# THE BORSUK-ULAM THEOREM ELUCIDATES CHAOTIC SYSTEMS

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**ABSTRACT**

Nonlinear chaotic dynamics are widespread, both in physical and biological systems. This form of dynamics is frequently studied through logistic maps equipped with bifurcations, where intervals are dictated by the Feigenbaum constants. In such a multifaceted framework, a concept from the far-flung branch of topology, namely the Borsuk-Ulam theorem, comes into play. The theorem tells us that a continuous mapping from antipodal points with matching feature values on an n-sphere to the same real value can always be found. Here we demonstrate that embracing nonlinearity in the framework of the Borsuk-Ulam theorem means that bifurcation transformations (the antipodal points) can be described as paths or trajectories on abstract spheres equipped with a Feigenbaum dimension. Such an approach allows the evaluation of nonlinear systems through linear techniques. In conclusion, we provide a general topological mechanism which explains the elusive chaotic phenomena, cast in a physical/biological fashion which has the potential of being operationalized.

Open systems maintain a non-equilibrium steady-state (i.e., homoeostasis or allostasis) in the face of environmental fluctuations (Friston). Such systems, widespread in physics and biology, have been described for countless phenomena, from population dynamics (Newman) to evolution (Tozzi), from diseases spread to brain function (Van de Ville). Complex, non-linear systems with non-equilibrium dynamics occur everywhere, characterized by a large number of interacting and inter-dependent components, random walks (Perkins) and circular causality (Fraiman), spontaneous self-organization and self-organized criticality (Bak) and emergent properties.

Some of these open systems are said to operate at the edge of chaos – a chaotic system displays dependence from initial conditions, positive Lyapunov exponents, and attractors –while others tend to live near a metastable state of second-phase transition - i.e., displaying infinite correlation length, countless dimensions (Afrahimovich), spontaneous avalanches and universal power laws (de Arcangelis; Beggs). Sequential, hierarchical self-organization on an increasing scale and with constitutional dynamics in non-equilibrium systems can lead to the emergence of novel features and properties at each level, organized in space as well as in time and passage beyond reversibility (Foffi). Crowding-induced changes in the structure and dynamics of physical and biological phenomena describe a scenario of systems capable of generating well-defined functional architectures, by self-assembling from their components, thus behaving as programmed systems (Taylor). To make an example, the recent idea of "supramolecular chemistry" suggests that complex chemical entities can be reversibly constructed from molecular components bound together by labile non-covalent interactions (Lehn). The novel concept involves the storage of information at the molecular level and its retrieval, transfer and processing at the supramolecular level, via transitory processes that are self-organized, self-assembled and dynamic.
In such a multifaceted framework, a concept from the far flung branch of algebraic topology, namely the Borsuk-Ulam theory, is able to elucidate (at least some) of the dynamics underlying nonlinear systems and to shed new light on the

puzzling phenomenon of the occurrence of chaos. This paper is an effort to explain some aspects of dynamic systems theory in terms of algebraic topology, and to evaluate the consequences of this strategy.

## MATERIALS AND METHODS

**Logistic maps and the Feigenbaum constant.** A logistic map is a one-dimensional nonlinear difference equation widely used to study equations in the field of dynamic systems theory. To make an example, take into account the coupled equations (Richardson):

$$X_{(t+1)} = r_1 x_{(t)} \left(1 - x_{1(t)}\right) + \alpha r_2 x_{2(t)} \frac{\left(1 - x_{2(t)}\right)}{1+\alpha},$$

$$X_{2(t+1)} = r_2 x_{2(t)} \left(1 - x_{2(t)}\right) + \alpha_1 r_1 x_{1(t)} \frac{\left(1 - x_{2(t)}\right)}{1+\alpha},$$

where $x$ is a generic variable representing some observable behaviour, $r$ is a fixed behavioral parameter (the phase parameter) and $t$ equals time from step 0 to step n. A logistic map may be simply plotted and visualized on a one-parameter bifurcation diagram, as a function of the scaled parameter $r$ (**Figure A**). It can be clearly observed that, at the edge of criticality, a Hopf bifurcation occurs.
In bifurcation theory, the first universal **Feigenbaum constant** is a trascendental number which expresses a ratio in a bifurcation diagram for a non-linear map (Smith). In general, every chaotic system that corresponds to a one-dimensional map with a single quadratic maximum will bifurcate at the same rate (Alligood).

The first Feigenbaum constant is the limiting ratio of each bifurcation interval to the next between every period doubling, of a one-parameter map:

$$x_{i+1} = f(x_i),$$

Where $f(x)$ is a function parameterized by the bifurcation parameter $\boldsymbol{a}$. The constant is given by the limit (Jordan):

$$\delta = \lim_{n\to\infty} \frac{a_{n-1} - 1 - a_{n-2}}{a_n - a_{n-1}} = 4.669201609\ldots$$

Where $a_n$ are discrete values of variable $a$ at the $n$th period doubling. The ratio $\frac{a_{n-1} - 1 - a_{n-2}}{a_n - a_{n-1}}$ converges to the Feigenbaum constant. The same number arises for the following logistic maps:

$$f(x) = ax(1-x),$$

and

$$f(x) = a - x^2.$$

***n*-spheres.** The notation $S^n$ denotes an n-sphere, which is a generalization of the circle (Weeks). An *n*-sphere, also called $S^n$, is a *n*-dimensional structure embedded in a *n*+1 Euclidean space *(*Moura*)*. For example, a 2-sphere ($S^2$) is the 2-dimensional surface of a 3-dimensional space (a beach ball's surface is a good illustration). An *n*-sphere is formed by points which are constant distance from the origin in (*n*+1)-dimensions (Marsaglia). For example, a 2-sphere (also called *glome* or *hypersphere*) of radius *r* (where r may be any positive real number) is defined as the set of points in a 3D Euclidean space at distance *r* from some fixed center point **c** (which may be any point in the 3D space) (Moura). A2-sphere is a simply connected 2-dimensional manifold of constant, positive curvature, which is enclosed in an Euclidean 3-dimensional space called a 3-ball. From a geometer's perspective, we have different n-spheres, starting with the perimeter of a circle ($S^1$) and advancing to $S^3$, which is the smallest hypersphere, embedded in a 4-ball (**Figure 1**).

**The Borsuk-Ulam theorem.** The Borsuk-Ulam Theorem (BUT) is a finding by K. Borsuk (Borsuk 1933) about Euclidean n-spheres and antipodal points. It states that (Dodson):
*Every continuous map $f : S^n \to R^n$ must identify a pair of antipodal points on $S^n$ map to same point in $R^n$.*

This means that diametrically opposite points (antipodes) on $S^n$ are mapped to a single point in *n*-dimensional Euclidean space $R^n$. Points on $S^n$ are *antipodal*, provided they are diametrically opposite (Krantz). Examples of antipodal points are the opposite points along the circumference of a circle, or the poles of a sphere. A point embedded in an $R^n$ manifold is projected to a pair of diametrically opposite points on a $S^{n+1}$-sphere, and vice versa. In effect, the Borsuk-Ulam theorem tells us that we can always find antipodal points on $S^n$ (which is embedded in $R^{n+1}$) project to the same point on $R^n$ (which contains $S^{n-1}$). In other words, if a sphere is mapped continuously into a plane set, there is at least one pair of antipodal points having the same image; that is, they are mapped in the same point of the plane (Beyer; Borsuk 1958-1959). Put simply, two opposite points on a sphere, when projected on a circumference, give rise to a single point with a description that matches the description of both antipodes. This means that the projection from a higher dimension (equipped with two antipodal points) to a lower one gives rise to a single point. Conversely, we have pullback in which a point on $S^{n-1}$ (embedded in $R^n$) projects back to two antipodal points on $R^{n+1}$ (which contains $S^n$). As a result, this also means that the projection from a lower dimension (equipped with just one point) to a higher dimensional space, leads us back to information-carriers in the form of antipodal points that explain the origin of the single signal value predicted by BUT. Note that, in the classical formulation of BUT, *n* must be a natural number (although we will see that this is not always the case). For other definitions of BUT and its countless proofs, see (Matoušek).

**Description of a signal through BUT.** In terms of activity, a feature vector $x \in R^n$ models the description of a signal. To elucidate the picture in the application of the BUT in signal analysis, we view the surface of a manifold as an *n*-sphere and the feature space for signals as a finite Euclidean topological space. The BUT states that for the description $f(-x)$ for a signal $x$, we expect to find an antipodal feature vector $f(-x)$ describing a signal on the opposite (antipodal) side of the manifold $S^n$. The pair of antipodal signals have matching descriptions on $S^n$.

Let *X* denote a nonempty set of points on the manifold's surface. A *topological structure* on a nonempty set *X* is defined by a family of sets $\tau$ on *X*, having the following properties:

(Str.1) Every union of sets in $\tau$ is a set in $\tau$.

(Str.2) Every finite intersection of sets in $\tau$ is a set in $\tau$.

The pair $(X, \tau)$ is a topological space. Usually, *X* by itself is also called a topological space, provided it has a topology $\tau$ on it. Let *X*,*Y* be topological spaces. Recall that a function or map $f : X \rightarrow Y$ on a set *X* to a set *Y* is a subset $X \times Y$ so that for each $x \in X$ there is a unique $y \in Y$ such that $(x,y) \in f$ (usually written $y = f(x)$). The mapping *f* is defined by a rule that tells us how to find $f(x)$ (Willard).

**Shapes and homotopies.** A mapping $f : X \rightarrow Y$ is *continuous*, provided, when $A \subset Y$ is open, then the inverse $f^{-1}(A) \subset X$ is also open (Krantz). In such a view of continuous mappings from the signal topological space $X$ on the manifold's surface to the signal feature space $R^n$, we consider not just one signal feature vector $x \in R^n$, but also mappings from *X* to a set of signal feature vectors $f(X)$. This expanded view of signals is noteworthy, since every connected set of feature vectors $f(X)$ has a shape. It means that signal shapes can be compared.

A consideration of $f(X)$ (set of signal descriptions for a region *X*) instead of $f(x)$ (description of a single signal *x*) leads to a region-based view of signals. This region-based view of signals (and manifolds) arises naturally in terms of a comparison of shapes produced by different mappings from *X* (object space) to the feature space $R^n$. Continuous mappings from object spaces to feature spaces lead into homotopy theory and the study of shapes.

Let $f, g : X \rightarrow Y$ be continuous mappings from *X* to *Y*. The continuous map $H : X \times [0,1] \rightarrow Y$ is defined by:

$H(x,0) = f(x),\ H(x,1) = g(x),$ for every $x \in X$.

The mapping *H* is a *homotopy*, provided there is a continuous transformation (called a deformation) from *f* to *g*. The continuous maps *f*, *g* are called homotopic maps, provided $f(X)$ continuously deforms into $g(X)$ (denoted by $f(X) \rightarrow g(X)$). The sets of points $f(X)$, $g(X)$ are called shapes (Manetti; Cohen).

For the mapping $H : X \times [0,1] \rightarrow R^n$, where $H(X,0)$ and $H(X,1)$ are *homotopic*, provided $f(X)$ and $g(X)$ and have the same shape. That is, $f(X)$ and $g(X)$ are homotopic, if:

$\| f(X) - g(X) \| < \| f(X) \|$, for all $x \in X$.

K. Borsuk first associated the geometric notion of shape and homotopies during the 1950s. The early work on n-spheres and antipodal points led to the study of retraction and homotopic mappings (Borsuk 1969) and to the geometry of shapes and shapes of space (Collins). A pair of connected planar subsets in Euclidean space $R^2$ have equivalent shapes, provided the planer sets have the same number of holes. In terms of signals, this means that the connected graph for $f(X)$ with, for example, an *e* shape, can be deformed into the 9 shape. A pair of antipodal points on a $S^n$-sphere are characterized by the same function, have matching descriptions and display similar features (Borsuk 1980). This suggests a useful application, in terms of signal analysis, of Borsuk's view of the transformation of a shape into another (Schleicher; Su). Sets of signals not only will have similar descriptions, but also dynamic character; further, the deformation of one signal shape into another occurs when they are descriptively near (Collins). In sum, the concept of antipodal points can be generalized to countless types of systems' signals (Peters 2014). Two antipodal points can be used not just for the description of simple topological points, but also of more complicated structures and systems, such as shapes of space (spatial patterns), shapes of time (temporal patterns), movements and trajectories (Peters 2015). If the collections of signals can be viewed as surface shapes, where one shape maps to another antipodal shape (Cohen), then different phenomena can be studied in terms of opposite points, when we consider them embedded in just one dimension higher than the usual one.

## RESULTS

We applied BUT and its extensions to chaotic logistic maps. The procedure we followed is shown in **figure A**. At first, we embedded nonlinear dynamics in a *n*-sphere, in order to be allowed to study chaotic behavior in terms of opposite topological points. However, this time the *n* number of the *n*-sphere did not stand for a spatial dimension or for a natural number as usual in Borsuk-Ulam theorem, but for the first Feigenbaum constant. Thus, our *n* number was a constant (not anymore a spatial dimension) and an irrational number (not anymore a natural one). Are we allowed to modify the BUT's "classical" exponent on an *n*-sphere, changing a natural number into an irrational one, to achieve a chaotic system equipped with two antipodal points? We here demonstrate that the answer is positive, by taking into account a Borsuk-Ulam theorem on *d*-spheres with the Hausdorff dimension *d*, which is a fraction between 0 and 1.
We used the following terminology:

1) **Metric space**: Let $X$ be a metric space with the metric $\mu_d(X)$ defined on it. This means that $\mu_d(X) \geq 0$ and $\mu_d$ has the usual symmetry and triangle inequality properties for all subsets of $X$.
2) **Hausdorff measure**: Let $d$ be either 0 or a positive real number in $R_0^+$. The Hausdorff measure $\mu_d(X)$ equals a real number for each number $d$ in $X = R^d$.
3) **Hausdorff dimension (informal)**: The threshold value of d denoted by $\dim_H(X)$ is the Hausdorff dimension of $X$, provided $\mu_d(X) = 0$, if d $>\dim_H(X)$, and $\mu_d(X) = \infty$, if $d<\dim_H(X)$.

**Hausdorff Dimension-** To arrive at the Hausdorff (fractional) dimension of a subset $X$ in a metric space, we need to consider the Hausdorff measure of $X$.
**Definition 1. Hausdorff measure**. Let $X$ be a subset of a metric space $M$ and let $d$ any real number in $R_0^+$ $\varepsilon \in R_0^+$(a real number that is either positive or zero) a nonempty subset of $X$, $U_i$, $i\in\{1, . . . ,n\}$ is a cover of $X$, i.e., $X$ is a subset of $X \subseteq C_i$ for all i (Schleicher 2007). Here n is any positive integer. Also, let diam($Ui$) <ε be the diameter of the cover $U_i$. The $d$-dimensional Hausdorff measure $\mu_d(X)$ is defined by:

$$\mu_d(X) = \lim_{\varepsilon \to 0}\left[\inf_{U_i \supseteq X} \sum_{i=1}^{n} n\left(diam\left(U_i\right)\right)^d\right].$$

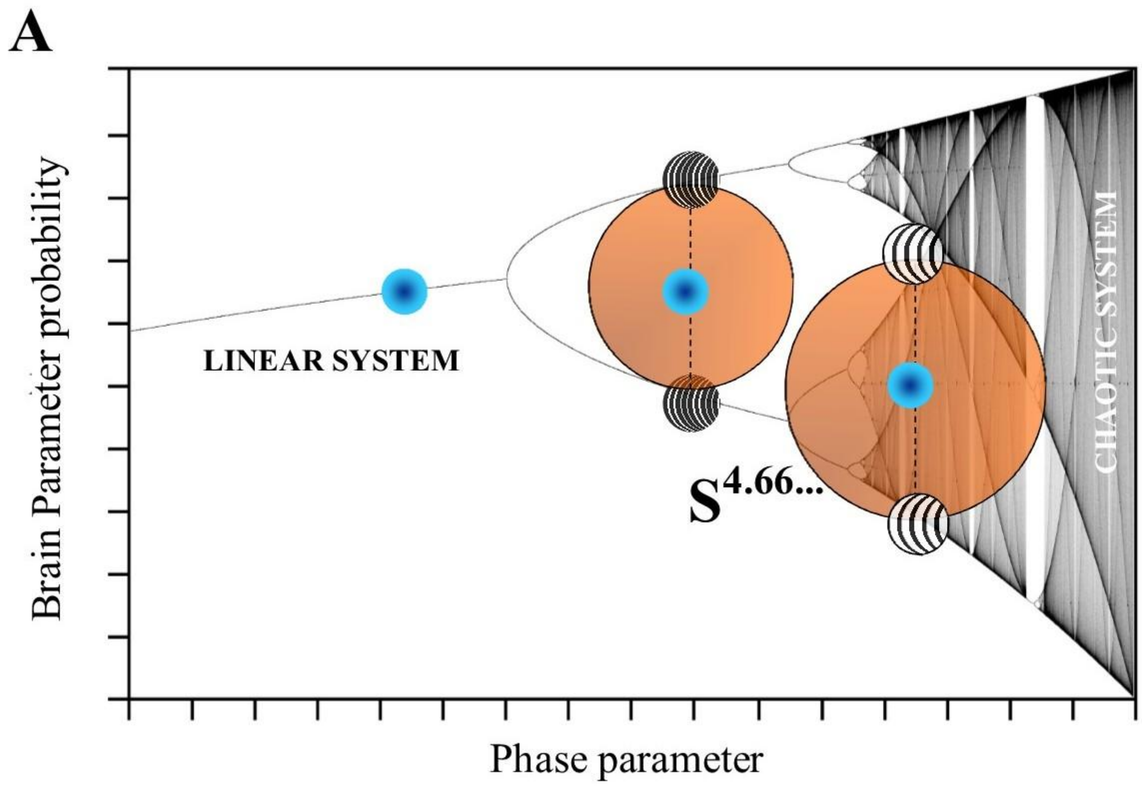
The basic idea is to cover $X$ with sets $U_i$ with small diameters and estimate the $d$-measure of $X$ as the sum of the $(\text{diam}(U_i))^d$ , i.e., the sum of the $U_i$ diameters raised to the power $d$.
**Lemma 1. Schleicher Lemma.** Let $d$ be any real number in $R_0^+$. For every bounded set $X$ in a metric space, there is a unique value of $d := \dim_H(X)$ in $R_0^+ \cup \{\infty\}$ such that:
$\mu_d^I(X) = 0$, if $d^I>d$.
$\mu_d^I(X) = \infty$, if $d^I<d$.
**Definition 2. Hausdorff dimension.** The value of $d= \dim_H(X)$ in $R_0^+$ called the Hausdorff dimension of $X$. With $d= \dim_H(X)$, the Hausdorff measure $\mu_d(X)$ may be zero, positive or infinite.
**Lemma 2. Schleicher Boundedness Lemma.** Let $d$ be any real number in $R_0^+$ and let $Y$ be a metric space. If $X \subset Y$, then:
$\dim_H(X) \leq \dim_H(Y)$.
Proof. Immediate from the definition of the Hausdorff dimension of a nonempty set.
Assume that $X$ is a nonempty subset (inner sphere) of an n-sphere and having the same center as $S^n$ with the Hausdorff measure $\mu_d(X)$ defined on it and assume that $\mu_d(X)$ satisfies the Schleicher Lemma 1 conditions. The inner sphere $S^d$ of an $n$-sphere $S^n$ can be any sub-sphere in $S^n$, including $S^n$ itself. Then, the inner sphere $S^d$ has dimension $d= \dim_H(X)$, $d\leq n$. In addition, assume that $R^d$ is a $d$-dimensional space which is a subset of the $n$-dimensional Euclidean space $R^n$, $d<n$. This gives us new form of the Borsuk-Ulam Theorem (Borsuk 1933).
**Theorem 1. Hausdorff-Borsuk-Ulam Theorem.** Let $S^d$ with Hausdoff dimension $d$ be an inner sphere of an $n$-sphere and let $f : S^d \rightarrow R^d$ be a continuous map. There exists a pair of antipodal points on $S^d$ that are mapped to the same point in $R^d$.

Proof. A direct proof of this theorem is symmetric with the proof of the Borsuk-Ulam Theorem is given by Su (1997), since we assume that $S^d$ is an inner sphere of $S^n$ symmetric about the center of $S^n$ and, from the Schleicher Boundedness Lemma 2, $\dim_H (S^d) \leq \dim_H (S^n)$.

In sum, we achieved a *n*-sphere with a *n* exponent corresponding to the Feigenbaum number. The next step was to locate the n-sphere on a logistic map, in order to embed the topological antipodal points into the bifurcation diagram of a chaotic phenomenon (**Figure A**). Once achieved two antipodal points for every bifurcation, we were allowed to map them in the concentric layers of another *n*-sphere (**Figure B**). We obtained a topological structure which summarizes the whole behaviour of a nonlinear, chaotic system. Note that the center of the sphere stands for a completely linear system, while, when we move along the circumference towards the sphere surface, we achieve antipodal points which represent a progressively more chaotic system. In such a vein, the knowledge of just the single central point and the first Feigenbaum constant leads to predictions about the temporal developement of otherwise chaotic phenomena.

## CONCLUSION

In conclusion, we demonstrated a close link between topological mappings and nonlinear chaotic systems. The Borsuk-Ulam theorem is a versatile tool displaying a very useful general feature which helps us explain a wide-range of phenomena: in our case, if we evaluate nonlinear dynamics instead of "signals", a collection of chaotic signals can be viewed as surface shapes (or sets of region signals) where one shape maps to another antipodal shape. When assessing physical and biological nonlinear dynamics as an alternative to "signals", the BUT leads naturally to the possibility of a region-based, not a simple point-based, geometry of logistic maps. Furthermore, the BUT can be used for the description of antipodal points on n-spheres equipped with a dimension *d* different from a natural number. Although BUT has been originally described as valid just in case of a natural number *n*, recent studies have shown that this theorem is also valid in the framework of more general conditions. In particular, it has been shown that this extension holds for the case of rationally independent numbers (Kim). Here we demonstrated that the *n* value in $S^n$ can be an irrational number, in this case the Feigenbaum constant. BUT is thus suitable to make use of irrational numbers, instead of integer ones, as *n* exponents in $S^n$. This suggests that the *n* parameter can be used as a versatile tool not just for the description of topological manifolds, but also for biological and physical chaotic systems.
What does a topologic reformulation bring to the table, in the evaluation of nonlinear chaotic systems? The opportunity to treat chaos as a topological structure gives us the invaluable chance to describe it through the powerful analytical tools of homology theory and functional analysis (Matoušek; Yang; Dol'nikov). The BUT perspective gives us a symmetry property located in the real space (the environment) to be translated to an abstract space and *vice-versa*, enabling us to achieve a map from one dynamical system to another (**Figure B**). Embracing nonlinearity in the framework of BUT means that bifurcation transformations (the antipodal points) can be described as paths or trajectories on "abstract" structures (called topological configuration space manifolds). Indeed, BUT makes it possible for us to evaluate a nonlinear system through linear techniques. It takes us into the powerful realm of algebraic topology, where the abstract metric space (a projection of the physical and biological melieu's real geometric space) is able to elucidate countless relationships of large scale structures, through correspondences from topological spaces to algebraic groups (Willard; Dodson).



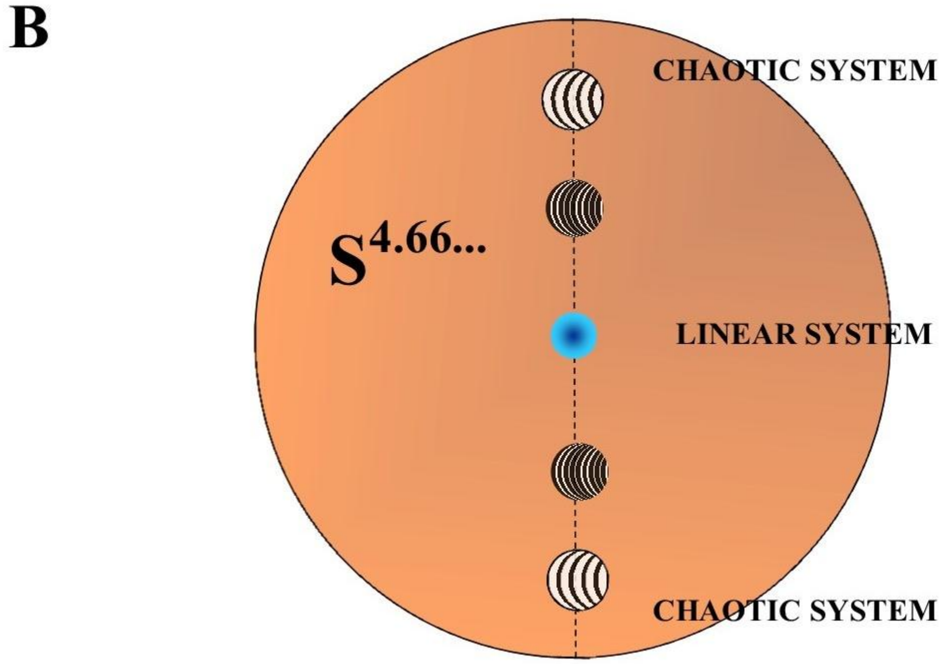




**Figure A.** Bifurcation diagram of a logistic map's nonlinear dynamical equation. The axis *x* displays the phase parameter. At the first Hopf bifurcation, the exponent *n* of the *n*-sphere corresponds to the first Feigenbaum constant. The same operation can be repeated at each of the following bifurcations: just the first and the second ones are displayed in Figure. Each sphere is equipped with two antipodal points which intersect the curves of the corresponding bifurcation and which display the same value on the axis *x*.

**Figure B**. The antipodal points described in Figure A are projected onto a single sphere, which stands for the Figure A's spheres casted in a concentric way, so that the center is the same for all of them.

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