ON A CONNECTED $T_{1/2}$ ALEXANDROFF TOPOLOGY AND $^*g\hat{\alpha}$-CLOSED SETS IN DIGITAL PLANE

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Abstract. The Khalimsky topology plays a significant role in the digital image processing. In this paper we define a topology κ_1 on the set of integers generated by the triplets of the form $\{2n, 2n+1, 2n+3\}$. We show that in this space $(\mathbb{Z}, \kappa_1)$, every point has a smallest neighborhood and hence this is an Alexandroff space. This topology is homeomorphic to Khalimskt topology. We prove, among others, that this space is connected and $T_{3/4}$. Moreover, we introduce the concept of $^*g\hat{\alpha}$-closed sets in a topological space and characterize it using $^*g\alpha o$-kernel and closure. We investigate the properties of $^*g\hat{\alpha}$-closed sets in digital plane. The family of all $^*g\hat{\alpha}$-open sets of $(\mathbb{Z}^2, \kappa^2)$, forms an alternative topology of $\mathbb{Z}^2$. We prove that this plane $(\mathbb{Z}^2, ^* g\hat{\alpha}O)$ is $T_{1/2}$. It is well known that the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$, even if $(\mathbb{Z}, \kappa)$ is $T_{1/2}$.

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1. Introduction

Digital topology is associated with the topological properties of digital images. The results in digital topology provide a mathematical foundation for image processing operations such as connected component labeling and counting, contour filling and thinning. The information required for a digital picture can be stored by specifying the colour at each pixel. If a digital picture is formed by simple closed curve, one can specify the pixels on the simple closed curves and then specify uniformly the colours for the insides and the outside. This method results in the reduction of computer memory usage significantly. This method employs the celebrated Jordan curve theorem, which states that every simple closed curve in the plane separates the plane into two connected components. A computer screen can be regarded as a finite rectangular array of lattice points and being a finite space, it components and hence no Jordan curve theorem. Therefore researchers, who applied topology to computer science, looked for connected non-T_1 topology on the set of integers. E. Khalimsky [17] defined a topology on set of integers which is generated by the triplets of the form $\{2n-1, 2n, 2n+1\}$ as subspace. Since this topology has been widely used in computer graphics. This topology is called the Khalimsky topology. In this paper, we define a topology κ_1 on the set of integers generated by the triplets of the form $\{2n, 2n+1, 2n+3\}$. We show that in this space $(\mathbb{Z}, \kappa_1)$, every point has a smallest neighbourhood and hence this is an Alexandroff space. This topology is homeomorphic to Khalimsky topology. We prove that this space is connected and $T_{3/4}$. Further, we also investigate some properties of this topology.In 1970, N. Levine [20] introduced and investigated the concept of generalized closed sets in a topological space. He studied most fundamental properties and also introduced a separation axiom $T_{1/2}$. The digital line is typical example of a $T_{1/2}$ space [10]. After Levine's works, many authors defined and investigated various notions to Levine's g-closed sets and related topics [7]. In 1970, E. Khalimsky [18] introduced digital line. In 1990, K. Kopperman and R. Meyer [17] developed finite analog of the Jordan curve theorem motivated by a problem in computer graphics (cf. [17, 19]). In this paper, we introduce the concept of $^*g\hat{\alpha}$-closed sets in a topological space and characterize it using $^*g\alpha o$-kernel and closure. Moreover, we investigate the properties of $^*g\hat{\alpha}$-closed sets in digital plane. We prove that this plane $(\mathbb{Z}^2, ^*g\hat{\alpha}O)$ is $T_{1/2}$. It is well known that the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$, even if $(\mathbb{Z}, \kappa)$ is $T_{1/2}$.

2. Preliminaries

M. H. Stone [28] introduced regular open sets and regular closed sets in 1937. Interior and closure operators play an important role in the definition of nearly open sets. In 1963, N. Levine [21] defined semi-open sets and introduced the concept of semi-continuity. In 1965, O. Njastad [27] studied some classes of nearly open sets. D. Andrijevic [3] defined and investigated semi-preopen sets in 1986. N. Levine [20] defined and studied $T_{1/2}$ spaces. Bhattacharya and Lahiri [6] introduced and studied semi-$T_{1/2}$ spaces using the semi-open sets defined by Levine. Dunham ([14], [15]) obtained some characterizations of $T_{1/2}$-spaces and semi-$T_{1/2}$ spaces respectively. P. Thangavelu [29] investigated some properties of subspace topologies of the Khalimsky topology. In this sequel, $int(A)$, $cl(A)$ and A^c respectively denote the interior, closure and the complement of the subset A of X. A topological space X is said to be an Alexandroff space if every point in X has a smallest neighbourhood, equivalently if arbitrary intersection of open set is open. A subset A of X pointwise dense if $X = \bigcup\{cl(\{x\}) | x \in A$ and $\{x\}$ is open$\}$. A space X is extremely disconnected if the closure of every open set is open. A space X is a door space if every subset of X is either open or closed. A space X is locally finite if each point in X lies in a finite open set and in a finite closed set. The derived set of a subset S of a topological space is the set of all limit points of S and boundary of S is defined by $bd(S) = cl(S) \setminus int(S)$. A space (X, τ) is said to be locally indiscrete [2] (also called a partition space) if and only if $cl(\{x\}) \in \tau$ for each $x \in X$. In this paper a set with nowhere dense boundary is called an AN-set. A topological space (X, τ) is submaximal if every dense subset is open.

Definition 2.1. *A subset A of a topological space (X, τ) is called*
(1) *α-open* [27] *if $A \subseteq int(cl(int(A)))$,*
(2) *semi-open* [21] *if $A \subseteq cl(int(A))$,*
(3) *preopen* [24] *if $A \subseteq int(cl(A))$,*
(4) *β-open* [1] *or semi-preopen* [3] *if $A \subseteq cl(int(cl(A)))$,*
(5) *regular open* [28] *if $A = int(cl(A))$.*

Moreover, A is said to be α-closed (resp. semi-closed, preclosed, β-closed or semi-preclosed and regular closed) if $X \backslash A$ is α-open (resp. semi-open, preopen, β-open or semi-preopen and regular open). The collection of all α-open subsets in (X, τ) is denoted by τ^α. The α-closure (resp. semi closure, preclosure and semi-preclosure) of a subset A is the smallest α-closed (resp. semi-closed, preclosed and semi-preclosed) sets containing A and this is denoted by τ^α-$cl(A)$ (resp. $scl(A)$, $pcl(A)$ and $spcl(A)$) in the present paper.

Definition 2.2. *A subset A of a topological space (X, τ) is called*
(1) *g-closed* [20] *if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an open set in (X, τ),*
(2) *sg-closed* [6] *if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is an semi-open set in (X, τ).*

Moreover, A is said to be g-open (res. sg-open) if $X \backslash A$ is g-closed (res. sg-closed).

Definition 2.3. A space (X, τ) is
(1) $T_{1/2}$ [20] if every g-closed set is closed. Equivalently, X is $T_{1/2}$ [15] if every singleton in X is either open or closed,
(2) T_0 if for each pair of distinct points x and y in X, there exists an open set containing one of them but not the other.
(3) semi-$T_{1/2}$ [11] if every sg-closed set is semi-closed. Equivalently , X is semi-$T_{1/2}$ if every singleton in X is either semi-open or semi-closed,
(4) R_0 [25] if $cl(\{x\}) \subseteq U$ whenever $x \in U$ and U is open.
(5) $T_{3/4}$ [10] if and only if every singleton $\{x\}$ in X is closed or regular open in (X, τ).
(6) semi-regular [12] if and only if for each semi-closed set A and $x \notin A$, there exist disjoint semi-open sets U and V such that $x \in U$ and $A \subseteq V$,
(7) semi-normal [13] if for every pair of disjoint semi-closed subsets F_1 and F_2 of X, there exist disjoint semi-open sets U and V such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

In an Alexandroff space X, $N(x)$ is the smallest neighborhood of x and $N[x]$ is the smallest closed set containing x. If B is a subset of X, then $N(B) = \bigcup\{N(x) : x \in B\}$ and $N[B] = \bigcup\{N[x] : x \in B\}$. Two distinct points x and y in X are said to be *adjacent* if the subspace $\{x, y\}$ is connected. Equivalently x and y are adjacent if and only if $y \in N(x)$ or $x \in N(y)$. But it is shown that $x \in N(y)$ is equivalent to $y \in N[x]$. Therefore x and y are *adjacent* if and only if $y \in N(x) \cup N[x]$. The *adjacency set* of a point x in X is the set of all points in X which are *adjacent* to x. It is denoted by $\mathcal{A}(x)$. Thus $\mathcal{A}(x) = N(x) \cup N[x] \setminus \{x\}$. The point adjacent to x is referred to as a *neighbor* of x. If B is subset of X, then (B) is the set of points not in B, but adjacent to some point in B. Thus $\mathcal{A}(B) = N(B) \cup N[B] \setminus B$. A point x in X is called *open* if the set $\{x\}$ is open and it is called *closed* if the set $\{x\}$ is closed.

3. A new topology on the set of integers

We introduce a new digital line is the set of the integers $\mathbb{Z}$, equipped with the topology κ_1 having $\mathcal{S} = \{\{2n, 2n+1, 2n+3\} | n \in \mathbb{Z}\}$ as a subbase. This is denoted by $(\mathbb{Z}, \kappa_1)$. Let $A, B \in \mathcal{S}$. Now, we can choose $A = \{2n, 2n+1, 2n+3\}, B = \{2m, 2m+1, 2m+3\}$, where $m, n \in \mathbb{Z}$. Then

$$A \cap B = \begin{cases} A & \text{if } m = n, \\ \{2n+3\} & \text{if } m = n+1 \\ \{2n+1\} & \text{if } m = n-1 \\ \emptyset & \text{otherwise} \end{cases}$$

Result 3.1. (1) *The smallest open set containing $x \in \mathbb{Z}$ is*

$$N(x) = \begin{cases} \{x\} & \textit{if } x \textit{ is odd} \\ \{x, x+1, x+3\} & \textit{if } x \textit{ is even} \end{cases}$$

(2) *The smallest closed set containing $x \in \mathbb{Z}$ is*

$$N[x] = \begin{cases} \{x\} & \textit{if } x \textit{ is even} \\ \{x-3, x-1, x\} & \textit{if } x \textit{ is odd} \end{cases}$$

Remark 3.2. .

(1) *Since every point in $(\mathbb{Z}, \kappa_1)$ has a smallest neighborhood, $(\mathbb{Z}, \kappa_1)$ is an Alexandroff space.*
(2) *If A is a subset of an Alexandroff space, then*

$$int(A) = \bigcup_{x \in A} N(x).$$

Hence A is open if and only if $N(x) \subseteq A$ for every $x \in A$.

(3) *If A is a subset of an Alexendroff space, then*
$$cl(A) = \bigcup_{x \in A} N[x].$$
Hence A is closed if and only if $N[x] \subseteq A$ for every $x \in A$.

Lemma 3.3. *A subset A of $(\mathbb{Z}, \kappa_1)$ is open if and only if $2m+1$ and $2m+3 \in A$ whenever $2m \in A$.*

Proof. **Necessity:** Let $2m \in A$. Since A is open, $N(2m) = \{2m, 2m+1, 2m+3\} \subseteq A$. **Sufficiency:** To prove A is open it is enough to prove that $A \subseteq int(A)$. Let $x \in A$. **Case 1:** $x = 2m$. By Hypothesis $2m+1$ and $2m+3 \in A$ and therefore $N(2m) \subseteq A \Rightarrow x \in int(A)$. **Case 2:** $x = 2m+1$. Since {2m+1} is an open subset of $\mathbb{Z}$, $x \in int(A)$. In both the cases $A \subseteq int(A)$. ☐

Lemma 3.4. *A subset A of $(\mathbb{Z}, \kappa_1)$ is closed if and only if $2n, 2n-2 \in A$ whenever $2n+1 \in A$.*

Proof. **Necessity:** Suppose A is closed. Then by Remark 3.2(3), for every $x \in A$, $N[x] \subseteq A$. If $x = 2n+1$, $N[x] = \{2n-2, 2n, 2n+1\} \subseteq A$. Therefore $2n-2, 2n \in A$. **Sufficiency:** Let $2m+1 \in A$. By assumption $2m$ and $2m-2 \in A$. i.e., for every $x \in A$, $N[x] \subseteq A$. Then by Remark 3.2(3), A is closed in $(\mathbb{Z}, \kappa_1)$. ☐

Theorem 3.5.
1. *If A is set of odd integers, then A is open in $(\mathbb{Z}, \kappa_1)$.*
2. *If B is set of even integers, then B is closed and nowhere dense in $(\mathbb{Z}, \kappa_1)$.*

Proof. **(1)** Let A be set of odd integers. If $x \in A$, then $\{x\}$ is open in $(\mathbb{Z}, \kappa_1)$. Therefore $A = \bigcup\{\{x\} : x \in A\}$ is open.
(2) Let B be set of even integers. If $x \in B$, then $\{x\}$ is closed in $(\mathbb{Z}, \kappa_1)$ and in an Alexondroff space arbitrary union of closed sets is closed. Therefore $B = \bigcup\{\{x\} : x \in B\}$ is closed. $cl(B) = B$, $int(cl(B)) = \emptyset$. Therefore B is nowhere dense. ☐

Result 3.6. *The rare sets and nowhere dense sets coincide in $(\mathbb{Z}, \kappa_1)$.*

Proof. Let A be a rare set in $(\mathbb{Z}, \kappa_1)$. Then $int(A) = \emptyset$. Which implies that A can not contain any odd integers and hence it is a set of even integers. Therefore by Theorem 3.5(2), A is nowhere dense in $(\mathbb{Z}, \kappa_1)$. We know that every nowhere dense set is rare. Thus, the rare sets and nowhere dense sets coincide in $(\mathbb{Z}, \kappa_1)$. ☐

Remark 3.7. *Every regular open set is semi-closed. The same is true for nowhere dense set.*

Result 3.8. .

(1) *If A is a set of even integers, then A is an AN-set.*
(2) *If A is open, then A is an AN-set.*

Proof. **(1)** Let A be set of even integers. Then A is closed and $int(A) = \emptyset$. Now,

$$\begin{aligned} bd(A) &= cl(A) \setminus int(A) = A \\ cl(bd(A)) &= A \\ int(cl(bd(A))) &= \emptyset \end{aligned}$$

Therefore, A is an AN-set.
(2) Let A is a open set. Then

$$\begin{aligned} bd(A) &= cl(A) \setminus A \\ cl(bd(A)) &= bd(A) \\ int(cl(bd(A))) &= int(bd(A)) = \emptyset \end{aligned}$$

Therefore, A is an AN-set. □

Remark 3.9. *Hence every semi-open set is an AN-set (see Theorem 2.1 in* [2]*).*

Theorem 3.10. *The dense subsets of $(\mathbb{Z}, \kappa_1)$ are precisely supersets of the set of odd integers.*

Proof. Let A be a dense subset of $(\mathbb{Z}, \kappa_1)$. Then, A must contain every odd integers. Suppose, A does not contain an odd integer x such, then $\{x\}$ is an open set that does intersect A. Therefore, $x \notin cl(A)$. Hence, $cl(A) \neq \mathbb{Z}$. This contradiction proves the theorem. □

Theorem 3.11. *$(\mathbb{Z}, \kappa_1)$ is submaximal.*

Proof. Suppose A is dense in $(\mathbb{Z}, \kappa_1)$. Then by Theorem 3.10, A contains every odd integer. The complement of A is a set of even integers and hence by Theorem 3.5(2), $X \setminus A$ is closed in $(\mathbb{Z}, \kappa_1)$. Therefore, A is open in $(\mathbb{Z}, \kappa_1)$. Thus every dense subset of $(\mathbb{Z}, \kappa_1)$ is open. Hence, $(\mathbb{Z}, \kappa_1)$ is submaximal. □

Theorem 3.12. *$(\mathbb{Z}, \kappa_1)$ is connected.*

Proof. Let $A \subseteq \mathbb{Z}$ be a nonempty set which is both open and closed in $(\mathbb{Z}, \kappa_1)$. Fix $x \in A$. **Case 1:** x is odd. Since A is closed, $x - 3$, $x - 1 \in A$. Since $x - 3$, $x - 1$ are even and A is open, $x - 3$, $x - 2$, $x - 1$, x, $x + 2 \in A$. Since $x + 2$ is odd and A is closed, $x - 3$, $x - 2$, $x - 1$, x,$x + 1$, $x + 2 \in A$. Proceeding like this, A contains every integer. Therefore $A = \mathbb{Z}$. **Case 2:** x is even. Since A is open, $x + 1$, $x + 3 \in A$, $x + 1$ is odd and $x + 1 \in A$. By Case 1, $A = \mathbb{Z}$. The only subsets of $\mathbb{Z}$ which are both open and closed in $(\mathbb{Z}, \kappa_1)$ are $\emptyset$ and $\mathbb{Z}$. Therefore $(\mathbb{Z}, \kappa_1)$ is connected. □

Theorem 3.13. *In the space $(\mathbb{Z}, \kappa_1)$, for any $n \in \mathbb{Z}$*

1. *$\{2n + 1\}$ is both regular open and β-closed AN-set.*
2. *$\{2n\}$ is α-closed, nowhere dense and preclosed AN-set.*

Proof. **(1)** $cl(\{2n+1\}) = \{2n-2, 2n, 2n+1\}$; $int(cl(\{2n+1\})) = \{2n+1\}$. Hence $\{2n+1\}$ is regular open and this also means that the set is preclosed AN-set. **(2)** $cl(\{2n\}) = \{2n\}$; $int(cl(\{2n\})) = \emptyset$. Hence $\{2n\}$ is nowhere dense. $cl(int(cl(\{2n\}))) = \emptyset$. Hence $\{2n\}$ is both α-closed and preclosed AN-set. □

Theorem 3.14. *In the space $(\mathbb{Z}, \kappa_1)$, for any $n \in \mathbb{Z}$.*

1. *$\{2n, 2n+1\}$ is semi-open and semi-closed, hence semi-regular.*
2. *$\{2n, 2n+3\}$ is semi-open and semi-closed, hence semi-regular.*
3. *$\{2n-1, 2n\}$ is semi-closed.*

Proof. **(1)** $int(\{2n, 2n+1\}) = \{2n+1\}$; $cl(int(\{2n, 2n+1\})) = \{2n-2, 2n, 2n+1\}$. Therefore $\{2n, 2n+1\}$ is semi-open. Also, $cl(\{2n, 2n+1\}) = \{2n-2, 2n, 2n+1\}$; $int(cl(\{2n, 2n+1\})) = \{2n+1\}$. Therefore $\{2n, 2n+1\}$ is semi-closed. Hence $\{2n, 2n+1\}$ is semi-regular.
(2) $int(\{2n, 2n+3\}) = \{2n+3\}$; $cl(int(\{2n, 2n+3\})) = \{2n, 2n+2, 2n+3\}$. Therefore $\{2n, 2n+3\}$ is semi-open. Also, $cl(\{2n, 2n+3\}) = \{2n, 2n+2, 2n+3\}$; $int(cl(\{2n, 2n+3\})) = \{2n+3\}$. Therefore $\{2n, 2n+3\}$ is semi-closed. Hence $\{2n, 2n+3\}$ is semi-regular.
(3) $cl(\{2n-1, 2n\}) = \{2n-4, 2n-2, 2n-1, 2n\}$; $int(cl(\{2n-1, 2n\})) = \{2n-1\}$. Therefore $\{2n-1, 2n\}$ is semi-closed. □

Remark 3.15. *Semi-regularity implies that the sets in Theorem 3.14 (1)and (2) are β-clopen sets. Moreover, the boundary of $A = \{2n, 2n+1\}$ is $\{2n, 2n-2\}$ and its derived set is $\{2n-2\}$.*

Theorem 3.16. *In the space $(\mathbb{Z}, \kappa_1)$, for any $n \in \mathbb{Z}$*

1. *$\{2n+1\}$ is regular open.*
2. *$\{2n\}$ is α-closed and nowhere dense.*

Proof. **(1)** $cl(\{2n+1\}) = \{2n-2, 2n, 2n+1\}$; $int(cl(\{2n+1\})) = \{2n+1\}$. Hence $\{2n+1\}$ is regular open.
(2) $cl(\{2n\}) = \{2n\}$; $int(cl(\{2n\})) = \emptyset$. Hence $\{2n\}$ is nowhere dense. $cl(int(cl(\{2n\}))) = \emptyset$. Hence $\{2n\}$ is α-closed. □

Theorem 3.17. *In the space $(\mathbb{Z}, \kappa_1)$, for any $n \in \mathbb{Z}$.*

1. *$\{2n, 2n+1, 2n+3\}$ is regular open.*
2. *$\{2n, 2n+1, 2n+2\}$ is semi-closed.*
3. *$\{2n-1, 2n, 2n+1\}$ is semi-open.*
4. *$\{2n-1, 2n, 2n+2\}$ is semi-closed.*
5. *$\{2n, 2n+2, 2n+3\}$ is regular closed.*
6. *$\{2n, 2n+2, 2n+4\}$ is nowhere dense.*

Proof. **(1)** $cl(\{2n, 2n+1, 2n+3\}) = \{2n-2, 2n, 2n+1, 2n+2, 2n+3\}$; $int(cl(\{2n, 2n+1, 2n+3\})) = \{2n, 2n+1, 2n+3\}$. Therefore $\{2n, 2n+1, 2n+3\}$ is regular open.
(2) $cl(\{2n, 2n+1, 2n+2\}) = \{2n-2, 2n, 2n+1, 2n+2\}$; $int(cl(\{2n, 2n+1, 2n+2\})) = \{2n+1\}$. Therefore $\{2n, 2n+1, 2n+2\}$

is semi-closed.
(3) $int(\{2n-1, 2n, 2n+1\}) = \{2n-1, 2n+1\}$; $cl(int(\{2n-1, 2n, 2n+1\})) = \{2n-4, 2n-2, 2n-1, 2n, 2n+1\}$. Therefore $\{2n-1, 2n, 2n+1\}$ is semi-open.
(4) $cl(\{2n-1, 2n, 2n+2\}) = \{2n-4, 2n-2, 2n-1, 2n, 2n+2\}$; $int(cl(\{2n-1, 2n, 2n+2\})) = \{2n-1\}$. Therefore $\{2n-1, 2n, 2n+2\}$ is semi-closed.
(5) $int(\{2n, 2n+2, 2n+3\}) = \{2n+3\}$; $cl(int(\{2n, 2n+2, 2n+3\})) = \{2n, 2n+2, 2n+3\}$. Therefore $\{2n, 2n+2, 2n+3\}$ is regular closed.
(6) $cl(\{2n, 2n+2, 2n+4\}) = \{2n, 2n+2, 2n+4\}$; $int(cl(\{2n, 2n+2, 2n+4\})) = \emptyset$. $\{2n, 2n+2, 2n+4\}$ is nowhere dense. □

Theorem 3.18. 1. *Every basic open set in $(\mathbb{Z}, \kappa_1)$ is regular open.*
2. *$(\mathbb{Z}, \kappa_1)$ is $T_{3/4}$. Hence $(\mathbb{Z}, \kappa_1)$ is $T_{1/2}$ and therefore T_0.*

Proof. **(1)** By Theorem 3.16(1), $\{2n+1\}$ is regular open. By Theorem 3.17(1), $\{2n-2, 2n-1, 2n+1\}$ is regular open. Thus every basis element is regular open.
(2) Let $x \in \mathbb{Z}$. If x is odd, $\{x\}$ is regular open. If x is even, $\{x\}$ is closed. Since $\{x\}$ is regular open or closed, $(\mathbb{Z}, \kappa_1)$ is $T_{3/4}$. Hence $(\mathbb{Z}, \kappa_1)$ is $T_{1/2}$. Therefore it is T_0. □

Theorem 3.19. *The set A of all odd integers is dense in $(\mathbb{Z}, \kappa_1)$.*

Proof. Let $x \in \mathbb{Z}$. If x is odd, $x \in A$. If x is even, $N(x) = \{x, x+1, x+3\}$ is the smallest neighborhood of x and it intersects A. That is $x \in \bar{A}$. Therefore $\bar{A} = \mathbb{Z}$. A is dense in $\mathbb{Z}$. □

Theorem 3.20. *$(\mathbb{Z}, \kappa_1)$ is neither R_0, nor T_1, nor locally indiscrete.*

Proof. The set $U = \{2n, 2n+1, 2n+3\}$ is open in $(\mathbb{Z}, \kappa_1)$ and $2n+1 \in U$. But $cl(\{2n+1\}) = \{2n-2, 2n, 2n+1\} \nsubseteq U$. Therefore $(\mathbb{Z}, \kappa_1)$ is not R_0. Since $T_1 = T_0 + R_0$, $(\mathbb{Z}, \kappa_1)$ is not T_1. Since $(\mathbb{Z}, \kappa_1)$ is not R_0, then the space is not locally indiscrete. □

Theorem 3.21. *In $(\mathbb{Z}, \kappa_1)$, every F_σ set is closed and every G_δ set is open.*

Proof. In any Alexandorff space, arbitrary intersection of open sets is open and arbitrary union of closed sets is closed. In particular, every G_δ set is open and every F_σ set is closed. Since $(\mathbb{Z}, \kappa_1)$ is an Alexandorff space, every F_σ set is closed and every G_δ set is open. □

Theorem 3.22. 1. *$(\mathbb{Z}, \kappa_1)$ is locally finite.*
2. *$(\mathbb{Z}, \kappa_1)$ is not a door space.*
3. *$(\mathbb{Z}, \kappa_1)$ is not extremely disconnected.*

Proof. **(1)** Let $x \in \mathbb{Z}$. If x is odd, $\{x\}$ is a finite open set containing x and $\{x-3, x-1, x\}$ is a finite closed set containing x. If x is even, $\{x, x+1, x+3\}$ is a finite open set containing x and $\{x\}$ is a finite closed set containing x. Therefore $(\mathbb{Z}, \kappa_1)$ is locally finite.

(2) $\{2n, 2n+1\}$ is a subset of $\mathbb{Z}$ which is neither open nor closed in $(\mathbb{Z}, \kappa_1)$. Therefore $(\mathbb{Z}, \kappa_1)$ is not a door space.
(3) The set $U = \{2n+1\}$ is open in $(\mathbb{Z}, \kappa_1)$. $cl(U) = \{2n-2, 2n, 2n+1\}$ is not open in $(\mathbb{Z}, \kappa_1)$. Therefore $(\mathbb{Z}, \kappa_1)$ is not extremely disconnected. □

From Theorem 3.14 (2), it follows that the semi-regularization of κ_1 is not a topology. Moreover since $(\mathbb{Z}, \kappa_1)$ is a locally finite Alexandroff space which is not T_1, then there exists two points $P_1 \neq P_2$ in $(\mathbb{Z}, \kappa_1)$ such that for every neighborhoods $U(P_1)$ and $U(P_2)$ of these points any injection $f : U(P_1) \to U(P_2)$ mapping P_1 in P_2 is not continuous (see [**?**]).

Theorem 3.23. *A subset A of $(\mathbb{Z}, \kappa_1)$ is semi-open if and only if $2n+1$ or $2n+3 \in A$ whenever $2n \in A$.*

Proof. **Necessity:** Let $A \subseteq \mathbb{Z}$ be semi-open and $2n \in A$. Suppose $2n+1$ and $2n+3 \notin A$. Then $int(A) \cap \{2n, 2n+1, 2n+3\} = \emptyset$ implies that $int(A) \subseteq G^c$ where $G = \{2n, 2n+1, 2n+3\}$ is open. This implies $cl(int(A)) \subseteq G^c$ and therefore $2n \notin cl(int(A))$ implies that $A \nsubseteq cl(int(A))$, a contradiction. **Sufficiency:** Let $x \in A$. **Case 1:** $x = 2m+1$. Then, $x \in int(A)$ and therefore $x \in cl(int(A))$. **Case 2:** $x = 2m$. Then $2m+3$ or $2m+1 \in A$ implies that $2m+3$ or $2m+1 \in int(A)$ and therefore $2m \in cl(2m+3) \subseteq cl(int(A))$ or $2m \in cl(2m+1) \subseteq cl(int(A))$. Hence $A \subseteq cl(int(A))$. □

Theorem 3.24. *A subset A of $(\mathbb{Z}, \kappa_1)$ is semi-closed if and only if $2n+1$ or $2n+3 \notin A$ whenever $2n \notin A$.*

Proof. **Necessity:** Let $A \subseteq \mathbb{Z}$ be semi-closed and $2n \notin A$. Suppose $2n+1$ and $2n+3 \in A$. Then $cl(\{2n+1, 2n+3\}) \subseteq cl(A)$. This implies that $int(cl(\{2n+1, 2n+3\})) \subseteq int(cl(A)) \subseteq A$ implies that $\{2n, 2n+1, 2n+3\} \subseteq A$, which implies that $2n \in A$, a contradiction. **Sufficiency:** Let $x \in int(cl(A))$. **Case 1:** $x = 2n+1$. $x \in int(cl(A)) \subseteq cl(A)$. Since $\{x\}$ is open, $x \in A$. **Case 2:** $x = 2m$. $x \in int(cl(A))$ implies that $\{2m, 2m+1, 2m+3\} \subseteq cl(A)$. Since $\{2m+3\}$ and $\{2m+1\}$ are open, $2m+3$ and $2m+1 \in A$. Then by assumption, $2m \in A$. Hence, $int(cl(A)) \subseteq A$. Therefore, A is semi-closed. □

Using the characterization of semi-closed subsets of $(\mathbb{Z}, \kappa_1)$, it can be proved that $(\mathbb{Z}, \kappa_1)$ is semi-regular and semi-normal.

Theorem 3.25. $(\mathbb{Z}, \kappa_1)$ *is semi-regular.*

Proof. Let A be a semi-closed subset of $(\mathbb{Z}, \kappa_1)$ and $x \notin A$. **Case 1:** $x = 2n$. Since $x = 2n \notin A$, by Theorem 3.24, $2n+3$ or $2n+1 \notin A$. Let $U = \{2n, 2n+3\}$ if $2n+3 \notin A$ and $U = \{2n, 2n+1\}$ if $2n+1 \notin A$. Let $V = \mathbb{Z} \backslash U$. **Case 2:** $x = 2n+1$. Let $U = \{2n+1\}$ and $V = \mathbb{Z} \backslash U$.

In each case U and V are disjoint semi-open sets such that $x \in U$ and $A \subseteq V$. Hence $(\mathbb{Z}, \kappa_1)$ is semi-regular. □

Theorem 3.26. *$(\mathbb{Z}, \kappa_1)$ is semi-normal.*

Proof. Let A and B be disjoint semi-closed subsets of $(\mathbb{Z}, \kappa_1)$. Let $A = C_1 \cup D_1$ and $B = C_2 \cup D_2$ where C_1 and C_2 are subsets of $2\mathbb{Z}+1$ and D_1 and D_2 are subsets of $2\mathbb{Z}$. Let us form the semi-open sets U and V as follows. If $2n \in D_1$, then $2n \notin B$. Since B is semi-closed $2n+1$ or $2n+3 \notin B$. Let

$$E_1 = \bigcup_{2n\in D_1, x\in\{2n+1,2n+3\}, x\notin B} \{2n, x\}$$

and $U = C_1 \cup E_1$. Similarly, let $V = C_2 \cup E_2$ where

$$E_2 = \bigcup_{2m\in D_2, x\in\{2m+1,2m+3\}, x\notin A} \{2m, x\}. \tag{1}$$

Then U and V are semi-open subsets of $(\mathbb{Z}, \kappa_1)$ containing A and B respectively. Also, $U \cap V = \emptyset$. □

4. Adjacency in the digital plane and digital space

If x is a point in the digital line $(\mathbb{Z}, \kappa_1)$,

$$\mathcal{A}(x) = \begin{cases} \{x-3, x-1\} & \text{if } x \text{ is odd} \\ \{x+1, x+3\} & \text{if } x \text{ is even.} \end{cases}$$

Now consider the digital plane $(\mathbb{Z}^2, {\kappa_1}^2)$ which is the topological product of two copies of digital line $(\mathbb{Z}, \kappa_1)$ where $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and ${\kappa_1}^2 = \kappa_1 \times \kappa_1$. We note that in the digital plane $(\mathbb{Z}^2, {\kappa_1}^2)$, the point (x, y) is open if both the coordinates are odd and is closed if both the coordinates are even. The point (x, y) is pure if x and y are of the same parity, otherwise it is mixed.

Result 4.1. *If x is a point in the digital plane $(\mathbb{Z}^2, {\kappa_1}^2)$, then the smallest neighborhood of x is*

$$\begin{aligned} N(x) &= \{(2n+1, 2m+1)\} \textit{ if } x = (2n+1, 2m+1) \\ &= \{2n+1\} \times \{2m, 2m+1, 2m+3\} \textit{ if } x = (2n+1, 2m) \\ &= \{2n, 2n+1, 2n+3\} \times \{2m+1\} \textit{ if } x = (2n, 2m+1) \\ &= \{2n, 2n+1, 2n+3\} \times \{2m, 2m+1, 2m+3\} \textit{ if } x = (2n, 2m) \end{aligned}$$

Result 4.2. *If x is a point in the digital plane $(\mathbb{Z}^2, {\kappa_1}^2)$, then the smallest closed set containing x is*

$$\begin{aligned} N[x] &= \{(2n, 2m)\} \textit{ if } x = (2n, 2m) \\ &= \{2n-2, 2n, 2n+1\} \times \{2m\} \textit{ if } x = (2n+1, 2m) \\ &= \{2n\} \times \{2m-2, 2m, 2m+1\} \textit{ if } x = (2n, 2m+1) \\ &= \{2n-2, 2n, 2n+1\} \times \{2m-2, 2m, 2m+1\} \textit{ if } x = (2n, 2m) \end{aligned}$$

Result 4.3. *The digital plane* $(\mathbb{Z}^2, {\kappa_1}^2)$ *is not* $T_{1/2}$.

Proof. In digital plane $(\mathbb{Z}^2, {\kappa_1}^2)$, the set $\{(2n, 2m+1)\}$ is neither open nor closed. Therefore, the plane $(\mathbb{Z}^2, {\kappa_1}^2)$ is not $T_{1/2}$. □

Result 4.4. *The digital plane* $(\mathbb{Z}^2, {\kappa_1}^2)$ *is semi-*$T_{1/2}$.

Proof. Choose a point x in $(\mathbb{Z}^2, {\kappa_1}^2)$.
Case 1: $x = (2n+1, 2m+1)$. Then $\{x\}$ is open in $(\mathbb{Z}^2, {\kappa_1}^2)$ and hence $\{x\}$ is semi-open.
Case 2: $x = (2n, 2m)$. Then $\{x\}$ is closed in $(\mathbb{Z}^2, {\kappa_1}^2)$ and hence $\{x\}$ is semi-closed.
Case 3: $x = (2n, 2m+1) \in \mathbb{Z}^2$. Now, $cl(\{x\}) = \{(2n, 2n-2), (2n, 2m), (2n, 2m+1)\}$ and $int(cl(\{x\})) = \emptyset$. That is $int(cl(\{x\})) \subseteq \{x\}$. Therefore, $\{x\}$ is semi-closed in $(\mathbb{Z}^2, {\kappa_1}^2)$. Similarly we can prove for $x = (2n+1, 2m)$.
In each case, $\{x\}$ is either semi-open or semi-closed in $(\mathbb{Z}^2, {\kappa_1}^2)$. Therefore, the plane $(\mathbb{Z}^2, {\kappa_1}^2)$ is semi-$T_{1/2}$. □

Result 4.5. *In the digital plane* $(\mathbb{Z}^2, {\kappa_1}^2)$*, pure points are 8-connected and mixed points are 4-connected.*

Proof. If $(x, y) \in \mathbb{Z}^2$, we find that

$$\begin{aligned}
\mathcal{A}(x,y) = \{&(x-3,y-3),(x-3,y-1),(x-3,y),(x-1,y-3),(x-1,y-1),\\
&(x-1,y),(x,y-3),(x,y-1)\} \text{ if x and y are odd}\\
= \{&(x,y+1),(x,y+3),(x+1,y),(x+1,y+1),(x+1,y+3),(x+3,y),\\
&(x+3,y+1),(x+3,y+3)\} \text{ if x and y are even}\\
= \{&(x,y+1),(x,y+3),(x-3,y),(x-1,y)\} \text{ if x is odd and y is even}\\
= \{&(x+1,y),(x+3,y),(x,y-3),(x,y-1)\} \text{ if x is even and y is odd.}
\end{aligned}$$

Therefore, the pure points are 8-connected and the mixed points are 4-connected. □

Theorem 4.6. *The spaces* $(\mathbb{Z}, \kappa)$ *and* $(\mathbb{Z}, \kappa_1)$ *are homeomorphic.*

Proof. Define $f : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa_1)$ by

$$f(x) = \begin{cases} x+2 & \text{if } x \text{ is odd} \\ x & \text{if } x \text{ is even.} \end{cases}$$

Then, f is a bijection. And

$$f^{-1}(x) = \begin{cases} x & \text{if } x \text{ is even} \\ x-2 & \text{if } x \text{ is odd.} \end{cases}$$

Now,

$$f^{-1}(\{2n+1\}) = \{2n-1\}$$
$$f^{-1}(\{2n, 2n+1, 2n+3\}) = \{2n-1, 2n, 2n+1\}$$

Since image of every basis element in $(\mathbb{Z}, \kappa_1)$ is open in $(\mathbb{Z}, \kappa)$. Therefore, f is continuous. And

$$f(\{2n+1\}) = \{2n+3\}$$
$$f(\{2n-1, 2n, 2n+1\}) = \{2n, 2n+1, 2n+3\}$$

Therefore, f is an open map. Thus, f is a homeomorphism. Thus, $(\mathbb{Z}, \kappa)$ and $(\mathbb{Z}, \kappa_1)$ are homeomorphic. □

Theorem 4.7. *The spaces $(\mathbb{Z}^2, \kappa^2)$ and $(\mathbb{Z}^2, {\kappa_1}^2)$ are homeomorphic.*

Proof. Define $h : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, {\kappa_1}^2)$ by $h(x, y) = (f(x), f(y))$, where f is defined as in Theorem 4.6. Then, h is continuous bijection and h^{-1} is also continuous. Therefore, h is a homeomorphism. Thus, $(\mathbb{Z}^2, \kappa^2)$ and $(\mathbb{Z}^2, {\kappa_1}^2)$ are homeomorphic. □

Corollary 4.8. *Let $h : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, {\kappa_1}^2)$ be a homeomorphism. Then,*

(1) *h maps connected subsets of $(\mathbb{Z}^2, \kappa^2)$ onto those of $(\mathbb{Z}^2, {\kappa_1}^2)$.*
(2) *h maps simple closed curves of $(\mathbb{Z}^2, \kappa^2)$ onto those of $(\mathbb{Z}^2, {\kappa_1}^2)$.*

Remark 4.9. *In the plane $(\mathbb{Z}^2, {\kappa_1}^2)$, Jordan curve theorem hold.*

5. $^*g\hat{\alpha}$-closed sets and its properties

we introduce the concept of $^*g\hat{\alpha}$-closed sets in a topological space and characterize it using$^*g\alpha o$-kernel and closure. Moreover, we investigate the properties of $^*g\hat{\alpha}$-closed sets in digital plane. We prove that this plane $(\mathbb{Z}^2, {}^*g\hat{\alpha}O)$ is $T_{1/2}$. It is well known that the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$, even if $(\mathbb{Z}, \kappa)$ is $T_{1/2}$.

Definition 5.1. A subset A of a topological space (X, τ) is called

(1) αg-closed [23] if $\tau^\alpha\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an open set in (X, τ),
(2) gs-closed [4] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is an open set in (X, τ),
(3) g^*-closed [30] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a g-open set in (X, τ),
(4) $^*g\alpha$-closed [26] if $\tau^\alpha\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an g-open set in (X, τ),
(5) gp-closed [5] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is an open set in (X, τ),
(6) gsp-closed [9] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is an open set in (X, τ),
(7) gpr-closed [16] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is a regular open set in (X, τ),
(8) $g\alpha$-closed [22] if $\tau^\alpha\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an α-open set in (X, τ).

Moreover, A is said to be g-open (res. αg-ope n, gs-open, g^*-open, $^*g\alpha$-open, gp-open, gsp-open, gpr-open and $g\alpha$-open) if $X\backslash A$ is g-closed (res. αg-closed, gs-closed, g^*-closed, $^*g\alpha$-closed, gp-closed, gsp-closed, gpr-closed and $g\alpha$-closed).

Lemma 5.2. [26] *For a subset A of (X, τ), the following conditions are equivalent:*

(1) *A is $^*g\alpha$-closed in (X, τ).*
(2) *τ^α-$cl(A) \subseteq go$-$Ker(A)$ holds.*

Lemma 5.3. [26] *Let a subset A of $(\mathbb{Z}^2, \kappa^2)$.*

(1) *go-$Ker(A) = U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2}$, where $U(A_{\mathcal{F}^2}) = \bigcup\{U(x) | x \in A_{\mathcal{F}^2}\}$.*
(2) *For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, a subset $\{x\} \cup (U(x))_{\kappa^2}$ is preopen and hence it is α-open in $(\mathbb{Z}^2, \kappa^2)$.*

Definition 5.4. A subset A of a space (X, τ) is called a $^*g\hat{\alpha}$-closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a $^*g\alpha$-open set in (X, τ). The class of $^*g\hat{\alpha}$-closed subsets of (X, τ) is denoted by $^*g\hat{\alpha}C(X, \tau)$.

Theorem 5.5. *Every closed set is $^*g\hat{\alpha}$-closed in (X, τ).*

Proof. Let $A \subseteq U$ and U is a $^*g\alpha$-open set in (X, τ). Since A is closed, $cl(A) = A \subseteq U$. Therefore A is $^*g\hat{\alpha}$-closed. □

The following example shows that the above implication is not reversible.

Example 5.6. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then, $\{b\}$ is $^*g\hat{\alpha}$-closed but it is not closed in (X, τ).*

Theorem 5.7. (1) *Every $^*g\hat{\alpha}$-closed set is g-closed set in (X, τ).*
(2) *Every $^*g\hat{\alpha}$-closed set is gs-closed set in (X, τ).*
(3) *Every $^*g\hat{\alpha}$-closed set is gp-closed set in (X, τ).*
(4) *Every $^*g\hat{\alpha}$-closed set is gsp-closed set in (X, τ).*
(5) *Every $^*g\hat{\alpha}$-closed set is gpr-closed set in (X, τ).*
(6) *Every $^*g\hat{\alpha}$-closed set is αg-closed set in (X, τ).*

Proof. (1) Let $A \subseteq U$ and U is an open set in (X, τ). Since every open set is $^*g\alpha$-open, U is $^*g\alpha$-open. Since A is $^*g\hat{\alpha}$-closed, $cl(A) \subseteq U$. Therefore A is g-closed.
(2) Let $A \subseteq U$ and U is an open set in (X, τ). Since every open set is $^*g\alpha$-open, U is $^*g\alpha$-open. Since A is $^*g\hat{\alpha}$-closed, $scl(A) \subseteq cl(A) \subseteq U$. Therefore A is gs-closed.
(3) Let $A \subseteq U$ and U is an open set in (X, τ). Since every open set is $^*g\alpha$-open, U is $^*g\alpha$-open. Since A is $^*g\hat{\alpha}$-closed, $pcl(A) \subseteq cl(A) \subseteq U$. Therefore A is gp-closed.
(4) Let $A \subseteq U$ and U is an open set in (X, τ). Since every open set is $^*g\alpha$-open, U is $^*g\alpha$-open. Since A is $^*g\hat{\alpha}$-closed, $spcl(A) \subseteq cl(A) \subseteq U$. Therefore A is gsp-closed.

(5) Let $A \subseteq U$ and U is an regular open set in (X, τ). Since every regular open set is $^*g\alpha$-open, U is $^*g\alpha$-open. Since A is $^*g\hat{\alpha}$-closed, $pcl(A) \subseteq cl(A) \subseteq U$. Therefore A is gpr-closed.
(6) Let $A \subseteq U$ and U is an open set in (X, τ). Since every open set is $^*g\alpha$-open, U is $^*g\alpha$-open. Since A is $^*g\hat{\alpha}$-closed, τ^α-$cl(A) \subseteq cl(A) \subseteq U$. Therefore A is αg-closed. □

Example 5.8. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then, $\{a\}$ is gpr-closed but it is not $^*g\hat{\alpha}$-closed in (X, τ).*

Example 5.9. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}\}$. Then, $\{a\}$ is both gp-closed and gsp-closed but it is not $^*g\hat{\alpha}$-closed in (X, τ).*

Example 5.10. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{a, b\}\}$. Then, $\{a\}$ is both αg-closed and gs-closed but it is not $^*g\hat{\alpha}$-closed in (X, τ).*

The following example shows that pre-closed sets and $^*g\hat{\alpha}$-closed sets are independent.

Example 5.11. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then, $\{a, b\}$ is $^*g\hat{\alpha}$-closed but it is not pre-closed in (X, τ). When $Y = \{a, b, c\}$ and $\sigma = \{Y, \emptyset, \{a, b\}\}$, the subset $\{a\}$ is pre-closed but it is not $^*g\hat{\alpha}$-closed in (Y, σ).*

The following example shows that semi-closed sets, $g\alpha$-closed sets and α-closed sets are independent form $^*g\hat{\alpha}$-closed sets.

Example 5.12. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then, $\{a, b\}$ is $^*g\hat{\alpha}$-closed but it is none of semi-closed, $g\alpha$-closed and α-cloded in (X, τ). When $Y = \{a, b, c\}$ and $\sigma = \{Y, \emptyset, \{b\}, \{a, b\}\}$, the subset $\{a\}$ is semi-closed, $g\alpha$-closed and α-cloded but it is not $^*g\hat{\alpha}$-closed in (Y, σ).*

Theorem 5.13. *Finite union of $^*g\hat{\alpha}$-closed sets is a $^*g\hat{\alpha}$-closed set in (X, τ).*

Proof. Let A_i's are $^*g\hat{\alpha}$-closed sets, where $i = 1, 2, 3, ..., n$ and $n \in \mathbb{N}$. Let $\bigcup_{i=1}^{n} A_i \subseteq U$, U is a $^*g\alpha$-open set xin (X, τ). Since A_i's are $^*g\hat{\alpha}$-closed sets, $cl(A_i) \subseteq U, \forall A_i \subseteq U$. This implies that $cl(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} cl(A_i) \subseteq U$. Therefore $\bigcup_{i=1}^{n} A_i$ is $^*g\hat{\alpha}$-closed. □

Remark 5.14. *Finite intersection of $^*g\hat{\alpha}$-open sets is a $^*g\hat{\alpha}$-open set in (X, τ).*

Proof. Proof is obvious, since $X \backslash A$ is $^*g\hat{\alpha}$-open, whenever A is $^*g\hat{\alpha}$-closed. □

The following example shows that intersection of two $^*g\hat{\alpha}$-closed sets need not be $^*g\hat{\alpha}$-closed in (X, τ).

Example 5.15. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then, $\{a, b\}$ and $\{a, c\}$ are $^*g\hat{\alpha}$-closed but their intersection $\{a\}$ is not $^*g\hat{\alpha}$-closed in (X, τ).*

Theorem 5.16. *Let A be a $^*g\hat{\alpha}$-closed set in (X,τ) if and only if $cl(A)\backslash A$ does not contain any non empty $^*g\alpha$-closed set.*

Proof. **Necessity:** Suppose that A is $^*g\hat{\alpha}$-closed and let F be an non-empty $^*g\alpha$-closed set with $F \subseteq cl(A)\backslash A$. Then $A \subseteq X\backslash F$ ans so $cl(A) \subseteq X\backslash cl(A)$. Henece $F \subseteq X - cl(A)$, a contradiction. **Sufficient:** Suppose A is a subset of (X,τ) such that $cl(A)\backslash A$ does not contain any non-empty $^*g\alpha$-closed set. Let U be a $^*g\alpha$-open set in (X,τ) such that $A \subseteq U$. If $cl(A) \nsubseteq U$, then $cl(A) \cap cl(U) \neq \emptyset$. Then $\emptyset \neq cl(A) \cap cl(U)$ is a $^*g\alpha$-closed set in (X,τ), since the intersection of two $^*g\alpha$-closed sets is again a $^*g\alpha$-closed set. □

Theorem 5.17. *Let (X,τ) be a space, A and B subsets.*

(1) *If A is $^*g\alpha$-open and $^*g\hat{\alpha}$-closed, then A is closed in (X,τ).*
(2) *If A is $^*g\hat{\alpha}$-closed set of (X,τ) such that $A \subseteq B \subseteq cl(A)$, then B is also $^*g\hat{\alpha}$-closed in (X,τ).*
(3) *For each $x \in X$, $\{x\}$ is $^*g\alpha$-closed or $X\backslash\{x\}$ is $^*g\hat{\alpha}$-closed in (X,τ).*
(4) *Every subset is $^*g\hat{\alpha}$-closed in (X,τ) if and only if every $^*g\alpha$-open set is closed.*

Proof. (1) Since $A \subseteq A$ and A is both $^*g\alpha$-open and $^*g\hat{\alpha}$-closed, $cl(A) \subseteq A$. Therefore A is closed.
(2) Let U be a $^*g\alpha$-open set such that $B \subseteq U$. Then we have that $cl(A) \subseteq U$ and $cl(B) \subseteq cl(A) \subseteq U$. Therefore, B is $^*g\hat{\alpha}$ closed in (X,τ).
(3) If $\{x\}$ is not $^*g\alpha$-closed, then $X\backslash\{x\}$ is not $^*g\alpha$-open. Therefore, $X\backslash\{x\}$ is $^*g\hat{\alpha}$-closed in (X,τ).
(4) **Necessity:** Let U be a $^*g\alpha$-open set. Then we have that $cl(U) \subseteq U$ and hence U is closed. **Sufficiency:** Let A be a subset and U a $^*g\alpha$-open set such that $A \subseteq U$. Then $cl(A) \subseteq cl(U) = U$ and hence A is $^*g\hat{\alpha}$-closed. □

Theorem 5.18. *Let X be a topological space. A subset A of (X,τ) is $^*g\hat{\alpha}$-open if and only if $U \subseteq int(A)$, whenever U is $^*g\alpha$-closed and $U \subseteq A$.*

Proof. Let A be a $^*g\hat{\alpha}$-open set and U is $^*g\alpha$-closed such that $U \subseteq A \Rightarrow X\backslash U \supseteq X\backslash A$ is $^*g\hat{\alpha}$-closed set. So $cl(X\backslash A) \subseteq X\backslash U \Rightarrow (X\backslash cl(X\backslash A)) \supseteq (X\backslash(X\backslash U)) = U$. But $(X\backslash cl(X\backslash A)) = int(A)$. Thus $U \subseteq int(A)$. Conversely, suppose A is a subset such that $U \subseteq int(A)$, whenever U is $^*g\alpha$-closed and $U \subseteq A$. Let $X\backslash A \subseteq V$, where V is $^*g\alpha$-open. Since $X\backslash A \subseteq V \Rightarrow X\backslash V \subseteq A$. By assumption, we must have $X\backslash V \subseteq int(A)$ or $X\backslash int(A) \subseteq V$. Now, $cl(X\backslash A) \subseteq V$ and $X\backslash A$ is $^*g\hat{\alpha}$-closed set, since $X\backslash int(A) = cl(X\backslash A)$. □

We have a characterization of $^*g\hat{\alpha}$-closed sets. We prepare some notations and a lemma. For a subset E of a space (X,τ), we define the

following subsets of E:
$E_\tau = \{x \in E|\{x\} \in \tau\}$, $E_{\mathcal{F}} = \{x \in E|\{x\} \in \tau^c\}$, $E_{^*g\alpha o} = \{x \in E|\{x\}$ is $^*g\alpha$-open in $(X,\tau)\}$, $E_{^*g\alpha c} = \{x \in E|\{x\}$ is $^*g\alpha$-closed in $(X,\tau)\}$, $E_{^*g\hat{\alpha}o} = \{x \in E|\{x\}$ is $^*g\hat{\alpha}$-open in $(X,\tau)\}$, $^*G\alpha O(X,\tau) = \{U|U$ is $^*g\alpha$-open in $(X,\tau)\}$ and $^*G\alpha O\text{-}ker(A) = \bigcap\{U|U \in {^*G\alpha O(X,\tau)}$ and $A \subseteq U\}$.

Theorem 5.19. *Any subset A is $^*g\hat{\alpha}$-closed if and only if $cl(A) \subseteq {^*G\alpha O\text{-}ker(A)}$ holds.*

Proof. **Necessary:** We know that $A \subseteq {^*G\alpha O\text{-}ker(A)}$. Since A is $^*g\hat{\alpha}$-closed, $cl(A) \subseteq {^*G\alpha O\text{-}ker(A)}$. **Sufficiency:** Let $A \subseteq U$ and U is $^*g\alpha$-open. Given that $cl(A) \subseteq {^*G\alpha O\text{-}ker(A)}$. If $cl(A) \nsubseteq U$, then $cl(A) \nsubseteq {^*G\alpha O\text{-}ker(A)}$, which is a contradiction. Therefore A is $^*g\hat{\alpha}$-closed. □

Lemma 5.20. *For any space (X,τ), $X = X_{^*g\alpha c} \cup X_{^*g\hat{\alpha}o}$ holds.*

Proof. Let $x \in X$. By Theorem 5.17(3), $\{x\} \in X_{^*g\alpha c}$ or $\{x\} \in X_{^*g\hat{\alpha}o}$. □

Theorem 5.21. *For a subset A of (X,τ), the following conditions are equivalent:*

(1) *A is $^*g\hat{\alpha}$-closed in (X,τ).*
(2) *$cl(A) \subseteq {^*G\alpha O\text{-}ker(A)}$ holds.*
(3) (a) *$cl(A) \cap X_{^*g\alpha c} \subseteq A$ and*
    (b) *$cl(A) \cap X_{^*g\hat{\alpha}o} \subseteq {^*G\alpha O\text{-}ker(A)}$ holds.*

Proof. (1) ⇒ (2) Let $x \in cl(A)$. Suppose that $x \notin {^*G\alpha O\text{-}ker(A)}$. Then there exists a $U \in {^*G\alpha O(X,\tau)}$ such that $A \subseteq U$ and $x \notin U$. We have that $cl(A) \subseteq U$ and so $x \notin cl(A)$. This is a contradiction.
(2) ⇒ (3) (a) It follows from (2) that $cl(A) \cap X_{^*g\alpha c} \subseteq {^*G\alpha O\text{-}ker(A)} \cap X_{^*g\alpha c}$. We claim that $^*G\alpha O\text{-}ker(A) \cap X_{^*g\alpha c} \subseteq A$. Let $x \in {^*G\alpha O\text{-}ker(A)} \cap X_{^*g\alpha c}$. Suppose that $x \notin A$. Then, $X\backslash\{x\} \in {^*G\alpha O(X,\tau)}$ and $A \subseteq X\backslash\{x\}$. We have $x \in cl(A) \subseteq {^*G\alpha O\text{-}ker(A)} \subseteq X\backslash\{x\}$. This is a contradiction. (b) It is obtained by (2).
(3) ⇒ (1) We note that $X = X_{^*g\alpha c} \cup X_{^*g\hat{\alpha}o}$, by Lemma 5.20. Then we have that $(cl(A) \cap X_{^*g\alpha c}) \cup (cl(A) \cap X_{^*g\hat{\alpha}o}) \subseteq {^*G\alpha O\text{-}ker(A)}$. Let U be any $^*g\alpha$-open set containing A. Then, $^*G\alpha O\text{-}ker(A) \subseteq U$ and so we have that $cl(A) \subseteq {^*G\alpha O\text{-}ker(A)} \subseteq U$. Therefore, A is $^*g\hat{\alpha}$-closed. □

The following is a theorem concerning of the behavior of $^*g\hat{\alpha}$-closed sets to a subspace. Let H be a subset of (X,τ). A subset B of H is called $^*g\hat{\alpha}$-closed relative to H, if B is $^*g\hat{\alpha}$-closed in a subspace $(H,\tau|H)$.

Theorem 5.22. *If U is $^*g\alpha$-open and H is clopen in (X,τ), then $U \cap H$ is $^*g\alpha$-open in $(H,\tau|H)$.*

Proof. Let F be $g\alpha$-closed in $(H, \tau|H)$ such that $F \subseteq U \cap H$. Since $U \cap H$ is $^*g\alpha$-open in (X, τ). Then we have $F \subseteq int(U \cap H)$ and so $F \subseteq (\tau|H)$-$int(U \cap H)$. Therefore $U \cap H$ is $^*g\alpha$-open in $(H, \tau|H)$. □

6. $^*g\hat{\alpha}$-closed sets in the digital plane

In the digital plane, we investigate explicite forms of $^*g\alpha o$-Kernal and Kernal of a subset. The digital line or the so called Khalimsky line is the set of the integers $\mathbb{Z}$, equipped with the topology κ having $\{\{2n-1, 2n, 2n+1\}|n \in \mathbb{Z}\}$ as a subbase. This is denoted by $(\mathbb{Z}, \kappa)$. Thus a subset U is open in $(\mathbb{Z}, \kappa)$ if and only if whenever $x \in U$ is an even integer, then $x-1, x+1 \in U$. Let $(\mathbb{Z}^2, \kappa^2)$ be the topological product of two digital lines $(\mathbb{Z}, \kappa)$, where $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $\kappa^2 = \kappa \times \kappa$. This space is called the digital plane in the present paper (cf. [17], [19]). We note that for each point $x \in \mathbb{Z}^2$ there exists the smallest open set containing x, say $U(x)$. For the case of $x = (2n+1, 2m+1), U(x) = \{2n+1\} \times \{2m+1\}$; for the case $x = (2n, 2m), U(x) = \{2n-1, 2n, 2n+1\} \times \{2m-1, 2m, 2m+1\}$; for the case $x = (2n, 2m+1), U(x) = \{2n-1, 2n, 2n+1\} \times \{2m+1\}$; for the case $x = (2n+1, 2m), U(x) = \{2n+1\} \times \{2m-1, 2m, 2m+1\}$, where $n, m \in \mathbb{Z}$. For a subset E of $(\mathbb{Z}^2, \kappa^2)$, we define the following three subsets as follows: $E_{\mathcal{F}^2} = \{x \in E|\{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)\}$; $E_{\kappa^2} = \{x \in E|\{x\}$ is open in $(\mathbb{Z}^2, \kappa^2)\}$; $E_{mix} = E \backslash (E_{\mathcal{F}^2} \cup E_{\kappa^2})$.

Lemma 6.1. *Let A and E be subsets of $(\mathbb{Z}^2, \kappa^2)$.*

(1) [8, Theorem 3.3(i)] *If E be non-empty $^*g\alpha$-closed set, then $E_{\mathcal{F}^2} \neq \emptyset$.*
(2) [8, Theorem 3.3(ii)]*If E is $^*g\alpha$-closed and $E \subseteq B_{mix} \cup B_{\kappa^2}$ holds for some subset B of $(\mathbb{Z}^2, \kappa^2)$, then $E = \emptyset$.*
(3) *The set $U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2}$ is a $^*g\hat{\alpha}$-open set containing A.*

Proof. (3) First we claim that $A_{mix} \cup A_{\kappa^2}$ is $^*g\hat{\alpha}$-open set. Let F be a non-empty $^*g\alpha$-closed set such that $F \subseteq A_{mix} \cup A_{\kappa^2}$. Then by (2), $F = \emptyset$. Thus, we have that $F \subseteq int(A_{mix} \cup A_{\kappa^2})$. Therefore $A_{mix} \cup A_{\kappa^2}$ is $^*g\hat{\alpha}$-open. Since every open set is $^*g\hat{\alpha}$-open, $U(A_{\mathcal{F}^2})$ is $^*g\hat{\alpha}$-open. Since union of two $^*g\hat{\alpha}$-open sets is $^*g\hat{\alpha}$-open, $U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2}$ is a $^*g\hat{\alpha}$-open set containing A. □

Theorem 6.2. [8] *Let E be a subset of $(\mathbb{Z}^2, \kappa^2)$.*

(1) *If E is a non-empty $^*g\alpha$-closed set, then $E_{\mathcal{F}^2} \neq \emptyset$.*
(2) *If E is a $^*g\alpha$-closed set and $E \subseteq B_{mix} \cup B_{\kappa}^2$ holds for some subset B of $(\mathbb{Z}^2, \kappa^2)$, then $E = \emptyset$.*

Theorem 6.3. [8] *Let E be a subset of $(\mathbb{Z}^2, \kappa^2)$.*

(1) $^*G\alpha O$-$ker(A) = U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2}$, $U(A_{\mathcal{F}^2}) = \bigcup\{U(x)|x \in A_{\mathcal{F}^2}\}$.
(2) $G\alpha O$-$ker(A) = U(A_{\mathcal{F}^2})$, $U(A_{\mathcal{F}^2}) = \bigcup\{U(x)|x \in A_{\mathcal{F}^2}\}$.

Lemma 6.4. *If A is a $^*g\hat{\alpha}$-closed set of $(\mathbb{Z}^2, \kappa^2)$ and $y \in A_{mix}$, then $cl(\{y\})\backslash\{y\} \subset A$.*

Proof. Since $y \in A_{mix}$, we can set $y = (2s, 2u+1)$ or $y = (2s+1, 2u)$, where $s, u \in \mathbb{Z}$. Then $cl(\{y\}) = \{2s\}\times\{2u, 2u+1, 2u+2\} = \{y, y^+, y^-\}$ if $y = (2s, 2u+1)$, where $y^+ = (2s, 2u+2)$ and $y^- = (2s, 2u)$; $cl(\{y\}) = \{2s, 2s+1, 2s+2\} \times \{2u\} = \{y^-, y, y^+\}$ if $y = (2s+1, 2u)$, where $y^+ = (2s+2, 2u)$ and $y^- = (2s, 2u)$. Thus, we have that $cl(\{y\})\backslash\{y\} = \{y^+, y^-\}$. It is noted that $\{y^+\}$ and $\{y^-\}$ are closed singletons and so $^*g\alpha$-closed singletons of $(\mathbb{Z}^2, \kappa^2)$. We suppose that $y^+ \notin A$ or $y^- \notin A$. If $y^+ \notin A$, then $y^+ \in cl(\{y\}) \subset cl(A)$ and so $y^+ \in cl(A)\backslash A$. Then, $cl(A)\backslash A$ contains a $^*g\alpha$-closed set $\{y^+\}$, this is a contradiction to Theorem 5.16. If $y^- \notin A$, then $y^- \in cl(\{y\}) \subset cl(A)$ and so $y^- \in cl(A)\backslash A$. Then, $cl(A)\backslash A$ contains a $^*g\alpha$-closed set $\{y^-\}$,, this is again a contradiction to Theorem 5.16. Therefore $cl(\{y\})\backslash\{y\} \subset A$. □

Theorem 6.5. *Let A be a subset in $(\mathbb{Z}^2, \kappa^2)$. If $(\mathbb{Z}^2)_{\mathcal{F}^2} \subseteq A$ holds, then A is $^*g\hat{\alpha}$-closed.*

Proof. Using Theorem 6.3(1) , we have $^*G\alpha O\text{-}ker(A) = U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2} = \mathbb{Z}^2$. Then, A is $^*g\hat{\alpha}$-closed set by Theorem 5.19. □

Theorem 6.6. *Let B be a non-empty subset of $(\mathbb{Z}^2, \kappa^2)$. If $B_{\mathcal{F}^2} = \emptyset$, then B is $^*g\hat{\alpha}$-open.*

Proof. Let F be a $^*g\alpha$-closed set such that $F \subseteq B$. Since $B_{\mathcal{F}^2} = \emptyset$, we have $B = B_{mix} \cup B^2_\kappa$. Then by Theorem 6.2(2), we get $F = \emptyset \Rightarrow F \subseteq int(B)$. Therefore, B is $^*g\hat{\alpha}$-open. □

Theorem 6.7. *Let B be a non-empty subset of $(\mathbb{Z}^2, \kappa^2)$ and $B_{\mathcal{F}^2} \neq \emptyset$. Then following are equivalent:*

(1) *The subset B is $^*g\hat{\alpha}$-open set of $(\mathbb{Z}^2, \kappa^2)$,*
(2) *$U(x) \subseteq B$ holds for each point $x \in B_{\mathcal{F}^2}$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in B_{\mathcal{F}^2}$. Since $\{x\}$ is closed, $\{x\}$ is $^*g\alpha$-closed set and $\{x\} \subseteq B$. By (1), $\{x\} \subseteq int(B)$. Namely, x is an interior point of the set B. Thus, we have that, for the smallest open set $U(x)$ containing x, $U(x) \subseteq B$. (2) $\Rightarrow$ (1) It follows from the assumption that, for each point $x \in B_{\mathcal{F}^2}$, $U(x) \subseteq B$ and so $\bigcup\{U(x)|x \in B_{\mathcal{F}^2}\} \subseteq B$. Put $V_B = \bigcup\{U(x)|x \in B_{\mathcal{F}^2}\}$ and so $V_B \neq \emptyset$, $V_B \subseteq B$. By definition of open sets, V_B is open. We have that $B = V_B \cup (B\backslash V_B) = V_B \cup \{(B\backslash V_B)_{\mathcal{F}^2} \cup (B\backslash V_B)_{\kappa^2} \cup (B\backslash V_B)_{mix}\} = V_B \cup (B\backslash V_B)_{\kappa^2} \cup (B\backslash V_B)_{mix}$, we note that, for a point $y \in (B\backslash V_B)_{mix}, U(y) \subseteq B$ or $U(y) \nsubseteq B$. we put $(B\backslash V_B)^1_{mix} = \{y \in (B\backslash V_B)_{mix}|U(y) \subseteq B\}$, $U((B\backslash V_B)^1_{mix}) = \bigcup\{U(y)|y \in (B\backslash V_B)^1_{mix}\}$, $(B\backslash V_B)^2_{mix} = \{y \in (B\backslash V_B)_{mix}|U(y) \nsubseteq B\}$. Then, $(B\backslash V_B)_{mix}$ is decomposed as $(B\backslash V_B)_{mix} = (B\backslash V_B)^1_{mix} \cup (B\backslash V_B)^2_{mix}$. Thus, we have that:

$(*^1)$ $B = V_B \cup (B\backslash V_B)_{\kappa^2} \cup (B\backslash V_B)^1_{mix} \cup (B\backslash V_B)^2_{mix}$. Here, V_B is open in $(\mathbb{Z}^2, \kappa^2)$; the set $(B\backslash V_B)_{\kappa^2}$ is open in $(\mathbb{Z}^2, \kappa^2)$; $U((B\backslash V_B)^1_{mix})$ is open

in $(\mathbb{Z}^2, \kappa^2)$. Thus, we have that:
$(*^2)$ the subset $V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix})$ is open in $(\mathbb{Z}^2, \kappa^2)$.
Moreover, we conclude that:
$(*^3)$ $B = V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix}) \cup (B\backslash V_B)^2_{mix}$ holds.
Proof of $(*^3)$**:** Since $(B\backslash V_B)^1_{mix} \subseteq U((B\backslash V_B)^1_{mix})$, it is shown that $B \subseteq V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix}) \cup (B\backslash V_B)^2_{mix}$(c.f $*^1$). Conversely we have that
$V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix}) \cup (B\backslash V_B)^2_{mix} \subseteq B$, because $U((B\backslash V_B)^1_{mix}) \subseteq B$, $V_B \subseteq B$, $(B\backslash V_B)_{\kappa^2} \subseteq B$ and $(B\backslash V_B)^2_{mix} \subseteq B$ hold. Thus, we have the required equality $(*^3)$. Let F be a nonempty $^*g\alpha$-closed set of $(\mathbb{Z}^2, \kappa^2)$ such that $F \subseteq B$. We claim that:
$(*^4)$ $F \cap ((B\backslash V_B)^2_{mix}) = \emptyset$ holds.
Proof of $(*^3)$**:** Suppose that there exists a point $y \in F \cap ((B\backslash V_B)^2_{mix})$. Then we have that:
$(**) y \in B_{mix}, y \in F_{mix}$ and $U(y) \nsubseteq B$.
By Theorem 2.4[31] for a $^*g\alpha$-closed set F and the point $y \in F_{mix}$, it is obtained that $cl(\{y\})\backslash\{y\} \subseteq F$. Since $y \in (\mathbb{Z}^2)_{mix}$, we may put $y = (2s, 2u+1)$(resp. $y = (2s+1, 2u)$), $y^+ = (2s, 2u+2)$(resp. $y^+ = (2s+2, 2u)$),$y^- = (2s, 2u)$(resp. $y^- = (2s, 2u)$), where $s, u \in \mathbb{Z}$. Then $cl(\{y\}) = \{y^+, y, y^-\}$. Thus, we have that $cl(\{y\})\backslash\{y\} = \{y^+, y^-\} \subseteq F$. Since $F \subseteq B$, we have that $y^+ \subseteq B_{\mathcal{F}^2}$ and $y^- \subseteq B_{\mathcal{F}^2}$. For the point y^+, it follows form the assumption (2) that $U(y^+) \subseteq B$ and so $U(y) \subseteq B$ which a contradiction to $(**)$. Thus, we have that $F \cap ((B\backslash V_B)^2_{mix}) = \emptyset$. By using $(*^3)$ and $(*^4)$, it is shown that, for the $^*g\alpha$-closed set F such that $F \subseteq B$, $F = B \cap F = [V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix}) \cup (B\backslash V_B)^2_{mix}] \cap F \subseteq V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix})$. We put $E = V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix})$ and so $F \subseteq E \subseteq B$ and E is open. Using $(*^2)$ and $(*^3)$, we have that $F \subseteq E \subseteq int(B)$ holds. Namely, B is $^*g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. □

Theorem 6.8. (1) *The union of any collection of* $^*g\hat{\alpha}$*-open sets of* $(\mathbb{Z}^2, \kappa^2)$ *is* $^*g\hat{\alpha}$*-open set in* $(\mathbb{Z}^2, \kappa^2)$.
(2) *The intersection of any collection of* $^*g\hat{\alpha}$*-closed sets of* $(\mathbb{Z}^2, \kappa^2)$ *is* $^*g\hat{\alpha}$*-closed set in* $(\mathbb{Z}^2, \kappa^2)$.

Proof. (1) Let $\{B_i | i \in J\}$ be a collection of $^*g\hat{\alpha}$-open sets of $(\mathbb{Z}^2, \kappa^2)$, where J is an index set and put $V = \bigcup\{B_i | i \in J\}$. First we assume that $V_{\mathcal{F}^2} \neq \emptyset$, there exists a point $x \in (B_j)_{\mathcal{F}^2}$ for some $j \in J$. By Theorem 6.7, it is obtained that $U(x) \subset B_j$ and hence $U(x) \subset V$. Again using Theorem 6.7, we conclude that V is $^*g\hat{\alpha}$-open. Finally we assume that $V_{\mathcal{F}^2} = \emptyset$. Then by Theorem 6.6, V is $^*g\hat{\alpha}$-open.
(2) We recall that a subset E is $^*g\hat{\alpha}$-closed if and only if the complement of E is $^*g\hat{\alpha}$-open. It follows from (1) and definition that the intersection of any collection of $^*g\hat{\alpha}$-closed sets is $^*g\hat{\alpha}$-closed in $(\mathbb{Z}^2, \kappa^2)$. □

Proposition 6.9. *Let* x *be a point of* $(\mathbb{Z}^2, \kappa^2)$. *The following properties on the singleton* $\{x\}$ *hold.*

(1) *If* $x \in (\mathbb{Z}^2)_{\kappa^2}$*, then* $\{x\}$ *is* $^*g\hat{\alpha}$*-open; it is not* $^*g\hat{\alpha}$*-closed in* $(\mathbb{Z}^2, \kappa^2)$.
(2) *If* $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$*, then* $\{x\}$ *is* $^*g\hat{\alpha}$*-closed; it is not* $^*g\hat{\alpha}$*-open in* $(\mathbb{Z}^2, \kappa^2)$.
(3) *If* $x \in (\mathbb{Z}^2)_{mix}$*, then* $\{x\}$ *is* $^*g\hat{\alpha}$*-is not* $^*g\hat{\alpha}$*-closed; it is not* $^*g\hat{\alpha}$*-open in* $(\mathbb{Z}^2, \kappa^2)$.

Proof. (1) It follows from the assumption that $\{x\}$ is open in $(\mathbb{Z}^2, \kappa^2)$ and so it is $^*g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. Since $\{x\}$ is $^*g\alpha$-open, then there exists a $^*g\alpha$-open set $U = \{x\}$ such that $cl(\{x\}) \nsubseteq \{x\}$. By Definition 5.4 $\{x\}$ is not $^*g\hat{\alpha}$-closed in $(\mathbb{Z}^2, \kappa^2)$.
(2) It follows from the assumption that $\{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)$ and so it is $^*g\hat{\alpha}$-closed in $(\mathbb{Z}^2, \kappa^2)$. Since $\{x\}$ is $^*g\alpha$-closed, then there exists a $^*g\alpha$-closed set $B = \{x\}$ such that $\{x\} \nsubseteq int(\{x\})$. Therefore $\{x\}$ is not $^*g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$.
(3) Let $x \in (\mathbb{Z}^2)_{mix}$, *i.e*, $x = (2s+1, 2u)$ such that $cl(\{x\}) = \{2s, 2s+1, 2s + 2\} \times \{2u, 2u+1, 2u+2\} \nsubseteq \{x\} = U$, U is $^*g\alpha$-open set. Therefore, $\{x\}$ is not $^*g\hat{\alpha}$-closed. Let $x = (2s+1, 2u)$ such that $F = \emptyset \subseteq (2s+1, 2u)$, where F is $^*g\alpha$-closed set$\Rightarrow \emptyset \subseteq int(\{x\}) = \emptyset$. Hence $\{x\}$ is $^*g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. Similarly we can prove this statement for $x = (2s, 2u+1)$. □

It is well known that the digital line $(\mathbb{Z}, \kappa)$ is $T_{1/2}$ but the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$. By Theorem 6.8 and Remark 5.14, we have a new topology, say $^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ of $\mathbb{Z}^2$.

Corollary 6.10. *Let* $^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ *be the family of all* $^*g\hat{\alpha}$*-open sets in* $(\mathbb{Z}^2, \kappa^2)$*. Then, the following properties hold.*

(1) *The family* $^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ *is a topology of* $\mathbb{Z}^2$.
(2) *Let* $(\mathbb{Z}^2, ^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ *be topological space obtained by changing the topology* κ^2 *of the digital plane* $(\mathbb{Z}^2, \kappa^2)$ *by* $^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$*. Then* $(\mathbb{Z}^2, ^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ *is a* $T_{1/2}$*-topological space.*

Proof. (1) It is obvious form Theorem 6.8 and Remark 5.14 that the family $^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ is topology of $\mathbb{Z}^2$.
(2) Let $(\mathbb{Z}^2, ^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ be topological space with new topology $^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$. Then, it is claimed that the topological space $(\mathbb{Z}^2, ^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$. By Proposition 6.9, a singleton set $\{x\}$ is open or closed in $(\mathbb{Z}^2, ^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ by Theorem 3.1(ii) [15]. Hence the space $(\mathbb{Z}^2, ^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$. □

Sometimes, we abbreviate the topology $^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ by $^*g\hat{\alpha}O$. For a subset A of $\mathbb{Z}^2$, we denote the closure of A, interior of A and the kernel of A with respect to $^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ by $^*g\hat{\alpha}O\text{-}cl(A)$, $^*g\hat{\alpha}O\text{-}int(A)$ and $^*g\hat{\alpha}O\text{-}ker(A)$ respectively. The kernel is defined by $^*g\hat{\alpha}O\text{-}ker(A) = \bigcap\{V | V \in ^* g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2), A \subset V\}$.

Proposition 6.11. *For the topological space* $(\mathbb{Z}^2, {}^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$, *we have the properties on the singletons as follows. Let x be a point of $\mathbb{Z}^2$ and $s, u \in \mathbb{Z}$.*

(1) (a) *If* $x \in (\mathbb{Z}^2)_{\kappa^2}$, *then* ${}^*g\hat{\alpha}O$-$ker(\{x\}) = \{x\}$ *and* ${}^*g\hat{\alpha}O$-$ker(\{x\}) \in {}^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.
(b) *If* $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, *then* ${}^*g\hat{\alpha}O$-$ker(\{x\}) = U(x) = \{2s-1, 2s, 2s+1\} \times \{2u-1, 2u, 2u+1\}$ *where* $x = (2s, 2u)$ *and* ${}^*g\hat{\alpha}O$-$ker(\{x\}) \in {}^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.
(c) *If* $x \in (\mathbb{Z}^2)_{mix}$, *then* ${}^*g\hat{\alpha}O$-$ker(\{x\}) = \{x\}$ *and* ${}^*g\hat{\alpha}O$-$ker(\{x\}) \in {}^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.

(2) (a) *If* $x \in (\mathbb{Z}^2)_{\kappa^2}$, *then* ${}^*g\hat{\alpha}O$-$cl(\{x\}) = cl(\{x\}) = \{2s, 2s+1, 2s+2\} \times \{2u, 2u+1, 2u+2\}$ *and hence* $\{x\}$ *is not closed in* $(\mathbb{Z}^2, {}^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$.
(b) *If* $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, *then* ${}^*g\hat{\alpha}O$-$cl(\{x\}) = \{x\}$.
(c) *If* $x \in (\mathbb{Z}^2)_{mix}$, *then* ${}^*g\hat{\alpha}O$-$cl(\{x\}) = cl\{x\}$.

(3) (a) *If* $x \in (\mathbb{Z}^2)_{\kappa^2}$, *then* ${}^*g\hat{\alpha}O$-$int(\{x\}) = \{x\}$.
(b) *If* $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, *then* ${}^*g\hat{\alpha}O$-$int(\{x\}) = \emptyset$.
(c) *If* $x \in (\mathbb{Z}^2)_{mix}$, *then* ${}^*g\hat{\alpha}O$-$int(\{x\}) = \{x\}$.

Proof. (1)(a) For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$, by Proposition 6.9(1), $\{x\}$ is ${}^*g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. Then, we have that ${}^*g\hat{\alpha}O$-$ker(\{x\}) = \{x\}$ and ${}^*g\hat{\alpha}O$-$ker(\{x\}) \in {}^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.
(1)(b) Let B be any ${}^*g\hat{\alpha}$-open set of $(\mathbb{Z}^2, \kappa^2)$ containing the point $x = (2s, 2u) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Then, by Theorem 6.7, $U(x) \subset B$ holds and $U(x) \in {}^*g\hat{\alpha}O$. Thus, we have that ${}^*g\hat{\alpha}O$-$ker(\{x\}) = U(x) \in {}^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.
(1)(c) Let B be any ${}^*g\hat{\alpha}$-open set of $(\mathbb{Z}^2, \kappa^2)$ containing the point $x = (2s+1, 2u) \in (\mathbb{Z}^2)_{mix}$ (res. $x = (2s, 2u+1) \in (\mathbb{Z}^2)_{mix}$). Then, by Proposition 6.9(3), $\{x\}$ is ${}^*g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. Then, we have that ${}^*g\hat{\alpha}O$-$ker(\{x\}) = \{x\}$ and ${}^*g\hat{\alpha}O$-$ker(\{x\}) \in {}^*g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.
(2)(a) Let $x \in (\mathbb{Z}^2)$. By (1), it is shown that, for a point $y \in \mathbb{Z}^2$, $y \in {}^*g\hat{\alpha}O$-$cl(\{x\})$ holds if and only if $x \in^* g\hat{\alpha}O$-$ker(\{y\})$ holds. For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$, we put $x = (2s+1, 2u+1)$, where $s, u \in \mathbb{Z}$. For a point $y \in {}^*g\hat{\alpha}O$-$cl(\{x\})$ holds (i.e., $(y \in {}^*g\hat{\alpha}O$-$cl(\{x\}))_{\kappa^2}$) if and only if $x \in {}^*g\hat{\alpha}O$-$ker(\{y\})$ holds (cf. (1)(a)). Thus we have that ${}^*g\hat{\alpha}O$-$cl(\{x\}))_{\kappa^2} = \{x\}$. For a point $y \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, $y \in {}^*g\hat{\alpha}O$-$cl(\{x\})$ holds (i.e., $y \in ({}^*g\hat{\alpha}O$-$cl(\{x\}))_{\mathcal{F}^2}$) if and only if $x \in {}^*g\hat{\alpha}O$-$ker(\{y\})$ holds (i.e., $x \in U(y)_{\kappa^2}$ and $x \neq y$ holds) (cf. (1)(b)). Thus, we have that $({}^*g\hat{\alpha}O$-$cl(\{x\}))_{\mathcal{F}^2} = \{y \in (\mathbb{Z}^2)_{\mathcal{F}^2} | x \in (U(y))_{\kappa^2}\} = W_x$, where $W_x = \{(2s, 2u), (2s, 2u+2), (2s+2, 2u), (2s+2, 2u+2)\}$ and $x = (2s+1, 2u+1)$. For a point $y \in (\mathbb{Z}^2)_{mix}$, $y \in {}^*g\hat{\alpha}O$-$cl(\{x\})$ holds (i.e., $y \in ({}^*g\hat{\alpha}O$-$cl(\{x\}))_{mix}$) if and only if $x \in {}^*g\hat{\alpha}O$-$ker(\{y\})$ holds (i.e., $x \in (U(y))$ and $x \neq y$ holds) (cf. (1)(c)). Thus, we have that

$(^*g\hat{\alpha}O\text{-}cl(\{x\}))_{mix} = \{y \in (\mathbb{Z}^2)_{mix} | x \in (U(y))\} = V_x$, where $V_x = \{(2s+1, 2u+2), (2s+1, 2u), (2s, 2u+1), (2s+2, 2u+1)\}$ and $x = (2s+1, 2u+1)$. Therefore, $^*g\hat{\alpha}O\text{-}cl(\{x\}) = (^*g\hat{\alpha}O\text{-}cl(\{x\}))_{mix} \cup (^*g\hat{\alpha}O\text{-}cl(\{x\}))_{\kappa^2} \cup (^*g\hat{\alpha}O\text{-}cl(\{x\}))_{\mathcal{F}^2} = cl(\{x\})$.
(2)(b) For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, by Proposition 6.9(2), it is obtained that $^*g\hat{\alpha}O\text{-}cl(\{x\}) = \{x\}$.
(2)(c) Let a point $x = (2s+1, 2u) \in (\mathbb{Z}^2)_{mix}$. Consider the subset $A = \{(2s, 2u), (2s+1, 2u)\} \supseteq \{x\}$. We claim that, A is not $^*g\hat{\alpha}$-closed. Now, $W = U(2s, 2u)$ is a $^*g\alpha$-open set containg A but $cl(A) = \{(2s, 2u)(2s+1, 2u)(2s+2, 2u)\} \nsubseteq W$. Therefore A is not $^*g\hat{\alpha}$-closed. Similarly we can prove for the subset $B = \{(2s+1, 2u), (2s+2, 2u)\} \supseteq \{x\}$. Therfore the smallest $^*g\hat{\alpha}$-closed set containg $\{x\}$ is $cl(\{x\}) = \{(2s, 2u), (2s+1, 2u), (2s+2, 2u)\}$. Similarly, we can prove for $x = (2s, 2u+1)$. (3) For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$ (res. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, $x \in (\mathbb{Z}^2)_{mix}$), by Proposition 6.9(1) (res. (2), (3)), it is shown that $^*g\hat{\alpha}O\text{-}int(\{x\}) = \{x\}$ (res. $^*g\hat{\alpha}O\text{-}cl(\{x\}) = \emptyset$, $^*g\hat{\alpha}O\text{-}cl(\{x\}) = \{x\}$) holds. □

Theorem 6.12. *If $x \in (\mathbb{Z}^2)_{mix}, i.e., x = (2s, 2u+1)$ or $(2s+1, 2u)$, then $\{x\}$ is neither regular open nor regualr closed, moreover $\{x\}$ is semi open in $(\mathbb{Z}^2, ^* g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$.*

Proof. Let $x \in (\mathbb{Z}^2)_{mix}$, by Proposition 6.11(2(c), 3(c)), $^*g\hat{\alpha}O\text{-}cl(^*g\hat{\alpha}O\text{-}int(\{x\})) = \{(2s, 2u), (2s+1, 2u), (2s+2, 2u)\} \supseteq \{x\}$, where $x = (2s+1, 2u)$. Therefore $\{x\}$ is not regular closed and hence it is semi-open. Let Let $x \in (\mathbb{Z}^2)_{mix}$, by Proposition 6.11(2(c) and 3(c)), $^*g\hat{\alpha}O\text{-}int(^*g\hat{\alpha}O\text{-}cl(\{x\})) = \emptyset$. Therefore, $\{x\}$ is not regular open. □

Theorem 6.13. *If $x \in (\mathbb{Z}^2)_{\kappa^2}, i.e., x = (2s+1, 2u+1)$, then $\{x\}$ is not regular closed, moreover $\{x\}$ is semi open and regular open in $(\mathbb{Z}^2, ^* g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$.*

Proof. Let $x \in (\mathbb{Z}^2)_{\kappa^2}$, by Proposition 6.11(2(a), 3(a)), $^*g\hat{\alpha}O\text{-}cl(^*g\hat{\alpha}O\text{-}int(\{x\})) =^* g\hat{\alpha}O\text{-}cl(\{x\}) \supseteq \{x\}$, where $x = (2s+1, 2u+1)$. Therefore $\{x\}$ is not regular closed and hence it is semi-open. Let Let $x \in (\mathbb{Z}^2)_{\kappa^2}$, by Proposition 6.11(2(a) and 3(a)), $^*g\hat{\alpha}O\text{-}int(^*g\hat{\alpha}O\text{-}cl(\{x\})) = \{x\}$. Therefore, $\{x\}$ is regular open in $(\mathbb{Z}^2, ^* g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$. □

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