CAN EINSTEIN TENSOR BE GENERALIZED?

TOMASZ KOBIERZYCKI

Abstract. In this short paper I will write a possible generalizations of Einstein tensor and energy momentum tensor that will lead to generalizations of Einstein field equations.

Date: April 24, 2024.

Contents

1. Einstein tensor

Einstein tensor [1] is basis of General Relativity, it has vacuum solutions equal to:

$$G^{\mu\nu} = 0 \tag{1.1}$$

Another property is that it is symmetric and it's covariant derivative is equal to zero from it follows that:

$$\nabla_{\nu} G^{\mu\nu} = 0 \tag{1.2}$$

$$G^{\mu\nu} = G^{\nu\mu} \tag{1.3}$$

It plays crucial role in Einstein field equations [2] as it is left side of field equation:

$$G^{\mu\nu} = \kappa T^{\mu\nu} \tag{1.4}$$

Where tensor itself is equal to:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \tag{1.5}$$

So field equations are equal to :

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \kappa T^{\mu\nu} \tag{1.6}$$

But in whole paper I will be using not contravariant form but covariant form of this tensor so $G_{\mu\nu}$. It will be same tensor but with covariant indexes, it will be equal to:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.7}$$

So field equation is same but with covariant indexes:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.8}$$

This is form of field equations I will use in whole paper.

2. Riemann tensor and generalized Einstein tensor

To build a generalized Einstein tensor I need to assume some kind of basis of deriving it. I will use Riemann tensor contractions are that basis, I want generalized tensor to have same contractions as Riemann tensor [3] [4]. It means that if i write Riemann tensor contractions they will be same as contractions of generalized Einstein tensor:

$$g^{\alpha\mu}R_{\alpha\mu\beta\nu} = 0 \tag{2.1}$$

$$g^{\alpha\beta}R_{\alpha\mu\beta\nu} = R_{\mu\nu} \tag{2.2}$$

$$g^{\alpha\nu}R_{\alpha\mu\beta\nu} = -R_{\mu\beta} \tag{2.3}$$

$$g^{\mu\beta}R_{\alpha\mu\beta\nu} = -R_{\alpha\nu} \tag{2.4}$$

$$g^{\mu\nu}R_{\alpha\mu\beta\nu} = R_{\alpha\beta} \tag{2.5}$$

$$g^{\beta\nu}R_{\alpha\mu\beta\nu} = 0 \tag{2.6}$$

So I can write down now same contractions but for generalized Einstein tensor $G_{\alpha\mu\beta\nu}$:

$$g^{\alpha\mu}G_{\alpha\mu\beta\nu} = 0 \tag{2.7}$$

$$g^{\alpha\beta}G_{\alpha\mu\beta\nu} = G_{\mu\nu} \tag{2.8}$$

$$g^{\alpha\nu}G_{\alpha\mu\beta\nu} = -G_{\mu\beta} \tag{2.9}$$

$$g^{\mu\beta}G_{\alpha\mu\beta\nu} = -G_{\alpha\nu} \tag{2.10}$$

$$g^{\mu\nu}G_{\alpha\mu\beta\nu} = G_{\alpha\beta} \tag{2.11}$$

$$g^{\beta\nu}G_{\alpha\mu\beta\nu} = 0 \tag{2.12}$$

From it comes another part of equations that is generalized energy momentum tensor [5], that will have same contraction properties as Riemann tensor and generalized Einstein tensor to follow a field equation:

$$g^{\alpha\mu}T_{\alpha\mu\beta\nu} = 0 \tag{2.13}$$

$$g^{\alpha\beta}T_{\alpha\mu\beta\nu} = T_{\mu\nu} \tag{2.14}$$

$$g^{\alpha\nu}T_{\alpha\mu\beta\nu} = -T_{\mu\beta} \tag{2.15}$$

$$g^{\mu\beta}T_{\alpha\mu\beta\nu} = -T_{\alpha\nu} \tag{2.16}$$

$$g^{\mu\nu}T_{\alpha\mu\beta\nu} = T_{\alpha\beta} \tag{2.17}$$

$$g^{\beta\nu}T_{\alpha\mu\beta\nu} = 0 \tag{2.18}$$

So from it comes that generalized Einstein tensor reduces either to plus-minus Einstein tensor or zero and generalized energy momentum tensor have to obey same rule to make it consistent with field equations.

3. Generalized Einstein tensor

I will first write generalized Einstein tensor and generalized energy momentum tensor, then will show that they indeed follow contractions properties. So those tensors are equal to:

$$G_{\alpha\mu\beta\nu} = R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right) \tag{3.1}$$

$$T_{\alpha\mu\beta\nu} = \frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right) \tag{3.2}$$

$$g^{\alpha\mu}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{3.3}$$

$$g^{\alpha\beta}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = G_{\mu\nu} \tag{3.4}$$

$$g^{\alpha\nu}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = -G_{\mu\beta} \tag{3.5}$$

$$g^{\mu\beta}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = -G_{\alpha\nu} \tag{3.6}$$

$$g^{\mu\nu}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = G_{\alpha\beta} \tag{3.7}$$

$$g^{\beta\nu}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{3.8}$$

$$g^{\alpha\mu}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{3.9}$$

$$g^{\alpha\beta}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = T_{\mu\nu} \tag{3.10}$$

$$g^{\alpha\nu}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = -T_{\mu\beta} \tag{3.11}$$

$$g^{\mu\beta}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = -T_{\alpha\nu} \tag{3.12}$$

$$g^{\mu\nu}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = T_{\alpha\beta} \tag{3.13}$$

$$g^{\beta\nu}\left(\frac{1}{2}\left(T_{\alpha\beta}g_{\mu\nu} + T_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}\left(T_{\mu\beta}g_{\alpha\nu} + T_{\alpha\nu}g_{\mu\beta}\right) - \frac{1}{6}T\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{3.14}$$

4. Conservation

It easy to derive that covariant derivative of extended Einstein tensor is equal to zero for all indexes. It means that:

$$g^{\alpha\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.1}$$

$$g^{\mu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.2}$$

$$g^{\beta\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.3}$$

$$g^{\nu\phi}\nabla_{\phi}G_{\alpha\mu\beta\nu} = 0 \tag{4.4}$$

Where i did use covariant derivative with upper index index raising, trick to do it is very simple, covariant derivative of metric tensor is equal to zero so I can treat metric tensor as a constant, from fact that tensor reduces to Einstein tensor so if I multiply one side of equation by metric tensor I will still get valid result and from fact that after using metric tensor to contract generalized Einstein tensor I will always arrive and zero:

$$g^{\mu\phi}\nabla_{\phi}g^{\alpha\beta}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{4.5}$$

$$g^{\mu\phi}\nabla_{\phi}G_{\mu\nu} = 0 \tag{4.6}$$

$$g^{\nu\phi}\nabla_{\phi}g^{\alpha\beta}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{4.7}$$

$$g^{\nu\phi}\nabla_{\phi}G_{\mu\nu} = 0 \tag{4.8}$$

$$g^{\alpha\phi}\nabla_{\phi}g^{\mu\nu}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{4.9}$$

$$g^{\alpha\phi}\nabla_{\phi}G_{\alpha\beta} = 0 \tag{4.10}$$

$$g^{\beta\phi}\nabla_{\phi}g^{\mu\nu}\left(R_{\alpha\mu\beta\nu} - \frac{1}{6}R\left(g_{\alpha\beta}g_{\mu\nu} - g_{\beta\mu}g_{\alpha\nu}\right)\right) = 0 \tag{4.11}$$

$$g^{\beta\phi}\nabla_{\phi}G_{\alpha\beta} = 0 \tag{4.12}$$

For generalized energy momentum tensor case is even simpler it's build form energy momentum tensor and metric tensors both of them will vanish when taking the covariant derivative.

5. Summary

In this short paper I showed possible generalization of Einstein tensor. This leads to generalized energy momentum tensor, that combined create a new field equation:

$$G_{\alpha\mu\beta\nu} = \kappa T_{\alpha\mu\beta\nu} \tag{5.1}$$

Contractions of this field equation lead to zero or plus-minus Einstein tensor. That gives new equation for space-time curvature and new vacuum equations that will be equal to:

$$G_{\alpha\mu\beta\nu} = 0 \tag{5.2}$$

Problem with this equation is that is really hard to solve, as its a four rank tensor. For example field equation will take form for simplest case of vacuum:

$$R_{\alpha\beta\alpha\beta} - \frac{1}{6}R\left(g_{\alpha\alpha}g_{\beta\beta} - g_{\alpha\beta}g_{\alpha\beta}\right) = 0 \tag{5.3}$$

From fact that independent components for Riemann tensor in case of spherical symmetric space-time are only six of them [6] and there are no cross terms for metric tensor I will get where I can write independent components:

$$R_{0101} - \frac{1}{6}Rg_{11}g_{00} = 0 \tag{5.4}$$

$$R_{0202} - \frac{1}{6}Rg_{22}g_{00} = 0 \tag{5.5}$$

$$R_{0303} - \frac{1}{6}Rg_{33}g_{00} = 0 \tag{5.6}$$

$$R_{1212} - \frac{1}{6}Rg_{11}g_{22} = 0 \tag{5.7}$$

$$R_{1313} - \frac{1}{6}Rg_{11}g_{33} = 0 \tag{5.8}$$

$$R_{2323} - \frac{1}{6}Rg_{22}g_{33} = 0 \tag{5.9}$$

From it follows clearly that vacuum solutions have non-vanishing Ricci tensor, even in simplest case.

References

[1] https://mathworld.wolfram.com/EinsteinTensor.html

[2] https://mathworld.wolfram.com/EinsteinFieldEquations.html

[3] https://mathworld.wolfram.com/RicciCurvatureTensor.html

[4] https://mathworld.wolfram.com/RiemannTensor.html

[5] https://www.astro.gla.ac.uk/users/martin/teaching/gr1/gr1_sec08.pdf

[6] http://web.mit.edu/klmitch/classes/8.033/Schwarzschild.pdf

Email address: kobierzyckitomasz@gmail.com