*

Proof. Since A is (i,j)-$\mathcal{I}$-open, $A \subset \tau_i$-Int(A_j^*). Then $A_j^* \subset (\tau_i\text{-Int}(A_j^*))_j^*$. Also we have τ_i-Int$(A_j^*) \subset A_j^*$, $(\tau_i\text{-Int}(A_j^*))^* \subset (A_j^*)^* \subset A_j^*$. Hence we have, $A_j^* = (\tau_i\text{-Int}(A_j^*))_j^*$. □

Definition 2.14. *A subset F of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is called (i,j)-$\mathcal{I}$-closed if its complement is (i,j)-$\mathcal{I}$-open.*

Theorem 2.15. *For $A \subset (X, \tau_1, \tau_2, \mathcal{I})$ we have $((\tau_i\text{-Int}(A))_j^*)^c \neq \tau_i$-Int$((A^c)_j^*)$ in general.*

Example 2.16. *Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then $((\tau_1\text{-Int}(\{a, b\}))_2^*)^c = (\{a, b\}_2^*)^c = X^c = \emptyset$ $(*)$ and τ_1-Int$((\{a, b\}^c)_2^*) = \tau_1$-Int$(\{c\}_2^*) = \tau_1$-Int$(X) = X$ $(**)$. Hence from $(*)$ and $(**)$, we get $((\tau_1\text{-Int}(\{a, b\}))_2^*)^c \neq \tau_1$-Int$((\{a, b\}^c)_2^*)$.*

Theorem 2.17. *If $A \subset (X, \tau_1, \tau_2, \mathcal{I})$ is (i,j)-$\mathcal{I}$-closed, then $A \supset (\tau_i$-Int$(A))_j^*$.*

Proof. Let A be (i,j)-$\mathcal{I}$-closed. Then $B = A^c$ is (i,j)-$\mathcal{I}$-open. Thus, $B \subset \tau_i$-Int(B_j^*), $B \subset \tau_i$-Int$(\tau_j$-Cl$(B))$, $B^c \supset \tau_j$-Cl$(\tau_i$-Int$(B^c))$, $A \supset \tau_j$-Cl$(\tau_i$-Int$(A))$. That is, τ_j-Cl$(\tau_i$-Int$(A)) \subset A$, which implies that $(\tau_i$-Int$(A))_j^* \subset \tau_j$-Cl$(\tau_i$-Int$(A)) \subset A$. Therefore, $A \supset (\tau_i$-Int$(A))_j^*$. □

Theorem 2.18. *Let $A \subset (X, \tau_1, \tau_2, \mathcal{I})$ and $(X \backslash (\tau_i\text{-Int}(A))_j^*) = \tau_i$-Int$((X \backslash A)_j^*)$. Then A is (i,j)-$\mathcal{I}$-closed if and only if $A \supset (\tau_i$-Int$(A))_j^*$.*

Proof. It is obvious. □

Theorem 2.19. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A, B \subset X$. Then:*

(i) *If $\{U_\alpha : \alpha \in \Delta\} \subset (i,j)$-$\mathcal{I}O(X)$, then $\bigcup\{U_\alpha : \alpha \in \Delta\} \in (i,j)$-$\mathcal{I}O(X)$.*
(ii) *If $A \in (i,j)$-$\mathcal{I}O(X)$, $B \in \tau_i$ and $A_j^* \cap B \subset (A \cap B)_j^*$, then $A \cap B \in (i,j)$-$\mathcal{I}O(X)$.*
(iii) *If $A \in (i,j)$-$\mathcal{I}O(X)$, $B \in \tau_i$ and $B \cap A_j^* = B \cap (B \cap A)_j^*$, then $A \cap B \subset \tau_i$-Int$(B \cap (B \cap A)_j^*)$.*

Proof. (i) Since $\{U_\alpha : \alpha \in \Delta\} \subset (i,j)$-$\mathcal{I}O(X)$, then $U_\alpha \subset \tau_i$-Int$((U_\alpha)_j^*)$, for every $\alpha \in \Delta$. Thus, $\bigcup(U_\alpha) \subset \bigcup(\tau_i\text{-Int}((U_\alpha)_j^*)) \subset \tau_i\text{-Int}(\bigcup(U_\alpha)_j^* \subset \tau_i\text{-Int}(\bigcup U_\alpha)_j^*$, for every $\alpha \in \Delta$. Hence $\bigcup\{U_\alpha : \alpha \in \Delta\} \in (i,j)$-$\mathcal{I}O(X)$.

(*ii*) Given $A \in (i,j)$-$\mathcal{IO}(X)$ and $B \in \tau_i$, that is $A \subset \tau_i$-Int(A_j^*). Then $A \cap B \subset \tau_i$-Int$(A_j^*) \cap B = \tau_i$-Int$(A_j^* \cap B)$. Since $B \in \tau_i$ and $A_j^* \cap B \subset (A \cap B)_j^*$, we have $A \cap B \subset \tau_i$-Int$((A \cap B)_j^*)$. Hence, $A \cap B \in (i,j)$-$\mathcal{IO}(X)$.
(*iii*) Given $A \in (i,j)$-$\mathcal{IO}(X)$ and $B \in \tau_i$, That is $A \subset \tau_i$-Int(A_j^*). We have to prove $A \cap B \subset \tau_i$-Int$(B \cap (B \cap A)_j^*)$. Thus, $A \cap B \subset \tau_i$-Int$(A_j^*) \cap B = \tau_i$-Int$(A_j^* \cap B) = \tau_i$-Int$(B \cap A_j^*)$. Since $B \cap A_j^* = B \cap (B \cap A)_j^*$. Hence $A \cap B \subset \tau_i$-Int$(B \cap (B \cap A)_j^*)$. □

Corollary 2.20. *The union of* (i,j)-$\mathcal{I}$-*closed set and* τ_j-*closed set is* (i,j)-$\mathcal{I}$-*closed.*

Proof. It is obvious. □

Theorem 2.21. *If* $A \subset (X, \tau_1, \tau_2, \mathcal{I})$ *is* (i,j)-$\mathcal{I}$-*open and* (i,j)-*semiclosed, then* $A = \tau_i$-Int(A_j^*).

Proof. Given A is (i,j)-$\mathcal{I}$-open. Then $A \subset \tau_i$-Int(A_j^*). Since (i,j)-semiclosed, τ_i-Int$(A_j^*) \subset \tau_i$-Int$(\tau_j$-Cl$(A)) \subset A$. Thus τ_i-Int$(A_j^*) \subset A$. Hence we have, $A = \tau_i$-Int(A_j^*). □

Theorem 2.22. *Let* $A \in (i,j)$-$\mathcal{IO}(X)$ *and* $B \in (i,j)$-$\mathcal{IO}(Y)$, *then* $A \times B \in (i,j)$-$\mathcal{IO}(X \times Y)$, *if* $A_j^* \times B_j^* = (A \times B)_j^*$.

Proof. $A \times B \subset \tau_i$-Int$(A_j^*) \times \tau_i$-Int$(B_j^*) = \tau_i$-Int$(A_j^* \times B_j^*)$, from hypothesis. Then $A \times B = \tau_i$-Int$((A \times B)_j^*)$; hence, $A \times B \in (i,j)$-$\mathcal{IO}(X \times Y)$. □

Theorem 2.23. *If* $(X, \tau_1, \tau_2, \mathcal{I})$ *is an ideal bitopological space,* $A \in \tau_i$ *and* $B \in (i,j)$-$\mathcal{IO}(X)$, *then there exists a* τ_i-*open subset* G *of* X *such that* $A \cap G = \emptyset$, *implies* $A \cap B = \emptyset$.

Proof. Since $B \in (i,j)$-$\mathcal{IO}(X)$, then $B \subset \tau_i$-Int(B_j^*). By taking $G = \tau_i$-Int(B_j^*) to be a τ_i-open set such that $B \subset G$. But $A \cap G = \emptyset$, then $G \subset X \backslash A$ implies that τ_i-Cl$(G) \subset X \backslash A$. Hence $B \subset (X \backslash A)$. Therefore, $A \cap B = \emptyset$. □

Definition 2.24. *A subset* A *of* $(X, \tau_1, \tau_2, \mathcal{I})$ *is said to be:*
(i) τ_i^*-*closed if* $A_i^* \subset A$.
(ii) τ_i-$*$-*perfect* $A_i^* = A$.

Theorem 2.25. *For a subset* $A \subset (X, \tau_1, \tau_2, \mathcal{I})$, *we have*

(i) If A is τ_j^*-closed and $A \in (i,j)$-$\mathcal{IO}(X)$, then τ_i-Int$(A) = \tau_i$-Int(A_j^*).
(ii) If A is τ_j-$*$-perfect, then $A = \tau_i$-Int(A_j^*) for every $A \in (i,j)$-$\mathcal{IO}(X)$.

Proof. (*i*) Let A be τ_j-$*$-closed and $A \in (i,j)$-$\mathcal{IO}(X)$. Then $A_j^* \subset A$ and $A \subset \tau_i$-Int(A_j^*). Hence $A \subset \tau_i$-Int$(A_j^*) \Rightarrow \tau_i$-Int$(A) \subset \tau_i$-Int$(\tau_i$-Int$(A_j^*)) \Rightarrow \tau_i$-Int$(A) \subset \tau_i$-Int$(A_j^*)$. Also, $A_j^* \subset A$. Then τ_i-Int$(A_j^*) \subset$

τ_i-Int(A). Hence τ_i-Int$(A) = \tau_i$-Int(A_j^*).
(ii) Let A be τ_j-$*$-perfect and $A \in (i,j)$-$\mathcal{I}O(X)$. We have, $A_j^* = A$, τ_i-Int$(A_j^*) = \tau_i$-Int(A), τ_i-Int$(A_j^*) \subset A$. Also we have $A \subset \tau_i$-Int(A_j^*). Hence we have, $A = \tau_i$-Int(A_j^*). □

Definition 2.26. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then*

(i) *x is called an (i,j)-$\mathcal{I}$-interior point of S if there exists $V \in (i,j)$-$\mathcal{I}O(X, \tau_1, \tau_2)$ such that $x \in V \subset S$.*
ii) *the set of all (i,j)-$\mathcal{I}$-interior points of S is called (i,j)-$\mathcal{I}$-interior of S and is denoted by (i,j)-$\mathcal{I}\,\mathrm{Int}(S)$.*

Theorem 2.27. *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:*

(i) (i,j)-$\mathcal{I}\,\mathrm{Int}(A) = \cup\{T : T \subset A$ *and* $A \in (i,j)$-$\mathcal{I}O(X)\}$.
(ii) (i,j)-$\mathcal{I}\,\mathrm{Int}(A)$ *is the largest (i,j)-$\mathcal{I}$-open subset of X contained in A.*
(iii) *A is (i,j)-$\mathcal{I}$-open if and only if $A = (i,j)$-$\mathcal{I}\,\mathrm{Int}(A)$.*
(iv) (i,j)-$\mathcal{I}\,\mathrm{Int}((i,j)$-$\mathcal{I}\,\mathrm{Int}(A)) = (i,j)$-$\mathcal{I}\,\mathrm{Int}(A)$.
(v) *If $A \subset B$, then (i,j)-$\mathcal{I}\,\mathrm{Int}(A) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(B)$.*
(vi) (i,j)-$\mathcal{I}\,\mathrm{Int}(A) \cup (i,j)$-$\mathcal{I}\,\mathrm{Int}(B) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(A \cup B)$.
(vii) (i,j)-$\mathcal{I}\,\mathrm{Int}(A \cap B) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(A) \cap (i,j)$-$\mathcal{I}\,\mathrm{Int}(B)$.

Proof. (i). Let $x \in \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$. Then, there exists $T \in (i,j)$-$\mathcal{I}O(X, x)$ such that $x \in T \subset A$ and hence $x \in (i,j)$-$\mathcal{I}\,\mathrm{Int}(A)$. This shows that $\cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\} \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(A)$. For the reverse inclusion, let $x \in (i,j)$-$\mathcal{I}\,\mathrm{Int}(A)$. Then there exists $T \in (i,j)$-$\mathcal{I}O(X, x)$ such that $x \in T \subset A$. we obtain $x \in \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$. This shows that (i,j)-$\mathcal{I}\,\mathrm{Int}(A) \subset \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$. Therefore, we obtain (i,j)-$\mathcal{I}\,\mathrm{Int}(A) = \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$.
The proof of (ii)-(v) are obvious.
(vi). Clearly, (i,j)-$\mathcal{I}\,\mathrm{Int}(A) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(A \cup B)$ and (i,j)-$\mathcal{I}\,\mathrm{Int}(B) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(A \cup B)$. Then by (v) we obtain (i,j)-$\mathcal{I}\,\mathrm{Int}(A) \cup (i,j)$-$\mathcal{I}\,\mathrm{Int}(B) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(A \cup B)$.
(vii). Since $A \cap B \subset A$ and $A \cap B \subset B$, by (v), we have (i,j)-$\mathcal{I}\,\mathrm{Int}(A \cap B) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(A)$ and (i,j)-$\mathcal{I}\,\mathrm{Int}(A \cap B) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(B)$. By (v) (i,j)-$\mathcal{I}\,\mathrm{Int}(A \cap B) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(A) \cap (i,j)$-$\mathcal{I}\,\mathrm{Int}(B)$. □

Definition 2.28. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then*

(i) *x is called an (i,j)-$\mathcal{I}$-cluster point of S if $V \cap S \neq \emptyset$ for every $V \in (i,j)$-$\mathcal{I}O(X, x)$.*
(ii) *the set of all (i,j)-$\mathcal{I}$-cluster points of S is called (i,j)-$\mathcal{I}$-closure of S and is denoted by (i,j)-$\mathcal{I}\,\mathrm{Cl}(S)$.*

Theorem 2.29. *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:*

(i) (i,j)-$\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F$ *and* $F \in (i,j)$-$\mathcal{I}C(X)\}$.
(ii) (i,j)-$\mathcal{I}\,\mathrm{Cl}(A)$ *is the smallest* (i,j)-$\mathcal{I}$*-closed subset of* X *containing* A.
(iii) A *is* (i,j)-$\mathcal{I}$*-closed if and only if* $A = (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$.
(iv) (i,j)-$\mathcal{I}\,\mathrm{Cl}((i,j)$-$\mathcal{I}\,\mathrm{Cl}(A) = (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$.
(v) *If* $A \subset B$*, then* (i,j)-$\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)$-$\mathcal{I}\,\mathrm{Cl}(B)$.
(vi) (i,j)-$\mathcal{I}\,\mathrm{Cl}(A \cup B) = (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A) \cup (i,j)$-$\mathcal{I}\,\mathrm{Cl}(B)$.
(vii) (i,j)-$\mathcal{I}\,\mathrm{Cl}(A \cap B) \subset (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A) \cap (i,j)$-$\mathcal{I}\,\mathrm{Cl}(B)$.

Proof. (i). Suppose that $x \notin (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$. Then there exists $F \in (i,j)$-$\mathcal{I}O(X)$ such that $V \cap S \neq \emptyset$. Since $X\backslash V$ is (i,j)-$\mathcal{I}$-closed set containing A and $x \notin X\backslash V$, we obtain $x \notin \cap\{F : A \subset F$ and $F \in (i,j)$-$\mathcal{I}C(X)\}$. Then there exists $F \in (i,j)$-$\mathcal{I}C(X)$ such that $A \subset F$ and $x \notin F$. Since $X\backslash V$ is (i,j)-$\mathcal{I}$-closed set containing x, we obtain $(X\backslash F) \cap A = \emptyset$. This shows that $x \notin (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain (i,j)-$\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F$ and $F \in (i,j)$-$\mathcal{I}C(X)$.
The other proofs are obvious. □

Theorem 2.30. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be an ideal bitopological space and* $A \subset X$*. A point* $x \in (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in (i,j)$-$\mathcal{I}O(X,x)$.

Proof. Suppose that $x \in (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$. We shall show that $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$\mathcal{I}O(X,x)$. Suppose that there exists $U \in (i,j)$-$\mathcal{I}O(X,x)$ such that $U \cap A = \emptyset$. Then $A \subset X\backslash U$ and $X\backslash U$ is (i,j)-$\mathcal{I}$-closed. Since $A \subset X\backslash U$, (i,j)-$\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)$-$\mathcal{I}\,\mathrm{Cl}(X\backslash U)$. Since $x \in (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$, we have $x \in (i,j)$-$\mathcal{I}\,\mathrm{Cl}(X\backslash U)$. Since $X\backslash U$ is (i,j)-$\mathcal{I}$-closed, we have $x \in X\backslash U$; hence $x \notin U$, which is a contradicition that $x \in U$. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$\mathcal{I}O(X,x)$. We shall show that $x \in (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$. Suppose that $x \notin (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$. Then there exists $U \in (i,j)$-$\mathcal{I}O(X,x)$ such that $U \cap A = \emptyset$. This is a contradicition to $U \cap A \neq \emptyset$; hence $x \in (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$. □

Theorem 2.31. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be an ideal bitopological space and* $A \subset X$*. Then the following propeties hold:*

(i) (i,j)-$\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$;
(i) (i,j)-$\mathcal{I}\,\mathrm{Cl}(X\backslash A) = X\backslash(i,j)$-$\mathcal{I}\,\mathrm{Int}(A)$.

Proof. (i). Let $x \in (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$. There exists $V \in (i,j)$-$\mathcal{I}O(X,x)$ such that $V \cap A \neq \emptyset$; hence we obtain $x \in (i,j)$-$\mathcal{I}\,\mathrm{Int}(X\backslash A)$. This shows that $X\backslash(i,j)$-$\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)$-$\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Let $x \in (i,j)$-$\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Since (i,j)-$\mathcal{I}\,\mathrm{Int}(X\backslash A) \cap A = \emptyset$, we obtain $x \notin (i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$; hence $x \in X\backslash(i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain (i,j)-$\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)$-$\mathcal{I}\,\mathrm{Cl}(A)$.
(ii). Follows from (i). □

Definition 2.32. *A subset B_x of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be an (i,j)-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists an (i,j)-$\mathcal{I}$-open set U such that $x \in U \subset B_x$.*

Theorem 2.33. *A subset of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is (i,j)-$\mathcal{I}$-open if and only if it is an (i,j)-$\mathcal{I}$-neighbourhood of each of its points.*

Proof. Let G be an (i,j)-$\mathcal{I}$-open set of X. Then by definition, it is clear that G is an (i,j)-$\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is (i,j)-$\mathcal{I}$-open. Conversely, suppose G is an (i,j)-$\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in (i,j)$-$\mathcal{I}O(X)$ such that $S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each S_x is (i,j)-$\mathcal{I}$-open and arbtrary union of (i,j)-$\mathcal{I}$-open sets is (i,j)-$\mathcal{I}$-open, G is (i,j)-$\mathcal{I}$-open in $(X, \tau_1, \tau_2, \mathcal{I})$. □

3. (i,j)-$\mathcal{I}$-CONTINUOUS FUNCTIONS

Definition 3.1. *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-$\mathcal{I}$-continuous if for every $V \in \sigma_i$, $f^{-1}(V) \in (i,j)$-$\mathcal{I}O(X)$.*

Remark 3.2. *Every (i,j)-$\mathcal{I}$-continuous function is (i,j)-precontinuous but the converse is not true,in general.*

Example 3.3. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b,c\}, X\}$, $\sigma_1 = \mathcal{P}(X)$, $\sigma_2 = \{\emptyset, \{a\}, \{a,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is $(1,2)$-precontinuous but not $(1,2)$-$\mathcal{I}$-continuous, because $\{c\} \in \sigma_1$, but $f^{-1}(\{c\}) = \{c\} \notin (1,2)$-$\mathcal{I}O(X)$.*

Remark 3.4. *It is clear that $(1,2)$-$\mathcal{I}$-continuity and τ_1-continuity (resp. τ_2-continuity) are independent notions.*

Example 3.5. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{b\}, X\}$, $\tau_2 = \{\emptyset, \{a,b\}, X\}$, $\sigma_1 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$, $\sigma_2 = \{\emptyset, \{a\}, \{a,b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is τ_1-continuous but not $(1,2)$-$\mathcal{I}$-continuous, because $\{b\} \in \sigma_1$, but $f^{-1}(\{b\}) = \{b\} \notin (1,2)$-$\mathcal{I}O(X)$.*

Example 3.6. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a,b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a,b\}, X\}$, $\sigma_1 = \{\emptyset, \{b\}, \{b,c\}, X\}$, $\sigma_2 = \{\emptyset, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is $(1,2)$-$\mathcal{I}$-continuous but not τ_1-continuous, because $f^{-1}(\{a\}) = \{a\} \in (1,2)$-$\mathcal{I}O(X)$, but $\{a\} \notin \sigma_1$.*

Example 3.7. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$, $\sigma_1 = \{\emptyset, \{b,c\}, X\}$, $\sigma_2 = \{\emptyset, \{b\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is τ_2-continuous but not $(1,2)$-$\mathcal{I}$-continuous, because $\{b\} \in \sigma_2$ but $f^{-1}(\{b\}) = \{b\} \notin (1,2)$-$\mathcal{I}O(X)$.*

Example 3.8. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{b, c\}, X\}$, $\sigma_1 = \{\emptyset, \{a, c\}, X\}$, $\sigma_2 = \{\emptyset, \{b, c\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{c\}\}$. *Then the identity function* $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ *is* $(1, 2)$-$\mathcal{I}$-*continuous but not* τ_2-*continuous, because* $\{a\} \notin \sigma_2$ *but* $f^{-1}(\{a\}) = \{a\} \in (1, 2)$-$\mathcal{I}O(X)$.

Theorem 3.9. *For a function* $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$, *the following statements are equivalent:*

(i) f *is pairwise* $\mathcal{I}$-*continuous;*
(ii) *For each point* x *in* X *and each* σ_j-*open set* F *in* Y *such that* $f(x) \in F$, *there is a* (i, j)-$\mathcal{I}$-*open set* A *in* X *such that* $x \in A$, $f(A) \subset F$;
(iii) *The inverse image of each* σ_j-*closed set in* Y *is* (i, j)-$\mathcal{I}$-*closed in* X;
(iv) *For each subset* A *of* X, $f((i, j)$-$\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_j$-$\mathrm{Cl}(f(A))$;
(v) *For each subset* B *of* Y, (i, j)-$\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_j$-$\mathrm{Cl}(B))$;
(vi) *For each subset* C *of* Y, $f^{-1}(\sigma_j$-$\mathrm{Int}(C)) \subset (i, j)$-$\mathcal{I}\,\mathrm{Int}(f^{-1}(C))$.

Proof. (i)⇒(ii): Let $x \in X$ and F be a σ_j-open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is (i, j)-$\mathcal{I}$-open in X. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$.
(ii)⇒(i): Let F be σ_j-open in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (ii), there is an (i, j)-$\mathcal{I}$-open set U_x in X such that $x \in U_x$ and $f(U_x) \subset F$. Then $x \in U_x \subset f^{-1}(F)$. Hence $f^{-1}(F)$ is (i, j)-$\mathcal{I}$-open in X.
(i)⇔(iii): This follows due to the fact that for any subset B of Y, $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$.
(iii)⇒(iv): Let A be a subset of X. Since $A \subset f^{-1}(f(A))$ we have $A \subset f^{-1}(\sigma_j$-$\mathrm{Cl}(f(A)))$. Now, (i, j)-$\mathcal{I}\,\mathrm{Cl}(f(A))$ is σ_j-closed in Y and hence $f^{-1}(\sigma_j$-$\mathrm{Cl}(A)) \subset f^{-1}(\sigma_j$-$\mathrm{Cl}(f(A)))$, for (i, j)-$\mathcal{I}\,\mathrm{Cl}(A)$ is the smallest (i, j)-$\mathcal{I}$-closed set containing A. Then $f((i, j)$-$\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_j$-$\mathrm{Cl}(f(A))$.
(iv)⇒(iii): Let F be any (i, j)-pre-$\mathcal{I}$-closed subset of Y. Then $f((i, j)$-$\mathcal{I}\,\mathrm{Cl}(f^{-1}(F))) \subset (i, j)$-$\sigma_i$-$\mathrm{Cl}(f(f^{-1}(F))) = (i, j)$-$\sigma_i$-$\mathrm{Cl}(F) = F$. Therefore, (i, j)-$\mathcal{I}\,\mathrm{Cl}(f^{-1}(F)) \subset f^{-1}(F)$. Consequently, $f^{-1}(F)$ is (i, j)-$\mathcal{I}$-closed in X.
(iv)⇒(v): Let B be any subset of Y. Now, $f((i, j)$-$\mathcal{I}\,\mathrm{Cl}(f^{-1}(B))) \subset \sigma_i$-$\mathrm{Cl}(f(f^{-1}(B))) \subset \sigma_i$-$\mathrm{Cl}(B)$. Consequently, (i, j)-$\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i$-$\mathrm{Cl}(B))$.
(v)⇒(iv): Let $B = f(A)$ where A is a subset of X. Then, (i, j)-$\mathcal{I}\,\mathrm{Cl}(A) \subset (i, j)$-$\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i$-$\mathrm{Cl}(B)) = f^{-1}(\sigma_i$-$\mathrm{Cl}(f(A)))$. This shows that $f((i, j)$-$\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_i$-$\mathrm{Cl}(f(A))$.
(i)⇒(vi): Let B be a σ_j-open set in Y. Clearly, $f^{-1}(\sigma_i$-$\mathrm{Int}(B)$ is (i, j)-$\mathcal{I}$-open and we have $f^{-1}(\sigma_i$-$\mathrm{Int}(B)) \subset (i, j)$-$\mathcal{I}\,\mathrm{Int}(f^{-1}\sigma_i$-$\mathrm{Int}(B)) \subset (i, j)$-$\mathcal{I}\,\mathrm{Int}(f^{-1}B)$.
(vi)⇒(i): Let B be a σ_j-open set in Y. Then σ_i-$\mathrm{Int}(B) = B$ and $f^{-1}(B) \backslash f^{-1}(\sigma_i$-$\mathrm{Int}(B)) \subset (i, j)$-$\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$. Hence we have $f^{-1}(B)$

$= (i,j)$-$\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is (i,j)-$\mathcal{I}$-open in X. □

Theorem 3.10. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ be (i,j)-$\mathcal{I}$-continuous and σ_i-open function, then the inverse image of each (i,j)-$\mathcal{I}$-open set in Y is (i,j)-preopen in X.*

Proof. Let A be (i,j)-$\mathcal{I}$-open. Then $A \subset \tau_i$-$\mathrm{Int}(A_j^*)$. We have to prove $f^{-1}(A)$ is (i,j)-preopen which implies $f^{-1}(A) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(f^{-1}(A)))$. For this, $f(A) = f(\tau_i\text{-}\mathrm{Int}(A_j^*)) = \tau_i\text{-}\mathrm{Int}(f(\tau_i\text{-}\mathrm{Int}(A_j^*))) \subset \tau_i\text{-}\mathrm{Int}(f(A_j^*))$, $A \subset f^{-1}(\tau_i\text{-}\mathrm{Int}(f(A_j^*))) \subset \tau_i\text{-}\mathrm{Int}(f^{-1}(\tau_i\text{-}\mathrm{Int}(f(A_j^*))))_j^* \subset \tau_i\text{-}\mathrm{Int}(A_j^*)_j^* \subset \tau_i\text{-}\mathrm{Int}(A_j^*) \subset \tau_i\text{-}\mathrm{Int}(A \cup A_j^*) = \tau_i\text{-}\mathrm{Int}(\tau_j\text{-}\mathrm{Cl}^*(A))$. Hence $f^{-1}(A) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(f^{-1}(A)))$. Therefore, $f^{-1}(A)$ is (i,j)-preopen in X. □

Theorem 3.11. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ be (i,j)-$\mathcal{I}$-continuous and $f^{-1}(V_j^*) \subset (f^{-1}(V))_j^*$, for each $V \subset Y$. Then the inverse image of each (i,j)-$\mathcal{I}$-open set is (i,j)-$\mathcal{I}$-open.*

Remark 3.12. *The composition of two (i,j)-$\mathcal{I}$-continuous functions need not be (i,j)-$\mathcal{I}$-continuous, in general.*

Example 3.13. *Let $X = \{a, b, c\}$, $\tau_i = \{\emptyset, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\sigma_1 = \{\emptyset, \{b\}, \{b, c\}, X\}$, $\sigma_2 = \{\emptyset, \{b, c\}, X\}$, $\gamma_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, $\gamma_2 = \{\emptyset, \{b, c\}, X\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$, $\mathcal{J} = \{\emptyset, \{c\}\}$ and let the function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$ and $g : (Y, \sigma_1, \sigma_2, \mathcal{J}) \to (Z, \gamma_1, \gamma_2)$ is defined by $g(a) = c$, $g(b) = a$ and $g(c) = a$. It is clear that both f and g are $(1,2)$-$\mathcal{I}$-continuous. However, the composition function $g \circ f$ is not $(1,2)$-$\mathcal{I}$-continuous, because $\{a\} \in \gamma_1$, but $(g \circ f)^{-1}(\{a\}) = \{c\} \notin (1,2)$-$\mathcal{I}O(X)$.*

Theorem 3.14. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2, \mathcal{J}) \to (Z, \mu_1, \mu_2)$. Then $g \circ f$ is (i,j)-$\mathcal{I}$-continuous, if f is (i,j)-$\mathcal{I}$-continuous and g is σ_j-continuous.*

Proof. Let $V \in \mu_j$. Since g is μ_j-continuous, then $g^{-1}(V) \in \sigma_j$. On the other hand, since f is (i,j)-$\mathcal{I}$-continuous, we have $f^{-1}(g^{-1}(V)) \in (i,j)$-$\mathcal{I}O(X)$. Since $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$, we obtain that $g \circ f$ is (i,j)-$\mathcal{I}$-continuous. □

4. (i,j)-$\mathcal{I}$-open and (i,j)-$\mathcal{I}$-closed functions

Definition 4.1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ is said to be:*

(i) *pairwise $\mathcal{I}$-open if $f(U)$ is a (i,j)-$\mathcal{I}$-open set of Y for every τ_i-open set U of X.*
(ii) *pairwise $\mathcal{I}$-closed if $f(U)$ is a (i,j)-$\mathcal{I}$-closed set of Y for every τ_i-closed set U of X.*

Proposition 4.2. *Every (i,j)-$\mathcal{I}$-open function is (i,j)-preopen function but the converse is not true in general.*

Example 4.3. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\sigma_1 = \{\emptyset, \{a\}, X\}$, $\sigma_2 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then the function* $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2, \mathcal{I})$ *is defined by* $f(a) = b$, $f(b) = a$ *and* $f(c) = c$ *is* $(1, 2)$*-preopen but not* $(1, 2)$*-*$\mathcal{I}$*-open, because* $\{a\} \notin \tau_1$*, but* $f(\{a\}) = \{b\} \notin (1, 2)\text{-}\mathcal{I}O(Y)$.

Remark 4.4. *Each of* (i, j)*-*$\mathcal{I}$*-open function and* τ_i*-open function are independent.*

Example 4.5. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{b\}, \{b, c\}, X\}$, $\tau_2 = \{\emptyset, \{b, c\}, X\}$,$\sigma_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\sigma_2 = \{\emptyset, \{a\}, \{a, c\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{b\}\}$ *on* Y. *Then the identity function* $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2, \mathcal{I})$ *is* $(1, 2)$*-*$\mathcal{I}$*-open function but not* τ_1*-open, because* $\{a\} \notin \tau_1$*, but* $f(\{a\}) = \{a\} \in (1, 2)\text{-}\mathcal{I}O(Y)$.

Example 4.6. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\tau_2 = \{\emptyset, \{b, c\}, X\}$, $\sigma_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, $\sigma_2 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{c\}\}$ *on* Y. *Then the identity function* $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2, \mathcal{I})$ *is defined by* $f(a) = b = f(b)$ *and* $f(c) = c$ *is* τ_1*-open but not* $(1, 2)$*-*$\mathcal{I}$*-open function, because* $\{a\} \in \tau_1$*, but* $f(\{a\}) = \{b\} \notin (1, 2)\text{-}\mathcal{I}O(Y)$.

Theorem 4.7. *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$*, the following statements are equivalent:*

(i) f *is pairwise* $\mathcal{I}$*-open;*
(ii) $f(\tau_i\text{-Int}(U)) \subset (i, j)\text{-}\mathcal{I}\,\text{Int}(f(U))$ *for each subset* U *of* X*;*
(iii) $\tau_i\text{-Int}(f^{-1}(V)) \subset f^{-1}((i, j)\text{-}\mathcal{I}\,\text{Int}(V))$ *for each subset* V *of* Y.

Proof. $(i) \Rightarrow (ii)$: Let U be any subset of X. Then $\tau_i\text{-Int}(U)$ is a τ_i-open set of X. Then $f(\tau_i\text{-Int}(U))$ is a (i, j)-$\mathcal{I}$-open set of Y. Since $f(\tau_i\text{-Int}(U)) \subset f(U)$, $f(\tau_i\text{-Int}(U)) = (i, j)\text{-}\mathcal{I}\,\text{Int}(f(\tau_i\text{-Int}(U))) \subset (i, j)\text{-}\mathcal{I}\,\text{Int}(f(U))$.
$(ii) \Rightarrow (iii)$: Let V be any subset of Y. Then $f^{-1}(V)$ is a subset of X. Hence $f(\tau_i\text{-Int}(f^{-1}(V))) \subset (i, j)\text{-}\mathcal{I}\,\text{Int}(f(f^{-1}(V))) \subset (i, j)\text{-}\mathcal{I}\,\text{Int}(V))$. Then $\tau_i\text{-Int}(f^{-1}(V)) \subset f^{-1}(f(\tau_i\text{-Int}(f^{-1}(V)))) \subset f^{-1}((i, j)\text{-}\mathcal{I}\,\text{Int}(V))$.
$(iii) \Rightarrow (i)$: Let U be any τ_i-open set of X. Then $\tau_i\text{-Int}(U) = U$ and $f(U)$ is a subset of Y. Now, $V = \tau_i\text{-Int}(V) \subset \tau_i\text{-Int}(f^{-1}(f(V))) \subset f^{-1}((i, j)\text{-}\mathcal{I}\,\text{Int}(f(V)))$. Then $f(V) \subset f(f^{-1}((i, j)\text{-}\mathcal{I}\,\text{Int}(f(V)))) \subset (i, j)\text{-}\mathcal{I}\,\text{Int}(f(V))$ and $(i, j)\text{-}\mathcal{I}\,\text{Int}(f(V)) \subset f(V)$. Hence $f(V)$ is a (i, j)-$\mathcal{I}$-open set of Y; hence f is pairwise $\mathcal{I}$-open. □

Theorem 4.8. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ *be a function. Then* f *is a pairwise* $\mathcal{I}$*-closed function if and only if for each subset* V *of* X*,* $(i, j)\text{-}\mathcal{I}\,\text{Cl}(f(V)) \subset f(\tau_i\,\text{Cl}(V))$.

Proof. Let f be a pairwise $\mathcal{I}$-closed function and V any subset of X. Then $f(V) \subset f(\tau_i\text{-Cl}(V))$ and $f(\tau_i\text{-Cl}(V))$ is a (i, j)-$\mathcal{I}$-closed set of Y. We have $(i, j)\text{-}\mathcal{I}\,\text{Cl}(f(V)) \subset (i, j)\text{-}\mathcal{I}\,\text{Cl}(f(\tau_i\text{-Cl}(V))) = f(\tau_i\text{-Cl}(V))$. Conversely, let V be a τ_i-open set of X. Then $f(V) \subset (i, j)\text{-}\mathcal{I}\,\text{Cl}(f(V)) \subset f(\tau_i\text{-Cl}(V)) = f(V)$; hence $f(V)$ is a (i, j)-$\mathcal{J}$-closed subset of Y. Therefore, f is a pairwise $\mathcal{I}$-closed function. □

Theorem 4.9. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a function. Then f is a pairwise $\mathcal{I}$-closed function if and only if for each subset V of Y, $f^{-1}((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(V)) \subset \tau_i\text{-}\mathrm{Cl}(f^{-1}(V))$.*

Proof. Let V be any subset of Y. Then by Theorem 4.8, $(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(V) \subset f(\tau_i\text{-}\mathrm{Cl}(f^{-1}(V)))$. Since f is bijection, $f^{-1}((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(V)) = f^{-1}((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f(f^{-1}(V)))) \subset f^{-1}(f(\tau_i\text{-}\mathrm{Cl}(f^{-1}(V)))) = \tau_i\text{-}\mathrm{Cl}(f^{-1}(V))$. Conversely, let U be any subset of X. Since f is bijection, $(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f(U)) = f(f^{-1}((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f(U))) \subset f(\tau_i\text{-}\mathrm{Cl}(f^{-1}(f(U)))) = f(\tau_i\text{-}\mathrm{Cl}(U))$. Therefore, by Theorem 4.8, f is a pairwise $\mathcal{I}$-closed function. □

Theorem 4.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a pairwise $\mathcal{I}$-open function. If V is a subset of Y and U is a τ_i-closed subset of X containing $f^{-1}(V)$, then there exists a (i,j)-$\mathcal{I}$-closed set F of Y containing V such that $f^{-1}(F) \subset U$.*

Proof. Let V be any subset of Y and U a τ_i-closed subset of X containing $f^{-1}(V)$, and let $F = Y \backslash (f(X \backslash V))$. Then $f(X \backslash V) \subset f(f^{-1}(X \backslash V)) \subset X \backslash V$ and $X \backslash U$ is a τ_i-open set of X. Since f is pairwise $\mathcal{I}$-open, $f(X \backslash U)$ is a (i,j)-$\mathcal{I}$-open set of Y. Hence F is an (i,j)-$\mathcal{I}$-closed set of Y and $f^{-1}(F) = f^{-1}(Y \backslash (f(X \backslash U)) \subset U$. □

Theorem 4.11. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a pairwise $\mathcal{I}$-closed function. If V is a subset of Y and U is a open subset of X containing $f^{-1}(V)$, then there exists (i,j)-$\mathcal{I}$-open set F of Y containing V such that $f^{-1}(F) \subset U$.*

Proof. The proof is similar to the Theorem 4.10. □

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