Separation Axioms in Ideal Bitopological Spaces

Dedicated to Professor Takashi Noiri on the occasion of his 70th birthday

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Abstract

The purpose of this paper is to introduce and study the notions $\mathcal{I}$-R_0, $\mathcal{I}$-R_1, $\mathcal{I}$-T_0, $\mathcal{I}$-T_1 and $\mathcal{I}$-T_2 in ideal bitopological space.

Key words: Ideal bitopological spaces, $\mathcal{I}$-closed set, $\mathcal{I}$-open set.
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1 Introduction

It is well known that while Topology in Computer Science can be used to model a given logic of programs it is no longer sufficient to model a contemporary

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software engineering environment. Multiple programs written in many different languages now need to work together through a global computer network to perform as if a single entity for their users. Those days of programming for a stand alone computer architecture have been superseded by networked interactive devices where it is routine to find numerous structurally diverse systems each accessing the same data set. And so, we look to a bitopology (X, τ_1, τ_2) as an inspirational template for studying how domain theory can be generalised to model contemporary programming paradigms characterised by their multiple interpretations. In 1990, Jankovic and Hamlett (See [4]) have defined the concept of $\mathcal{I}$-open set via local function which was given by Vaidyanathaswamy (See [7]). The latter concept was also established utilizing the concept of an ideal whose topic in general topological spaces was treated in the classical text by Kuratowski (See [5]). In 1992, Abd El-Monsef et al. (See [1]) studied a number of properties of $\mathcal{I}$-open sets as well as $\mathcal{I}$-closed sets and $\mathcal{I}$-continuous functions and investigated several of their properties. Recently, the authors introduce and studied ideal bitopological spaces. In this paper, $\mathcal{I}$-open sets are used to define some weak separation axioms and study some of their basic properties.

2 Preliminaries

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X, and is denoted by $(X, \tau, \mathcal{I})$, where the ideal is defined as a nonempty collection of subsets of X satisfying the following two conditions. (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$; (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. For a subset $A \subset X$, $A^*(\mathcal{I}) = \{x \in X |\ U \cap A \notin \mathcal{I}$ for each neighbourhood U of $x\}$ is called the local function of A with respect to $\mathcal{I}$ and τ (See [4]). Where there is no chance of confusion, $A^*(\mathcal{I})$ is denoted by A^*. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists topology $\tau^*(\mathcal{I})$, finer than τ, generated by the base $\beta(\mathcal{I}, \tau) = \{U \backslash I \mid U \in \tau$ and $I \in \mathcal{I}\}$, but in general $\beta(\mathcal{I}, \tau)$ is not always a topology (See [4]). Observe additionally that τ_i-$\mathrm{Cl}^*(A) = A \cup A_i^*(\tau_i, \mathcal{I})$ defines a Kuratowski closure operator for $\tau^*(\mathcal{I})$. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-open (See [1]) if $S \subset \mathrm{Int}(S^*)$. The complement of an $\mathcal{I}$-open set is called an $\mathcal{I}$-closed set. The intersection of all $\mathcal{I}$-closed sets containing S is called the $\mathcal{I}$-closure (See [6]) of S and is denoted by $\mathcal{I}$-$\mathrm{Cl}(S)$. A set S is $\mathcal{I}$-closed if and only if $\mathcal{I}$-$\mathrm{Cl}(A) = A$. The $\mathcal{I}$-interior (See [3]) of S is defined by the union of all $\mathcal{I}$-open sets of $(X, \tau, \mathcal{I})$ contained in S and is denoted by $\mathcal{I}$-$\mathrm{Int}(S)$. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$). The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $\mathcal{I}O(X, x)$ (resp. $\mathcal{I}C(X, x)$). A subset B_x of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-neighbourhood of a point x∈X if there exists an $\mathcal{I}$-open set U such that x $\in$ U $\subset B_x$. An ideal bitopological space (See [2]) is a bitopological space

(X, τ_1, τ_2) with an ideal $\mathcal{I}$ on X, and is denoted by $(X, \tau_1, \tau_2, \mathcal{I})$.

Theorem 1. An ideal topological space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_1 if and only if each singleton is $\mathcal{I}$-closed.

3 Pairwise $\mathcal{I}$-R_i (i = 0,1) spaces

Definition 2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then the $\mathcal{I}$-kernel of A, denoted by $\mathcal{I}$-Ker(A) is defined to be the set $\mathcal{I}$-Ker$(A) = \cap \{G \in \mathcal{I}O(X) \mid A \subset G\}$.

Theorem 3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $x \in X$. Then, $y \in \mathcal{I}$-Ker$(\{x\})$ if and only if $x \in \mathcal{I}$-Cl$(\{y\})$.

Proof. Suppose that $y \notin \mathcal{I}$-Ker$(\{x\})$. Then there exists $U \in \mathcal{I}O(X, x)$ such that $y \notin U$. Therefore, we have $x \notin \mathcal{I}$-Cl$(\{y\})$. The proof of the converse case can be done similarly.

Lemma 4. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A a subset of X. Then, $\mathcal{I}$-Ker$(A) = \{x \in X \mid \mathcal{I}$-Cl$(\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \in \mathcal{I}$-Ker(A) and $\mathcal{I}$-Cl$(\{x\}) \cap A = \emptyset$. Hence $x \notin X \setminus \mathcal{I}$-Cl$(\{x\})$ which is an $\mathcal{I}$-open set containing A. This is impossible, since $x \in \mathcal{I}$-Ker(A). Consequently, $\mathcal{I}$-Cl$(\{x\}) \cap A \neq \emptyset$. Next, let $x \in X$ such that $\mathcal{I}$-Cl$(\{x\}) \cap A \neq \emptyset$ and suppose that $x \notin \mathcal{I}$-Ker(A). Then, there exists an $\mathcal{I}$-open set U containing A and $x \notin U$. Let $y \in \mathcal{I}$-Cl$(\{x\}) \cap A$. Hence, U is a $\mathcal{I}$-neighbourhood of y which does not contains x. By this contradiction $x \in \mathcal{I}$-Ker(A) and hence the claim.

Definition 5. An ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0 if for each τ_i-$\mathcal{I}$-open set G, $x \in G$ implies τ_j-$\mathcal{I}$-Cl$(\{x\}) \subset G$, where $i, j = 1, 2$ and $i \neq j$.

Example 6. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is a $(1, 2)$-$\mathcal{I}$-R_0.

Example 7. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is not a $(1, 2)$-$\mathcal{I}$-R_0 space.

Theorem 8. In an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, the following statements are equivalent:

(i) $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0.
(ii) For any τ_i-$\mathcal{I}$-closed set F and a point $x \notin F$, there exists a $U \in \mathcal{I}O(X, \tau_j)$ such that $x \notin U$ and $F \subset U$ for $i, j = 1, 2$ and $i \neq j$.

(iii) For any τ_i-$\mathcal{I}$-closed set F and $x \notin F$, τ_j-$\mathcal{I}$-Cl$(\{x\})$ $\cap F = \emptyset$, for $i, j = 1, 2$ and $i \neq j$.

Proof. $(i) \Rightarrow (ii)$: Let F be a τ_i-$\mathcal{I}$-closed set and $x \notin F$. Then by (i) τ_j-$\mathcal{I}$-Cl$(\{x\}) \subset X \backslash F$, where $i, j = 1, 2$ and $i \neq j$. Let $U = X \backslash \tau_j$-$\mathcal{I}$-Cl$(\{x\})$, then $U \in \mathcal{I}O(X, \tau_j)$ and also $F \subset U$ and $x \notin U$. $(ii) \Rightarrow (iii)$: Let F be a τ_i-$\mathcal{I}$-closed set and a point $x \notin F$. Suppose the given conditions hold. Since $U \in \mathcal{I}O(X, \tau_j)$, $U \cap \tau_j$-$\mathcal{I}$-Cl$(\{x\}) = \emptyset$. Then $F \cap \tau_j$-$\mathcal{I}$-Cl$(\{x\}) = \emptyset$, where $i, j = 1, 2$ and $i \neq j$. $(iii) \Rightarrow (i)$: Let $G \in \mathcal{I}O(X, \tau_i)$ and $x \in G$. Now $X \backslash G$ is τ_j-$\mathcal{I}$-closed and $x \notin X \backslash G$. By (iii), τ_j-$\mathcal{I}$-Cl$(\{x\})$ $\cap (X \backslash G) = \emptyset$ and hence τ_j-$\mathcal{I}$-Cl$(\{x\}) \subset G$ for $i, j = 1, 2$ and $i \neq j$. Therefore, the space $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0.

Theorem 9. An ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0 if and only if for each pair x, y of distinct points in X, τ_1-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_2$-$\mathcal{I}$-Cl$(\{y\})$ $= \emptyset$ or $\{x, y\} \subset \tau_1$-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_2$-$\mathcal{I}$-Cl$(\{y\})$.

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be pairwise $\mathcal{I}$-R_0. Suppose that τ_1-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_2$-$\mathcal{I}$-Cl$(\{y\}) \neq \emptyset$ and $\{x, y\} \subseteq \tau_1$-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_2$-$\mathcal{I}$-Cl$(\{y\})$. Let S $\in \tau_1$-$\mathcal{I}$-Cl$(\{x\})$ $\cap$ τ_2-$\mathcal{I}$-Cl$(\{y\})$ and x $\notin \tau_1$-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_2$-$\mathcal{I}$-Cl$(\{y\})$. Then x $\notin \tau_2$-$\mathcal{I}$-Cl$(\{y\})$ and $x \in X \backslash \tau_2$-$\mathcal{I}$-Cl$(\{y\}) \in \mathcal{I}O(X, \tau_2)$. But τ_1-$\mathcal{I}$-Cl$(\{x\}) \subseteq X \backslash (\tau_2$-$\mathcal{I}$-Cl$(\{y\}))$, which contradicts the definition of pairwsise $\mathcal{I}$-R_0-ness. Hence for each pair of distinct points x, y in X, τ_2-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_2$-$\mathcal{I}$-Cl$(\{y\}) = \emptyset$ or $\{x, y\} \subset \tau_1$-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_2$-$\mathcal{I}$-Cl$(\{y\})$. Next assume that the given conditions hold. Let U be a τ_1-$\mathcal{I}$-open set and x $\in$ U. Suppose τ_2-$\mathcal{I}$-Cl$(\{x\}) \subseteq$ U. So there is a point y $\in \tau_2$-$\mathcal{I}$-Cl$(\{x\})$ such that y $\notin$ U and τ_1-$\mathcal{I}$-Cl$(\{y\}) \cap$ U $= \emptyset$, because of the fact that $X \backslash U$ is τ_1-$\mathcal{I}$-closed and $y \in X \backslash U$. Hence, $\{x, y\} \subseteq \tau_1$-$\mathcal{I}$-Cl$(\{y\})$ $\cap$ τ_2-$\mathcal{I}$-Cl$(\{x\})$ and thus τ_1-$\mathcal{I}$-Cl$(\{y\}) \cap \tau_2$-$\mathcal{I}$-Cl$(\{x\}) \neq \emptyset$.

Theorem 10. In an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ the following statements are equivalent:

(i) $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0.
(ii) For any $x \in X$, τ_i-$\mathcal{I}$-Cl$(\{x\}) = \tau_j$-$\mathcal{I}$-Ker$(\{x\})$, for $i, j = 1, 2$ and $i \neq j$.
(iii) For any $x \in X$, τ_j-$\mathcal{I}$-Cl$(\{x\}) \subset \tau_j$-$\mathcal{I}$-Ker$(\{x\})$, for $i, j = 1, 2$ and $i \neq j$.
(iv) For any $x, y \in X$ and $y \in \tau_i$-$\mathcal{I}$-Cl$(\{x\})$ if and only if $x \in \tau_j$-$\mathcal{I}$-Cl$(\{y\})$, for $i, j = 1, 2$ and $i \neq j$.
(v) For any τ_j-$\mathcal{I}$-closed F, $F = \bigcap$ $\{G | G$ is a τ_j-$\mathcal{I}$-open set and $F \subset$G$\}$, for $i, j = 1, 2$ and $i \neq j$.
(vi) For any τ_i-$\mathcal{I}$-open set G, $G = \bigcup \{F | F$ is a τ_i-$\mathcal{I}$-closed set and F$\subset$G$\}$.
(vii) For every nonempty set A and each $G \in \mathcal{I}$O(X, τ_i) such that $A \cap G \neq \emptyset$, there exists a τ_j-$\mathcal{I}$-closed set F such that $F \subset G$ and $A \cap F \neq \emptyset$.

Proof. $(i) \Rightarrow (ii)$: Let $x, y \in X$. Then by Theorems 3 and 3, $y \in \tau_j$-$\mathcal{I}$-Ker$(\{x\})$ $\Leftrightarrow x \in \tau_j$-$\mathcal{I}$-Cl$(\{y\}) \Leftrightarrow y \in \tau_i$-$\mathcal{I}$-Cl$(\{x\})$. Hence τ_i-$\mathcal{I}$-Cl$(\{x\}) = \tau_j$-$\mathcal{I}$-Ker$(\{x\})$ for $i, j = 1, 2$ and $i \neq j$. $(ii) \Rightarrow (iii)$: Straightforward. $(iii) \Rightarrow (iv)$: For $x, y \in X$, if $y \in \tau_i$-$\mathcal{I}$-Cl$(\{x\})$, then by (iii), $y \in \tau_j$-$\mathcal{I}$-Ker$(\{x\})$ and hence, by

Theorem 3, $x \in \tau_j$-$\mathcal{I}$-Cl($\{y\}$) for $i = 1, 2$ and $i \neq j$. $(iv) \Rightarrow (v)$: Let F be a τ_i-$\mathcal{I}$-closed set and $H = \bigcap\{$G$|G$ is a τ_i-$\mathcal{I}$-open set and F⊂G$\}$. Clearly, $F \subset H$. To prove the reverse inclusion, we proceed as follows. Let $x \notin F$. Then for any $y \in F$ implies that τ_i-$\mathcal{I}$-Cl($\{y\}$) $\subset F$. It follows that $x \notin \tau_i$-$\mathcal{I}$-Cl($\{y\}$). Now by (iv), $x \notin \tau_i$-$\mathcal{I}$-Cl($\{y\}$) implies that y$\notin \tau_j$-$\mathcal{I}$-Cl($\{x\}$). There exists a τ_i-$\mathcal{I}$-open set G_y such that $y \in G_y$ and $x \notin G_y$. Let $G = \bigcup_{y \in F}\{\ G_y \mid G_y$ is τ_j-$\mathcal{I}$-open, $y \in G_y$ and $x \notin G_y\}$. Since any union of $\mathcal{I}$-open sets is $\mathcal{I}$-open, G is τ_j-$\mathcal{I}$-open. Then there exists a τ_j-$\mathcal{I}$-open set G such that $x \notin G$ and $F \subset G$. Hence, $x \notin H$. Therefore $F = H$. $(v) \Rightarrow (vi)$: Obvious. $(vi) \Rightarrow (vii)$: Let A be a nonempty set and G be a τ_i-$\mathcal{I}$-open set and x $\in A \cap G$. By (vi), $G = \bigcup\{$F$|F$ is a τ_j-$\mathcal{I}$-closed and F ⊂G$\}$. It follows that there is a τ_i-$\mathcal{I}$-closed set F such that $x \in F \subset G$. Hence $A \cap F \neq \emptyset$. $(vii) \Rightarrow (i)$: Let G be a τ_i-$\mathcal{I}$-open set and $x \in G$, then $\{x\}$∩G$\neq \emptyset$. Therefore, by (vii), there exists a τ_j-$\mathcal{I}$-closed set F such that $x \in F \subset G$ and $\{x\}$∩F$\neq \emptyset$, which implies $\{x\} \cap \tau_j$-$\mathcal{I}$-Cl($\{x\}$) $\neq \emptyset$. Then τ_j-$\mathcal{I}$-Cl($\{x\}$) $\subset G$, where $i, j = 1, 2$ and $i \neq j$. Therefore, $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0.

Remark 11. For each x∈X, we define (τ_1, τ_2)-$\mathcal{I}$-Cl($\{x\}$) $= \tau_1$-$\mathcal{I}$-Cl($\{x\}$) $\cap$ τ_2-$\mathcal{I}$-Cl($\{x\}$) and (τ_1, τ_2)-$\mathcal{I}$-Ker($\{x\}$) $= \tau_1$-$\mathcal{I}$-Ker($\{x\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{x\}$).

Theorem 12. For any x, y ∈ X in a pairwise $\mathcal{I}$-R_0 space $(X, \tau_1, \tau_2, \mathcal{I})$ we have either (τ_1, τ_2)-$\mathcal{I}$-Cl($\{x\}$) $= (\tau_1, \tau_2)$-$\mathcal{I}$-Cl($\{y\}$) or (τ_1, τ_2)-$\mathcal{I}$-Cl($\{x\}$) $\cap$ (τ_1, τ_2)-$\mathcal{I}$-Cl($\{y\}$) $= \emptyset$.

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a pairwise $\mathcal{I}$-R_0 space. Suppose that (τ_1, τ_2)-$\mathcal{I}$-Cl($\{x\}$) $\neq (\tau_1, \tau_2)$-$\mathcal{I}$-Cl($\{y\}$) and (τ_1, τ_2)-$\mathcal{I}$-Cl($\{x\}$) $\cap$ (τ_1, τ_2)-$\mathcal{I}$-Cl($\{y\}$) $\neq \emptyset$. Let s $\in (\tau_1, \tau_2)$-$\mathcal{I}$-Cl($\{x\}$) $\cap$ (τ_1, τ_2)-$\mathcal{I}$-Cl($\{y\}$) and x $\notin (\tau_1, \tau_2)$-$\mathcal{I}$-Cl($\{y\}$) $= \tau_1$-$\mathcal{I}$-Cl($\{y\}$) $\cap$ τ_2-$\mathcal{I}$-Cl($\{y\}$). Then $x \notin \tau_i$-$\mathcal{I}$-Cl($\{y\}$) and x ∈ X − τ_i-$\mathcal{I}$-Cl($\{y\}$) $\in \mathcal{I}O(X, \tau_i)$. But τ_j-$\mathcal{I}$-Cl($\{x\}$) $\subseteq$ X−τ_i-$\mathcal{I}$-Cl($\{y\}$), because $s \in (\tau_1, \tau_2)$-$\mathcal{I}$-Cl($\{x\}$) $\cap$ (τ_1, τ_2)-$\mathcal{I}$-Cl($\{y\}$). Which in its turn, contradicts the hypothesis of pairwise $\mathcal{I}$-R_0-ness of X. Hence we have either (τ_1, τ_2)-$\mathcal{I}$-Cl($\{x\}$) $= (\tau_1, \tau_2)$-$\mathcal{I}$-Cl($\{y\}$) or (τ_1, τ_2)-$\mathcal{I}$-Cl($\{x\}$) $\cap$ (τ_1, τ_2)-$\mathcal{I}$-Cl($\{y\}$) $= \emptyset$.

Theorem 13. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a pairwise $\mathcal{I}$-R_0 space. Then for any point x, y ∈ X, (τ_1, τ_2)-$\mathcal{I}$-Ker($\{x\}$) $\neq (\tau_1, \tau_2)$-$\mathcal{I}$-Ker($\{y\}$) implies (τ_1, τ_2)-$\mathcal{I}$-Ker($\{x\}$) $\cap$ (τ_1, τ_2)-$\mathcal{I}$-Ker($\{y\}$) $= \emptyset$.

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a pairwise $\mathcal{I}$-R_0 space. Suppose that (τ_1, τ_2)-$\mathcal{I}$-Ker($\{x\}$) $\cap$ (τ_1, τ_2)-$\mathcal{I}$-Ker($\{y\}$) $\neq \emptyset$ and s $\in \tau_1$-$\mathcal{I}$-Ker($\{x\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{x\}$) $\cap$ τ_1-$\mathcal{I}$-Ker($\{y\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{y\}$). Also by Theorem 3, s $\in \tau_1$-$\mathcal{I}$-Ker($\{x\}$) implies that x $\in \tau_1$-$\mathcal{I}$-Ker($\{s\}$) which in its turn by Theorem 3 (iv) implies that x $\in$ τ_2-$\mathcal{I}$-Ker($\{s\}$). Hence τ_2-$\mathcal{I}$-Ker($\{x\}$) $\subset \tau_2$-$\mathcal{I}$-Ker($\{s\}$) $\subset \tau_2$-$\mathcal{I}$-Ker($\{y\}$). Thus s $\in \tau_1$-$\mathcal{I}$-Ker($\{x\}$) implies that τ_2-$\mathcal{I}$-Ker($\{x\}$) $\subset \tau_2$-$\mathcal{I}$-Ker($\{y\}$). Similarly, s $\in$ τ_2-$\mathcal{I}$-Ker($\{x\}$) implies τ_2-$\mathcal{I}$-Ker($\{x\}$) $\subset \tau_2$-$\mathcal{I}$-Ker($\{y\}$) and s $\in \tau_1$-$\mathcal{I}$-Ker($\{y\}$) implies τ_1-$\mathcal{I}$-Ker($\{y\}$) $\subset \tau_1$-$\mathcal{I}$-Ker($\{x\}$) and s $\in \tau_2$-$\mathcal{I}$-Ker($\{y\}$) implies τ_2-$\mathcal{I}$-Ker($\{y\}$) $\subset \tau_2$-$\mathcal{I}$-Ker($\{x\}$). Therefore, τ_1-$\mathcal{I}$-Ker($\{x\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{x\}$) $\subset$

τ_1-$\mathcal{I}$-Ker($\{y\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{y\}$) and τ_1-$\mathcal{I}$-Ker($\{y\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{y\}$) $\subset$ τ_1-$\mathcal{I}$-Ker($\{x\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{x\}$). Hence, τ_1-$\mathcal{I}$-Ker($\{y\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{y\}$) $=$ τ_1-$\mathcal{I}$-Ker($\{x\}$) $\cap$ τ_2-$\mathcal{I}$-Ker($\{x\}$). Therefore, (τ_1, τ_2)-$\mathcal{I}$-Ker($\{x\}$) $=$ (τ_1, τ_2)-$\mathcal{I}$-Ker($\{y\}$).

Corollary 14. For any pair of points x and y in a pairwise $\mathcal{I}$-R_0 space $(X, \tau_1, \tau_2, \mathcal{I})$, the following statements are equivalent:

(i) $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0.
(ii) For any τ_i-$\mathcal{I}$-closed set F$\subset$X, F$\subset$ τ_j-$\mathcal{I}$-Ker(F), where i, j = 1,2 and i$\neq$j.
(iii) For any τ_i-$\mathcal{I}$-closed set F$\subset$X and x$\in$F, τ_j-$\mathcal{I}$-Ker($\{x\}$) $\subset F$, where i, j = 1,2 and i$\neq$j.
(iv) For any x$\in$X, τ_j-$\mathcal{I}$-Ker($\{x\}$) $\subset$ τ_i-$\mathcal{I}$-Cl($\{x\}$), where i, j = 1, 2 and i$\neq$j.

Proof. $(i) \Rightarrow (ii)$: Let F be τ_i-$\mathcal{I}$-closed set and x$\notin$F. Then X–F is τ_i-$\mathcal{I}$-open contianing x. Since $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0, τ_j-$\mathcal{I}$-Cl($\{x\}$) $\subset$ X–F where i, j =1, 2 and i$\neq$j. Therefore, τ_j-$\mathcal{I}$-Cl($\{x\}$)$\cap$F=$\emptyset$ and by Lemma 3 x $\notin$ τ_j-$\mathcal{I}$-Ker(F). Hence τ_j-$\mathcal{I}$-ker(F)) $\subset$ F. Again by the definition of $\mathcal{I}$-kernel, F $\subset$ τ_j-$\mathcal{I}$-Ker(F), so F=τ_j-$\mathcal{I}$-ker(F), where i, j =1,2 and i$\neq$j. $(ii) \Rightarrow (iii)$: Let F be a τ_i-$\mathcal{I}$-closed set containing x. Then $\{x\}$ $\subset$ F and τ_j-$\mathcal{I}$-Ker($\{x\}$) $\subset$ τ_j-$\mathcal{I}$-Ker(F). From (ii), it follows that τ_j-$\mathcal{I}$-Ker($\{x\}$) $\subset$ F, where i, j = 1,2 and i$\neq$j. $(iii) \Rightarrow (iv)$: Since x$\in$ τ_i-$\mathcal{I}$-Cl($\{x\}$) and τ_i-$\mathcal{I}$-Cl($\{x\}$) is τ_i-$\mathcal{I}$-closed in X, which in turn ensures by (iii), that τ_j-$\mathcal{I}$-Ker($\{x\}$) $\subset$ τ_i-$\mathcal{I}$-Cl($\{x\}$), where i, j =1,2 and i$\neq$j. $(iv) \Rightarrow (i)$: It follows from Theorem 3.

Definition 15. An ideal bitopological space (X, τ_1, τ_2) is said to be pairwise $\mathcal{I}$-R_1 if for each x, y $\in$ X, τ_i-$\mathcal{I}$-Cl($\{x\}$) $\neq$ τ_j-$\mathcal{I}$-Cl($\{y\}$), there exist disjoint sets U $\in$ $\mathcal{I}$O(X, τ_j) and V $\in$ $\mathcal{I}$O(X, τ_i) such that τ_i-$\mathcal{I}$-Cl($\{x\}$) $\subset$ U and τ_j-$\mathcal{I}$-Cl($\{y\}$) $\subset$ V where i, j = 1,2 and i$\neq$j.

Example 16. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset,\ \{a\},\ \{c\},\ \{a, c\},\ \{b, c\},\ X\}$, $\tau_2 = \{\emptyset,\ \{a\},\ \{c\},\ \{a, c\},\ X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is a $(1, 2)$-$\mathcal{I}$-R_1 space.

Example 17. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset,\ \{a\},\ \{b\},\ \{a, b\},\ X\}$, $\tau_2 = \{\emptyset,\ \{b\},\ \{c\},\ \{b, c\},\ X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is not a $(1, 2)$-$\mathcal{I}$-R_1 space.

Theorem 18. If $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_1, then it is pairwise $\mathcal{I}$-R_0.

Proof. Suppose that $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_1. Let U be a τ_i-$\mathcal{I}$-open set and x$\in$U. If y$\notin$U, then y$\in$X–U and x $\notin$ τ_i-$\mathcal{I}$-Cl($\{y\}$). Therefore, for each point y$\in$X–U, τ_j-$\mathcal{I}$-Cl($\{x\}$) $\neq$ τ_i-$\mathcal{I}$-Cl($\{y\}$). Since $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_1, there exist a τ_i-$\mathcal{I}$-open set U_y and a τ_j-$\mathcal{I}$-open set V_y such that τ_j-$\mathcal{I}$-Cl($\{x\}$) $\subset U_y$, τ_i-$\mathcal{I}$-Cl($\{y\}$) $\subset V_y$ and $U_y \cap V_y = \emptyset$ where i,j=1,2 and i$\neq$j. Let A = $\bigcup\{V_y | y \in X - U\}$, then X–U$\subset$A, x$\notin$A and A is τ_j-$\mathcal{I}$-open set. Therefore, τ_j-$\mathcal{I}$-Cl($\{x\}$) $\subset$ X–A $\subset$ U. Hence $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0.

Remark 19. The ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ as in Example 3 is $(1,2)$-$\mathcal{I}$-R_0 but not $(1,2)$-$\mathcal{I}$-R_1.

Theorem 20. An ideal bitopological space (X, τ_1, τ_2, I) is pairwise $\mathcal{I}$-R_1 if and only if for every pair of points x and y of X such that τ_i-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_j$-$\mathcal{I}$-Cl$(\{y\})$, there exists a τ_i-$\mathcal{I}$-open set U and τ_j-$\mathcal{I}$-open set V such that x∈V, y∈U and U∩V$\neq \emptyset$, where i, j=1,2 and i≠j.

Proof. Suppose that $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_1. Let x, y be points of X such that τ_i-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_j$-$\mathcal{I}$-Cl$(\{y\})$, where i, j=1,2 and i≠j. Then there exist a τ_i-$\mathcal{I}$-open set U and τ_j-$\mathcal{I}$ open set V such that x $\in \tau_i$-$\mathcal{I}$-Cl$(\{x\}) \subset$ V and y $\in \tau_j$-$\mathcal{I}$-Cl$(\{y\}) \subset$ U and it follows that U∩V $= \emptyset$, where i,j=1,2 and i≠j. On the other hand, suppose there exist a τ_i-$\mathcal{I}$-open set U and a τ_j-$\mathcal{I}$-open set V such that x ∈ V, y ∈ U and U∩V=∅, where i,j=1,2 and i≠j. Since every pairwise $\mathcal{I}$-R_1 space is every pairwise $\mathcal{I}$-R_0, τ_j-$\mathcal{I}$-Cl$(\{x\}) \subset$ V and τ_i-$\mathcal{I}$-Cl$(\{y\})$ $\subset$ U, from which we infer that τ_i-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_j$-$\mathcal{I}$-Cl$(\{y\})$, for i=1,2 and i≠j.

Theorem 21. A pairwise $\mathcal{I}$-R_0 space $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_1 if for each pair of points x and y of X with τ_i-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_j$-$\mathcal{I}$-Cl$(\{y\}) = \emptyset$, there exist disjoint sets U$\in \mathcal{I}O(X, \tau_i)$ and V$\in \mathcal{I}O(X, \tau_j)$ such that x∈U and y∈V where i,j=1,2 and i≠j.

Proof. It follows directly from Theorems 3 and 3.

Theorem 22. In an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ the following statements are equivalent:

(i) $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_1.
(ii) For any two distinct points x, y ∈ X, τ_i-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_j$-$\mathcal{I}$-Cl$(\{y\})$ implies that there exist a τ_i-$\mathcal{I}$-closed set F_1 and a τ_j-$\mathcal{I}$-closed set F_2 such that x$\in F_1$, $y \in F_2$, $x \notin F_2$, $y \notin F_1$ and X=$F_1 \cup F_2$, i, j =1,2 and i≠j.

Proof. $(i) \Rightarrow (ii)$: Suppose that $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_1. Let x, y ∈ X such that τ_i-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_j$-$\mathcal{I}$-Cl$(\{y\})$. By Theorem 3, then there exist disjoint sets V$\in \mathcal{I}O(X, \tau_i)$, U$\in \mathcal{I}O(X, \tau_j)$ such that x∈U and y∈V where i,j=1,2 and i≠j. Then F_1 = X–V is a $\mathcal{I}$-closed set and F_2=X–U is a τ_j-$\mathcal{I}$-closed set such that x$\in F_1$, $x \notin F_2$, $y \notin F_1$, $y \in F_2$ and X=$F_1 \cup F_2$ where i,j=1,2 and i≠j. $(ii) \Rightarrow (i)$: Let x, y ∈ X such that τ_i-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_j$-$\mathcal{I}$-Cl$(\{y\})$ where i, j=1,2 and i≠j. Hence, for any two distinct points x, y of X, τ_i-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_j$-$\mathcal{I}$-Cl$(\{y\}) = \emptyset$, where i, j=1,2 and i≠j. Then by Theorem 3, $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0. By (ii), there exists a τ_i-$\mathcal{I}$-closed set F_1 and a τ_j-$\mathcal{I}$-closed set F_2 such that X=$F_1 \cup F_2$, $x \in F_1$, $y \in F_2$, $x \notin F_2$, $y \notin F_1$. Therefore, x∈X–F_2 = U$\in I$(X, τ_j) and y∈X–F_1 = V$\in \mathcal{I}O(X, \tau_j)$ which implies that τ_i-$\mathcal{I}$-Cl$(\{x\})$ $\subset$ U, τ_j-$\mathcal{I}$-Cl$(\{y\}) \subset$ V and U∩V=∅ where i, j = 1,2 and i≠j.

4 Pairwise $\mathcal{I}$-T_i (i = 0,1,2) spaces

Definition 23. An ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be:

(a) a pairwise $\mathcal{I}$-T_0 (resp. pairwise $\mathcal{I}$-T_1) if for any pair of distinct points x and y in X, there exists a τ_i-$\mathcal{I}$-open set which contains one of them but not the other i = 1 or 2 (resp. there exist τ_i-$\mathcal{I}$-open set U and τ_j-$\mathcal{I}$-open set V such that x∈U, y∉U and y∈V, x∉V, i,j = 1,2, i≠j).

(b) a pairwise $\mathcal{I}$-T_2 if for any pair of distinct points x and y in X, there exist τ_i-$\mathcal{I}$-open set U and τ_j-$\mathcal{I}$-open set V such that x∈U, y∈V and U∩V=∅, i,j=1,2, i≠j.

Example 24. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset,\ X\}$, $\tau_2 = \{\emptyset,\ \{a\},\ X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is a $(1, 2)$-$\mathcal{I}$-T_0 space but not a $(1, 2)$-$\mathcal{I}$-T_1 space.

Example 25. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset,\ \{a\},\ X\}$, $\tau_2 = \{\emptyset,\ \{b\},\ X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is a $(1, 2)$-$\mathcal{I}$-T_1 space but not a $(1, 2)$-$\mathcal{I}$-T_2 space.

Example 26. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset,\ \{a\},\ \{a, b\},\ \{a, c\},\ X\}$, $\tau_2 = \{\emptyset,\ \{b\},\ \{a, b\},\ \{b, c\},\ X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is a $(1, 2)$-$\mathcal{I}$-T_2 space.

Theorem 27. For an ideal bitopological $(X, \tau_1, \tau_2, \mathcal{I})$, the following are equivalent:

(i) $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-T_0.
(ii) For every x∈X, $\{x\}$=τ_1-$\mathcal{I}$-Cl($\{x\}$) ∩ τ_2-$\mathcal{I}$-Cl($\{x\}$).
(iii) For each x∈X, the intersection of all τ_1-$\mathcal{I}$-neighbourhoods of x and all τ_2-$\mathcal{I}$-neighbourhoods of x is $\{x\}$.

Proof. $(i) \Rightarrow (ii)$: Suppose y≠x in X. There exists a τ_1-$\mathcal{I}$-open set V containing x but not y or τ_2-$\mathcal{I}$-open set U containing y but not x. In otherwords, either x $\notin \tau_1$-$\mathcal{I}$-Cl($\{y\}$) or y $\notin \tau_2$-$\mathcal{I}$-Cl($\{x\}$). Hence for a point x,y $\notin \tau_1$-$\mathcal{I}$-Cl($\{x\}$) ∩ τ_2-$\mathcal{I}$-Cl($\{x\}$). Thus, $\{x\}$ = τ_1-$\mathcal{I}$-Cl($\{x\}$) ∩ τ_2-$\mathcal{I}$-Cl($\{x\}$). $(ii) \Rightarrow (iii)$: Straightforward. $(iii) \Rightarrow (i)$: Let x≠y in X. By (iii), $\{x\}$ = the intersection of all τ_1-$\mathcal{I}$-neighbourhoods and τ_2-$\mathcal{I}$-neighbourhoods of x. Hence, there exists either one τ_1-neighbourhood of y but not containing x or a τ_2-neighbourhood of y but not containing x. Therefore, $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-T_0.

Theorem 28. Every pairwise $\mathcal{I}$-T_0 pairwise $\mathcal{I}$-R_0 space is pairwise $\mathcal{I}$-T_1.

Proof. Let $x, y \in X$ and $x \neq y$. Since X is pairwise $\mathcal{I}$-T_0, there is a set which is either τ_1-$\mathcal{I}$-open or τ_2-$\mathcal{I}$-open containing one of the points but not the other. Let G be τ_1-$\mathcal{I}$-open and $x \in G$ but $y \notin G$. Since X is pairwise $\mathcal{I}$-R_0, τ_2-$\mathcal{I}$-Cl($\{x\}$) $\subset G$. Then $X \backslash \tau_2$-$\mathcal{I}$-Cl($\{x\}$) is a τ_2-$\mathcal{I}$-open set containing the point y but not x. Consequently X is pairwise $\mathcal{I}$-T_1.

Theorem 29. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a pairwise $\mathcal{I}$-R_0 space. If for any x∈X, τ_i-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_j$-$\mathcal{I}$-Ker$(\{x\}) = \{x\}$, i, j =1,2 and i≠j, then (X, τ_i) is $\mathcal{I}$-T_i for i=1,2.

Proof. Suppose that $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_0 and for any point x∈X, τ_i-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_j$-$\mathcal{I}$-Ker$(\{x\}) = \{x\}$, where i,j = 1,2 and i≠j. By Theorem 3(ii), it follows that τ_i-$\mathcal{I}$-Cl$(\{x\}) \cap \tau_i$-$\mathcal{I}$-Cl$(\{x\}) = \{x\}$ where i=1,2. Therefore, τ_i-$\mathcal{I}$-Cl$(\{x\})= \{x\}$, where i=1,2. Hence each singletons is $\mathcal{I}$-closed in (X, τ_i), where i=1,2. Hence by Theorem 2, (X, τ_i) is $\mathcal{I}$-T_i for i=1,2.

Theorem 30. If an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-T_2, then it is pairwise $\mathcal{I}$-R_1.

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be pairwise $\mathcal{I}$-T_2. Then for any two distinct points x, y of X, their exist a τ_i-$\mathcal{I}$-open set U and a τ_j-$\mathcal{I}$-open set V such that x∈U, y∈V and U∩V=∅ where i,j=1,2 and i≠j. If $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-T_1, then $\{x\}$ = τ_j-$\mathcal{I}$-Cl$(\{x\})$ and $\{y\}$ = τ_i-$\mathcal{I}$-Cl$(\{y\})$ and thus τ_i-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_j$-$\mathcal{I}$-Cl$(\{y\})$ i,j=1,2 and i≠j. Thus, for any distinct pair of points x,y of X such that τ_j-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_j$-$\mathcal{I}$-Cl$(\{y\})$ where i,j=1,2 and i≠j, there exist a τ_i-$\mathcal{I}$-open set U and τ_j-$\mathcal{I}$-open set V such that x∈V, y∈U and U∩V=∅ where i,j=1,2 and i≠j. Hence $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-R_1.

The following example shows that the converse of Theorem 4 is not true in general.

Example 31. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset,\ \{c\},\ X\}$, $\tau_2 = \{\emptyset,\ \{a, c\},\ X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is a $(1, 2)$-$\mathcal{I}$-R_1 space but not a $(1, 2)$-$\mathcal{I}$-T_2 space.

Theorem 32. Every pairwise $\mathcal{I}$-T_1 pairwise $\mathcal{I}$-R_1 space is pairwise $\mathcal{I}$-R_2.

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be pairwise $\mathcal{I}$-T_1 and pairwise $\mathcal{I}$-R_1. Let x, y be two distinct points of X. Since X is pairwise $\mathcal{I}$-T_1, $\{x\}$ is τ_2-$\mathcal{I}$-closed and $\{y\}$ is τ_1-$\mathcal{I}$-closed. Hence τ_2-$\mathcal{I}$-Cl$(\{x\}) \neq \tau_1$-$\mathcal{I}$-Cl$(\{y\})$. Since X is pairwise $\mathcal{I}$-R_1, there exists a τ_1-$\mathcal{I}$-open set U and τ_2-$\mathcal{I}$-open set V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Hence X is pairwise $\mathcal{I}$-T_2.

Corollary 33. An ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-T_2 if and only if it is pairwise $\mathcal{I}$-T_1 and pairwise $\mathcal{I}$-R_1.

Conclusion

This paper introduces and develops some new separation axioms known as pairwise $\mathcal{I}$-R_0, pairwise $\mathcal{I}$-R_1, pairwise $\mathcal{I}$-T_0, pairwise $\mathcal{I}$-T_1 and pairwise $\mathcal{I}$-T_2. Consequently, separation axioms finds its application in the study of relations between various spaces. Although it is classified as pure mathematics, when converted into Bitopology, Fuzzy topology and Digital topology, it becomes application oriented around. Hence this paper will serve as the basis which leads to many applications in Science and Technology.

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