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ON $\mathcal{I}$-OPEN SETS AND $\mathcal{I}$-CONTINUOUS FUNCTIONS IN IDEAL MINIMAL SPACES

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Abstract. The aim of this paper is to introduce and characterize the concepts of $\mathcal{I}$-open sets and their related notions in ideal minimal spaces.

In [6], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of m-continuous functions as a function defined between a set with a minimal structure and a topological space. They showed that the m-continuous functions have properties similar to those of continuous functions between topological spaces. Let X be a topological space and $A \subset X$. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subfamily m of the power set $P(X)$ of a nonempty set X is called a minimal structure [6] on X if $\emptyset$ and X belong to m. By (X, m), we denote a nonempty set X with a minimal structure m on X. The members of the minimal structure m are called m-open sets [6], and the pair (X, m) is called an m-space. The complement of an m-open set is said to be m-closed [6]. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [1] and Vaidyanathasamy [8]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a minimal space (X, m) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*_m$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local minimal function [7] of A with respect to m and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*_m(m, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in m(x)\}$, where $m(x) = \{U \in m | x \in U\}$. The set operator $m\,\mathrm{Cl}^*(.)$ is called a minimal $*$-closure and is defined as $m\,\mathrm{Cl}^*(A) = A \cup A^*_m$ for $A \subset X$. The minimal structure $m^*(\mathcal{I}, m) = \{U \subset X \mid m\,\mathrm{Cl}^*(X \setminus U) = X \setminus U\}$ called the $*$-minimal structure is finer than m and $m\,\mathrm{Int}^*(A)$ denotes the m^*-interior of A in $m^*(m, \mathcal{I})$.

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1. Preliminaries

Definition 1.1 ([2]). *Given $A \subset X$, the m-interior of A and the m-closure of A are defined by $m\,\mathrm{Int}(A) = \cup\{W/W \in m, W \subset A\}$ and $m\,\mathrm{Cl}(A) = \cap\{F/A \subset F, X \setminus F \in m\}$, respectively.*

Theorem 1.2 ([2]). *Let (X, m) be an m-space, and A, B subsets of X. Then $x \in m\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \varnothing$ for every $U \in m$ containing x. And the following properties hold:*

(i) $m\,\mathrm{Cl}(m\,\mathrm{Cl}(A)) = m\,\mathrm{Cl}(A)$.
(ii) $m\,\mathrm{Int}(m\,\mathrm{Int}(A)) = m\,\mathrm{Int}(A)$.
(iii) $m\,\mathrm{Int}(X \setminus A) = X \setminus m\,\mathrm{Cl}(A)$.
(iv) $m\,\mathrm{Cl}(X \setminus A) = X \setminus m\,\mathrm{Int}(A)$.
(v) *If* $A \subset B$, *then* $m\,\mathrm{Cl}(A) \subset m\,\mathrm{Cl}(B)$.
(vi) $m\,\mathrm{Cl}(A \cup B) \supset m\,\mathrm{Cl}(A) \cup m\,\mathrm{Cl}(B)$.
(vii) $A \subset m\,\mathrm{Cl}(A)$ *and* $m\,\mathrm{Int}(A) \subset A$.

Definition 1.3. *A subset A of a minimal space (X, m) is said to be*

(i) *m-preopen* [3] *if* $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(A))$.
(ii) *m-semiclosed* [4] *if* $m\,\mathrm{Int}(m\,\mathrm{Cl}(A)) \subset A$

Definition 1.4. *A function $f : (X, m) \to (Y, \tau)$ is said to be m-precontinuous* [3] *if the inverse image of every open set of Y is m-preopen in (X, m).*

Lemma 1.5 ([7]). *Let $(X, m, \mathcal{I})$ be an ideal generalized space and A, B subsets of X. Then we have the following:*

(1) *If* $A \subset B$, *then* $A^*_m \subset B^*_m$.
(2) $A^*_m = m\,\mathrm{Cl}(A^*_m) \subset m\,\mathrm{Cl}(A)$.
(3) $(A^*_m)^*_m \subset A^*_m$.
(4) $(A \cup B)^*_m \subset A^*_m \cup B^*_m$.

2. m-$\mathcal{I}$-open sets

Definition 2.1. *A subset A of an ideal minimal space $(X, m, \mathcal{I})$ is said to be m-$\mathcal{I}$-open if $A \subset m\,\mathrm{Int}(A^*_m)$.*
The family of all m-$\mathcal{I}$-open subsets of $(X, m, \mathcal{I})$ is denoted by $\mathcal{I}O(X, m)$. The family of all m-$\mathcal{I}$-open sets of $(X, m, \mathcal{I})$ containing the point x is denoted by $m\mathcal{I}O(X, x)$.

Remark 2.2. *It is clear that m-$\mathcal{I}$-openness and m-openness are independent notions.*

Example 2.3. *Let $X = \{a, b, c\}$, $m = \{\emptyset, \{a\}, \{b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the set $\{a\}$ is m-open but not m-$\mathcal{I}$-open.*

Example 2.4. *Let $X = \{a, b, c\}$, $m = \{\emptyset, \{a\}, \{b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then $m\,\mathrm{Int}(\{a, b\}^*_m) = m\,\mathrm{Int}(X) = X \supset \{a, b\}$. Therefore, $\{a, b\}$ is an m-$\mathcal{I}$-open set but it is not m-open.*

Proposition 2.5. *Every m-$\mathcal{I}$-open set is m-preopen.*

Proof. Let A be an m-$\mathcal{I}$-open set. Then $A \subset m\,\mathrm{Int}(A^*_m) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(A))$. Therefore, A is m-preopen. □

The following example shows that the converse of Proposition 2.5 is not true in general.

Example 2.6. *Let $X = \{a, b, c\}$, $m = \{\emptyset, \{a\}, \{b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the set $\{a\}$ is m-preopen but not m-$\mathcal{I}$-open.*

Theorem 2.7. *For an ideal minimal space $(X, m, \mathcal{I})$ and $A \subset X$, we have:*

(1) *If $\mathcal{I} = \{\emptyset\}$, then $A^*_m(\mathcal{I}) = m\,\mathrm{Cl}(A)$ and hence each m-preopen set is m-$\mathcal{I}$-open set.*
(2) *If $\mathcal{I} = \mathcal{P}(X)$, then $A^*_m(\mathcal{I}) = \emptyset$ and hence A is m-$\mathcal{I}$-open if and only if $A = \emptyset$.*

Theorem 2.8. *For any m-$\mathcal{I}$-open set A of an ideal minimal space $(X, m, \mathcal{I})$, we have $A^*_m = (m\,\mathrm{Int}(A^*_m))^*_m$.*

Proof. Since A is m-$\mathcal{I}$-open, $A \subset m\,\mathrm{Int}(A^*_m)$. Then $A^*_m \subset (m\,\mathrm{Int}(A^*_m))^*_m$. Also we have $m\,\mathrm{Int}(A^*_m) \subset A^*_m$, $(m\,\mathrm{Int}(A^*_m))^* \subset (A^*_m)^* \subset A^*_m$. Hence we have, $A^*_m = (m\,\mathrm{Int}(A^*_m))^*_m$. □

Theorem 2.9. *If $\{U_\alpha : \alpha \in \Delta\} \subset \mathcal{I}O(X, m)$, then $\bigcup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{I}O(X, m)$.*

Proof. Since $\{U_\alpha : \alpha \in \Delta\} \subset \mathcal{I}O(X, m)$, then $U_\alpha \subset m\,\mathrm{Int}((U_\alpha)^*_m)$, for every $\alpha \in \Delta$. Thus, $\bigcup U_\alpha \subset \bigcup(m\,\mathrm{Int}((U_\alpha)^*_m)) \subset m\,\mathrm{Int}(\bigcup(U_\alpha)^*_m) \subset mInt((\bigcup U_\alpha)^*_m)$. Hence $\bigcup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{I}O(X, m)$. □

Theorem 2.10. *If $A \subset (X, m, \mathcal{I})$ is m-$\mathcal{I}$-open and m-semiclosed, then $A = m\,\mathrm{Int}(A^*_m)$.*

Proof. Given A is m-$\mathcal{I}$-open. Then $A \subset m\,\mathrm{Int}(A^*_m)$. Since A is m-semiclosed, by Lemma 1.5 $m\,\mathrm{Int}(A^*_m) \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(A)) \subset A$. Thus $m\,\mathrm{Int}(A^*_m) \subset A$. Hence we have, $A = m\,\mathrm{Int}(A^*_m)$. □

Definition 2.11. *A subset F of an ideal minimal space $(X, m, \mathcal{I})$ is called m-$\mathcal{I}$-closed if its complement is m-$\mathcal{I}$-open.*

Remark 2.12. *For $A \subset (X, m, \mathcal{I})$ we have $X \backslash (m\,\mathrm{Int}(A))^*_m \neq m\,\mathrm{Int}((X \backslash A)^*_m)$ in general.*

Example 2.13. *Let* $X = \{a, b, c\}$, $m = \{\emptyset, \{a\}, \{b\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then* $X\backslash(m\operatorname{Int}(\{a\}))_m = X\backslash\{a\}^*_m = X\backslash\emptyset = X$ $(*)$ *and* $m\operatorname{Int}((X\backslash\{a\})^*_m) = m\operatorname{Int}(\{b, c\}^*_m) = m\operatorname{Int}\{b, c\} = b$ $(**)$. *Hence from* $(*)$ *and* $(**)$, *we get* $X\backslash(m\operatorname{Int}(A))^*_m \neq m\operatorname{Int}((X\backslash A)^*_m)$.

Theorem 2.14. *If* $A \subset (X, m, \mathcal{I})$ *is* m-$\mathcal{I}$*-closed, then* $A \supset (m\operatorname{Int}(A))^*_m$.

Proof. Let A be m-$\mathcal{I}$-closed. Then $B = A^c$ is m-$\mathcal{I}$-open. Thus, by Lemma 1.5 $B \subset m\operatorname{Int}(B^*_m) \subset m\operatorname{Int}(m\operatorname{Cl}(B))$ and $B^c \supset m\operatorname{Cl}(m\operatorname{Int}(B^c))$. That is, $m\operatorname{Cl}(m\operatorname{Int}(A)) \subset A$, which implies that $(m\operatorname{Int}(A))^*_m \subset m\operatorname{Cl}(m\operatorname{Int}(A)) \subset A$. Therefore, $A \supset (m\operatorname{Int}(A))^*_m$. □

Theorem 2.15. *Let* $A \subset (X, m, \mathcal{I})$ *and* $(X\backslash(m\operatorname{Int}(A))^*_m) = m\operatorname{Int}((X\backslash A)^*_m)$. *Then* A *is* m-$\mathcal{I}$*-closed if and only if* $A \supset (m\operatorname{Int}(A))^*_m$.

Proof. It is obvious. □

Definition 2.16 ([7]). *A subset* A *of an ideal minimal space* $(X, m, \mathcal{I})$ *is said to be:*

(i) $m*$*-closed if* $A^*_m \subset A$.
(ii) $m*$*-perfect if* $A^*_m = A$.

Theorem 2.17. *For a subset* $A \subset (X, m, \mathcal{I})$, *we have*

(i) *If* A *is* $m*$*-closed and* $A \in \mathcal{I}O(X, m)$, *then* $m\operatorname{Int}(A) = m\operatorname{Int}(A^*_m)$.
(ii) *If* A *is* $m*$*-perfect, then* $A = m\operatorname{Int}(A^*_m)$ *for every* $A \in \mathcal{I}O(X, m)$.

Proof. (i) Since A is $m*$-closed and $A \in \mathcal{I}O(X, m)$, $A^*_m \subset A$ and $A \subset m\operatorname{Int}(A^*_m)$. Then $A \subset m\operatorname{Int}(A^*_m)$ and $m\operatorname{Int}(A) \subset m\operatorname{Int}(m\operatorname{Int}(A^*_m)) \subset m\operatorname{Int}(A^*_m)$. Also, $A^*_m \subset A$. Then $m\operatorname{Int}(A^*_m) \subset m\operatorname{Int}(A)$. Hence $m\operatorname{Int}(A) = m\operatorname{Int}(A^*_m)$.
(ii) Let A be $m*$-perfect and $A \in \mathcal{I}O(X, m)$. We have, $A^*_m = A$, $m\operatorname{Int}(A^*_m) = m\operatorname{Int}(A)$, $m\operatorname{Int}(A^*_m) \subset A$. Also we have $A \subset m\operatorname{Int}(A^*_m)$. Hence we have, $A = m\operatorname{Int}(A^*_m)$. □

Definition 2.18. *Let* $(X, m, \mathcal{I})$ *be an ideal minimal space,* S *a subset of* X *and* x *a point of* X. *Then*

(i) x *is called an* m-$\mathcal{I}$*-interior point of* S *if there exists* $V \in \mathcal{I}O(X, m)$ *such that* $x \in V \subset S$.
ii) *the set of all* m-$\mathcal{I}$*-interior points of* S *is called the* m-$\mathcal{I}$*-interior of* S *and is denoted by* $m\mathcal{I}\operatorname{Int}(S)$.

Theorem 2.19. *Let* A *and* B *be subsets of* $(X, m, \mathcal{I})$. *Then the following properties hold:*

(i) $m\mathcal{I}\operatorname{Int}(A) = \cup\{T : T \subset A$ *and* $T \in \mathcal{I}O(X, m)\}$.
(ii) $m\mathcal{I}\operatorname{Int}(A)$ *is the largest* m-$\mathcal{I}$*-open subset of* X *contained in* A.
(iii) A *is* m-$\mathcal{I}$*-open if and only if* $A = m\mathcal{I}\operatorname{Int}(A)$.

(iv) $m\mathcal{I}\,\mathrm{Int}(m\mathcal{I}\,\mathrm{Int}(A)) = m\mathcal{I}\,\mathrm{Int}(A)$.
(v) *If* $A \subset B$*, then* $m\mathcal{I}\,\mathrm{Int}(A) \subset m\mathcal{I}\,\mathrm{Int}(B)$.
(vi) $m\mathcal{I}\,\mathrm{Int}(A) \cup m\mathcal{I}\,\mathrm{Int}(B) \subset m\mathcal{I}\,\mathrm{Int}(A \cup B)$.
(vii) $m\mathcal{I}\,\mathrm{Int}(A \cap B) \subset m\mathcal{I}\,\mathrm{Int}(A) \cap m\mathcal{I}\,\mathrm{Int}(B)$.

Proof. (i). Let $x \in \cup\{T : T \subset A \text{ and } T \in \mathcal{I}O(X,m)\}$. Then, there exists $T \in m\mathcal{I}O(X,x)$ such that $x \in T \subset A$ and hence $x \in m\mathcal{I}\,\mathrm{Int}(A)$. This shows that $\cup\{T : T \subset A \text{ and } T \in \mathcal{I}O(X,m)\} \subset m\mathcal{I}\,\mathrm{Int}(A)$. For the reverse inclusion, let $x \in m\mathcal{I}\,\mathrm{Int}(A)$. Then there exists $T \in m\mathcal{I}O(X,x)$ such that $x \in T \subset A$. We obtain $x \in \cup\{T : T \subset A \text{ and } T \in \mathcal{I}O(X,m)\}$. Then $m\mathcal{I}\,\mathrm{Int}(A) \subset \cup\{T : T \subset A \text{ and } T \in \mathcal{I}O(X,m)\}$. Therefore, we obtain $m\mathcal{I}\,\mathrm{Int}(A) = \cup\{T : T \subset A \text{ and } T \in \mathcal{I}O(X,m)\}$.
The proofs of (ii)-(v) are obvious.
(vi). Clearly, $m\,\mathrm{Int}(A) \subset m\,\mathrm{Int}(A \cup B)$ and $m\,\mathrm{Int}(B) \subset m\,\mathrm{Int}(A \cup B)$. Then we obtain $m\,\mathrm{Int}(A) \cup m\,\mathrm{Int}(B) \subset m\,\mathrm{Int}(A \cup B)$.
(vii). Since $A \cap B \subset A$ and $A \cap B \subset B$, by (v), we have $m\,\mathrm{Int}(A \cap B) \subset m\,\mathrm{Int}(A)$ and $m\,\mathrm{Int}(A \cap B) \subset m\,\mathrm{Int}(B)$. Then $m\,\mathrm{Int}(A \cap B) \subset m\,\mathrm{Int}(A) \cap m\,\mathrm{Int}(B)$. □

Definition 2.20. *Let* $(X, m, \mathcal{I})$ *be an ideal minimal space,* S *a subset of* X *and* x *a point of* X*. Then*
(i) x *is called an m-*$\mathcal{I}$*-cluster point of* S *if* $V \cap S \neq \emptyset$ *for every* $V \in m\mathcal{I}O(X,x)$.
(ii) *the set of all m-*$\mathcal{I}$*-cluster points of* S *is called the m-*$\mathcal{I}$*-closure of* S *and is denoted by* $m\mathcal{I}\,\mathrm{Cl}(S)$.

Theorem 2.21. *Let* $(X, m, \mathcal{I})$ *be an ideal minimal space and* $A \subset X$*. A point* $x \in m\mathcal{I}\,\mathrm{Cl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m\mathcal{I}O(X,x)$.

Proof. It follows easily from Definition 2.20. □

Theorem 2.22. *Let* $(X, m, \mathcal{I})$ *be an ideal minimal space and* $A \subset X$*. Then the following propeties hold:*
(i) $m\mathcal{I}\,\mathrm{Int}(X \backslash A) = X \backslash m\mathcal{I}\,\mathrm{Cl}(A)$;
(ii) $m\mathcal{I}\,\mathrm{Cl}(X \backslash A) = X \backslash m\mathcal{I}\,\mathrm{Int}(A)$.

Proof. (i) Let $x \notin m\mathcal{I}\,\mathrm{Cl}(A)$. There exists $V \in m\mathcal{I}O(X,x)$ such that $V \cap A = \emptyset$; hence $x \in V \subset X \setminus A$. Thus, we obtain $x \in m\mathcal{I}\,\mathrm{Int}(X \backslash A)$. This shows that $X \backslash m\mathcal{I}\,\mathrm{Cl}(A) \subset m\mathcal{I}\,\mathrm{Int}(X \backslash A)$. Let $x \in m\mathcal{I}\,\mathrm{Int}(X \backslash A)$. Since $m\mathcal{I}\,\mathrm{Int}(X \backslash A) \cap A = \emptyset$, we obtain $x \notin m\mathcal{I}\,\mathrm{Cl}(A)$; hence $x \in X \backslash m\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain $m\mathcal{I}\,\mathrm{Int}(X \backslash A) = X \backslash m\mathcal{I}\,\mathrm{Cl}(A)$.
(ii) follows from (i). □

Theorem 2.23. *Let* A *and* B *be subsets of* $(X, m, \mathcal{I})$*. Then the following properties hold:*
(i) $m\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F \text{ and } F \in \mathcal{I}C(X,m)\}$.

(ii) $m\mathcal{I}\,\mathrm{Cl}(A)$ *is the smallest* m-$\mathcal{I}$*-closed subset of* X *containing* A.
(iii) A *is* m-$\mathcal{I}$*-closed if and only if* $A = m\mathcal{I}\,\mathrm{Cl}(A)$.
(iv) $m\mathcal{I}\,\mathrm{Cl}(m\mathcal{I}\,\mathrm{Cl}(A)) = m\mathcal{I}\,\mathrm{Cl}(A)$.
(v) *If* $A \subset B$, *then* $m\mathcal{I}\,\mathrm{Cl}(A) \subset m\mathcal{I}\,\mathrm{Cl}(B)$.
(vi) $m\mathcal{I}\,\mathrm{Cl}(A \cup B) \supset m\mathcal{I}\,\mathrm{Cl}(A) \cup m\mathcal{I}\,\mathrm{Cl}(B)$.
(vii) $m\mathcal{I}\,\mathrm{Cl}(A \cap B) \subset m\mathcal{I}\,\mathrm{Cl}(A) \cap m\mathcal{I}\,\mathrm{Cl}(B)$.

Proof. The proofs follows from Theorems 2.19 and 2.22. □

Definition 2.24. *A subset* B_x *of an ideal minimal space* $(X, m, \mathcal{I})$ *is said to be an* m-$\mathcal{I}$*-neighbourhood of a point* $x \in X$ *if there exists an* m-$\mathcal{I}$*-open set* U *such that* $x \in U \subset B_x$.

Theorem 2.25. *A subset of an ideal minimal space* $(X, m, \mathcal{I})$ *is* m-$\mathcal{I}$*-open if and only if it is an* m-$\mathcal{I}$*-neighbourhood of each of its points.*

Proof. Let G be an m-$\mathcal{I}$-open set of X. Then by definition, it is clear that G is an m-$\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is m-$\mathcal{I}$-open. Conversely, suppose G is an m-$\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in \mathcal{I}O(X, m)$ such that $S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each S_x is m-$\mathcal{I}$-open and an arbitrary union of m-$\mathcal{I}$-open sets is m-$\mathcal{I}$-open, G is m-$\mathcal{I}$-open in $(X, m, \mathcal{I})$. □

3. m-$\mathcal{I}$-CONTINUOUS FUNCTIONS

Definition 3.1. *A function* $f : (X, m, \mathcal{I}) \to (Y, \tau)$ *is said to be* m-$\mathcal{I}$*-continuous if for every* $V \in \tau$, $f^{-1}(V) \in \mathcal{I}O(X, m)$.

Remark 3.2. *Every* m-$\mathcal{I}$*-continuous function is* m*-precontinuous but the converse is not true,in general.*

Example 3.3. *Let* $X = \{a, b, c\}$, $m = \{\emptyset, \{a\}, \{b\}, X\}$, $\tau = \{\emptyset, \{a\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then the identity function* $f : (X, m, \mathcal{I}) \to (X, \tau)$ *is* m*-precontinuous but not* m-$\mathcal{I}$*-continuous.*

Remark 3.4. *It is clear that* m-$\mathcal{I}$*-continuity and* m*-continuity are independent notions.*

Example 3.5. *Let* $(X, m, \mathcal{I})$ *be the ideal minimal space in Example 2.3,* $\tau = \{\emptyset, \{a\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then the identity function* $f : (X, m, \mathcal{I}) \to (X, \tau)$ *is* m*-continuous but not* m-$\mathcal{I}$*-continuous.*

Example 3.6. *Let* $(X, m, \mathcal{I})$ *be the ideal minimal space in Example 2.4 and* $\tau = \{\emptyset, \{a\}, X\}$. *Then the identity function* $f : (X, m, \mathcal{I}) \to (X, \tau)$ *is* m-$\mathcal{I}$*-continuous but not* m*-continuous.*

Theorem 3.7. *For a function $f : (X, m, \mathcal{I}) \to (Y, \tau)$, the following statements are equivalent:*

(i) *f is m-$\mathcal{I}$-continuous;*
(ii) *For each point x in X and each open set F of Y such that $f(x) \in F$, there is an m-$\mathcal{I}$-open set A in X such that $x \in A$, $f(A) \subset F$;*
(iii) *The inverse image of each closed set of Y is m-$\mathcal{I}$-closed in X;*
(iv) *For each subset A of X, $f(m\mathcal{I}\,\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A))$;*
(v) *For each subset B of Y, $m\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$;*
(vi) *For each subset C of Y, $f^{-1}(\mathrm{Int}(C)) \subset m\mathcal{I}\,\mathrm{Int}(f^{-1}(C))$.*

Proof. The proof is clear. □

Definition 3.8. *The graph $G(f)$ of a function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is said to be m-$\mathcal{I}$-closed in $X \times Y$ if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in mS\mathcal{I}O(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 3.9. *The graph $G(f)$ of a function $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is m-$\mathcal{I}$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in m\mathcal{I}O(X, x)$ and an open set V of Y containing y such that $f(U) \cap V = \emptyset$.*

Proof. The proof is an immediate consequence of Definition 3.8. □

Theorem 3.10. *If $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is an m-$\mathcal{I}$-continuous function and (Y, τ) is T_2, then $G(f)$ is m-$\mathcal{I}$-closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since Y is T_2, there exist disjoint open sets V, W of Y such that $f(x) \in W$ and $y \in V$. Since f is m-$\mathcal{I}$-continuous, there exists $U \in m\mathcal{I}O(X, x)$ such that $f(U) \subset W$. Therefore, $f(U) \cap V = \emptyset$. Therefore, by Lemma 3.9, $G(f)$ is m-$\mathcal{I}$-closed. □

Definition 3.11. *An ideal minimal space $(X, m, \mathcal{I})$ is called an m-$\mathcal{I}$-T_2 space if for each pair of distinct points $x, y \in X$, there exist $U, V \in m\mathcal{I}O(X)$ containing x and y, respectively, such that $U \cap V = \emptyset$.*

Theorem 3.12. *If $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is an m-$\mathcal{I}$-continuous injective function and Y is a T_2 space, then $(X, m, \mathcal{I})$ is an m-$\mathcal{I}$-T_2 space.*

Proof. The proof follows from the definitions. □

Theorem 3.13. *If $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is an injective m-$\mathcal{I}$-continuous function with an m-$\mathcal{I}$-closed graph, then X is an m-$\mathcal{I}$-T_2 space.*

Proof. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. Since the graph $G(f)$ is m-$\mathcal{I}$-closed, there exist an m-$\mathcal{I}$-open set U containing x_1 and $V \in \tau$ containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since f is m-$\mathcal{I}$-continuous, $f^{-1}(V)$ is an m-$\mathcal{I}$-open set containing x_2 such that $U \cap f^{-1}(V) = \emptyset$. Hence X is m-$\mathcal{I}$-T_2. □

Definition 3.14. *An ideal minimal space $(X, m, \mathcal{I})$ is said to be m-$\mathcal{I}$-connected if X cannot be expressed as the union of two nonempty disjoint m-$\mathcal{I}$-open sets.*

Theorem 3.15. *A m-$\mathcal{I}$-continuous image of an m-$\mathcal{I}$-connected space is connected.*

Proof. The proof is clear. □

Lemma 3.16 ([5]). *For any function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, $f(\mathcal{I})$ is an ideal on Y.*

Definition 3.17. *A subset K of an ideal minimal space $(X, m, \mathcal{I})$ is said to be m-$\mathcal{I}$-compact relative to X, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of K by m-$\mathcal{I}$-open sets of X, there exists a finite subset Λ_0 of Λ such that $K \setminus \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X, m, \mathcal{I})$ is said to be m-$\mathcal{I}$-compact if X is an m-$\mathcal{I}$-compact subset of X.*

Definition 3.18. *A subset K of an ideal minimal space $(X, m, \mathcal{I})$ is said to be countably m-$\mathcal{I}$-compact relative to X, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of K by countable m-$\mathcal{I}$-open sets of X, there exists a finite subset Λ_0 of Λ such that $K \setminus \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X, m, \mathcal{I})$ is said to be countably m-$\mathcal{I}$-compact if X is a countably m-$\mathcal{I}$-compact subset of X.*

Definition 3.19. *A subset K of an ideal minimal space $(X, m, \mathcal{I})$ is said to be m-$\mathcal{I}$-Lindelöf relative to X, if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of K by m-$\mathcal{I}$-open sets of X, there exists a countable subset Λ_0 of Λ such that $K \setminus \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{I}$. The space $(X, m, \mathcal{I})$ is said to be m-$\mathcal{I}$-Lindelöf if X is an m-$\mathcal{I}$-Lindelöf subset of X.*

Theorem 3.20. *If $f : (X, m, \mathcal{I}) \to (Y, \tau)$ is an m-$\mathcal{I}$-continuous surjection and $(X, m, \mathcal{I})$ is m-$\mathcal{I}$-compact, then $(Y, \tau, f(\mathcal{I}))$ is $f(\mathcal{I})$-compact.*

Proof. Let $\{V_\lambda : \lambda \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is an m-$\mathcal{I}$-open cover of X and hence, there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup\{f^{-1}(V_\lambda) : \lambda \in \Lambda_0\} \in \mathcal{I}$. Since f is surjective, $Y \setminus \bigcup\{V_\lambda : \lambda \in \Lambda_0\} = f(X \setminus \bigcup\{f^{-1}(V_\lambda) : \lambda \in \Lambda_0\}) \in f(\mathcal{I})$. Therefore, $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$-compact. □

Theorem 3.21. *If $f : (X, m, \mathcal{I}) \to (Y, \sigma)$ is an m-$\mathcal{I}$-continuous surjection and $(X, m, \mathcal{I})$ is m-$\mathcal{I}$-Lindelöf, then $(Y, \sigma, f(\mathcal{I}))$ is $f(\mathcal{I})$-Lindelöf.*

Proof. The proof is similar to the previous theorem. □

Theorem 3.22. *If $f : (X, m, \mathcal{I}) \to (Y, \sigma)$ is an m-$\mathcal{I}$-continuous surjection and $(X, m, \mathcal{I})$ is countably m-$\mathcal{I}$-compact, then $(Y, \sigma, f(\mathcal{I}))$ is countably $f(\mathcal{I})$-compact.*

Proof. The proof is similar to the previous theorem. □

We close with the following:

Are there proper examples showing the relationships of m-$\mathcal{I}$-compactness and m-compactness, countably m-$\mathcal{I}$-compactness and countably m-compactness, and m-$\mathcal{I}$-Lindelöfness and m-Lindelöfness?

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References

[1] K. Kuratowski, Topology, *Academic Press, New York*, (1966).

[2] H. Maki, K. C. Rao, A. Nagoor Gani, On generalizing semi-open and preopen sets, *Pure Appl. Math. Sci.*, **49(1999)**, 17-29.

[3] W. K. Min and Y. K. Kim, m-preopen sets and m-precontinuity on spaces with minimal structures, *Advances in Fuzzy Sets and Systems*, **4 (3) (2009)**, 237-245.

[4] W. K. Min, m-semiopen sets and M-semicontinuous functions on spaces with minimal structures, *Honam Math. J.*, **31 (2) (2009)**, 239-245.

[5] R. L. Newcomb, Topologies which are compact modulo an ideal, Ph.D. Thesis, University of California, USA (1967).

[6] V. Popa and T. Noiri, On the definitions of some generalized forms of continuity under minimal conditions, *Mem. Fac. Sci. Kochi. Univ. Ser. Math.*, **22 (2001)**, 9-18.

[7] O. B. Ozbakir and E. D. Yildirim, On some closed sets in ideal minimal spaces, *Acta Math. Hungar.*, **125(3) (2009)**, 227-235.

[8] R. Vaidyanathaswamy, The localisation theory in set topology, *Proc. Indian Acad. Sci.*, **20**(1945), 51-61.

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