On the Vector Space Representation of the Cumulative Hierarchy

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Abstract The cumulative hierarchy is an ordinal gradation of the class of all sets. It has numerous downstream applications in the foundations of mathematics. In order to constructively define topologies and algebraic structures on the hierarchy, we represent it using vector spaces.

Let 0 and 1 be defined by $0 = \phi$ and $1 = \{\phi\}$. We equip the product of the family of sets $\mathcal{A} = \{\prime, \infty\}_{\rangle \in \alpha \in \mathcal{OR}}$ with a vector space structure.

We recall that the product of a family of sets $\mathcal{F} = \{A_i\}_{i \in I}$ is given by $\prod_{i \in I} A_i = \{f : I \to \cup_{i \in I} A_i \mid f(i) \in A_i \ \forall i \in I\}$.

For the family $\mathcal{A}$, the union is $\cup_{i \in \alpha}\{0,1\} = \{0,1\}$. Therefore, the product of the family is given by,

$$\{0,1\}^\alpha = \prod_{i \in \alpha}\{0,1\} = \{f : \alpha \to \{0,1\} \mid f(i) \in \{0,1\} \ \forall i \in \alpha\}$$

We take $F_2 = \{0,1\}$ as the base field, and define addition as the logical 'OR' operator ($\vee$), and scalar multiplication as the logical 'AND' operator ($\wedge$).

When adding two vectors of dimensions α and β, we pad the vector having dimension $min(\alpha, \beta)$ with $|\alpha - \beta|$ zeros to the right. Thus, $(\{0,1\}^\alpha, +, \cdot)$ is a vector space for all $\alpha \in OR$.

The basis of this vector space is given by,

$$\{0,1\}^\alpha_B = \{f : \alpha \to \{0,1\} \mid f(i) = 1, f(j) = 0, \ j \neq i \ \forall i \in \alpha\}$$

Define the family of sets $\mathcal{B} = \{\{0,1\}^\alpha_B\}_{\alpha \in OR}$ and the family of vector spaces $\mathcal{F} = \{\{0,1\}^\alpha\}_{\alpha \in OR}$.

We get a compact representation of $\{0,1\}^\alpha_B$ and $\{0,1\}^\alpha$ by stacking the vectors in the sets vertically to form a matrix.

For $\{0,1\}_B^\alpha$, we get the matrix representation I_α, the identity matrix.

Similarly for $\{0,1\}^\alpha$, we get the matrix representation P_α, where each row belongs to the spanning set of $\{0,1\}_B^\alpha$.

Define the family of matrices $\mathcal{I} = \{I_\alpha\}_{\alpha\in OR}$, $\mathcal{P} = \{P_\alpha\}_{\alpha\in OR}$.

Given a family of sets $\mathcal{A} = \{A_\alpha\}_{\alpha\in I}$, we define the following bijective functions:

- $\psi_V : \mathcal{A} \to \mathcal{V}$, $A_\alpha \to \{0,1\}_B^\alpha$, $A_{2^\alpha} \to \{0,1\}^\alpha$
- $\psi_M : \mathcal{V} \to \mathcal{M}$, $\{0,1\}_B^\alpha \to I_\alpha$, $\{0,1\}^\alpha \to P_\alpha$

where, $\mathcal{V} = \mathcal{B} \cup \mathcal{F}$, and $\mathcal{M} = \mathcal{I} \cup \mathcal{P}$

The representation of the ordinal numbers are given by the sums,

$$\psi_V(OR) = \cup_{\{0,1\}_B^\alpha \in \mathcal{B}} \{0,1\}_B^\alpha$$
$$\psi_M(\psi_V(OR)) = \sum_{I_\alpha \in \mathcal{I}} I_\alpha$$

We recall the construction of the cumulative hierarchy. The cumulative hierarchy is given by $V = \cup_{\alpha\in OR} V_\alpha$, where:

- $V_0 = \phi$
- $V_{\alpha+1} = \mathcal{P}\ (V_\alpha)$
- $V_\alpha = \cup_{\beta\alpha} V_\beta$, if α is a limit ordinal

The representation of the cumulative hierarchy are given by the sums,

$$\psi_V(V) = \cup_{\{0,1\}^\alpha \in \mathcal{F}} \{0,1\}^\alpha$$
$$\psi_M(\psi_V(V)) = \sum_{P_\alpha \in \mathcal{P}} P_\alpha$$

The relationships between the objects are given by the diagram,

$$\begin{array}{ccccc}
OR & \to^{\psi_V} & \cup_{\{0,1\}_B^\alpha \in \mathcal{B}} \{0,1\}_B^\alpha & \to^{\psi_M} & \sum_{I_\alpha \in \mathcal{I}} I_\alpha \\
\downarrow^{\mathcal{P}} & & \downarrow^{span} & & \downarrow^{span} \\
V & \to^{\psi_V} & \cup_{\{0,1\}^\alpha \in \mathcal{F}} \{0,1\}^\alpha & \to^{\psi_M} & \sum_{P_\alpha \in \mathcal{P}} P_\alpha
\end{array}$$

1 References

1. Charles C. Pinter. *A Book of Set Theory.* Dover Publications, Inc., 2014.

2. Steven Roman. *Advanced Linear Algebra.* University of California, Irvine. Springer, 2005.