The Photon Gas in the Quantized Friedmann Universe

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Abstract: A semiclassical fundamental equation of quantum gravitation is applied to a flat Friedmann universe containing a photon gas. The mean energy of a photon is determined by the method of canonical partition and the classical one-particle state function for the ideal photon gas. For very high temperatures, there is a deviation from the classical equipartition theorem.

Keywords: Semiclassical quantum gravitation, Friedmann universe, Planck´s radiation law, statistical canonical partition

1 Introduction

If, on the smallest spatial and temporal scales, the universe has a grainy structure, Planck's electromagnetic radiation law ought to have a modified structure, too. Radiation propagating over a space-time range that has a lattice structure will not be able to fall below a bottom limit of the particle wavelength and, thus, will also have to maintain an upper limit of the particle energy.

For the theoretical implementation of this idea there are several approaches. The most difficult problem is encountered in supporting these investigations by a generalized special theory of relativity that would permit an invariant limit for a minimum limit unit of length or a maximum of particle energy. We refer to [5,7,12,14] and, especially for the thermodynamics of the photon gas, to [3,5,8,9,10,17]. To extricate ourselves us from these principal problems, let us first look for a first correction of Planck`s radiation formula as it would appear, e.g., in a quantized Friedmann universe.

2 The Hamiltonian operator

Let our investigations start from the Hamiltonian operator of matter in a flat Friedmann universe. In [1] or [2] C. Kiefer and T. P. Singh have developed a semiclassical quantum gravitation from Wheeler-DeWitt's equation of quantum gravitation (also see [6] and [11]) and applied this to the Friedmann universe. For the motion of matter in this universe, a corrected Schrödinger equation of the following type is given:

$$i\hbar\frac{\partial\chi}{\partial t}=\hat{H}_m\,\chi+\begin{cases}\frac{G}{3\pi c^4 a}\hat{H}_m^2\,\chi & (curved\ universe),\\ \frac{G}{2c^4 a}\hat{H}_m^2\,\chi & (flat\ universe).\end{cases} \tag{2.1}$$

From this fundamental equation, therefore, we derive a non-linear semiclassical Hamiltonian of the following structure for the motion of matter in the flat Friedmann universe

$$\hat{H}=\hat{H}_m\left(1+\frac{G}{2\,c^4 a}\hat{H}_m\right)\ , \tag{2.2}$$

or also, with the scale factor $a=L_*\cdot\alpha$,

$$\hat{H}=\hat{H}_m\left(1+\frac{1}{2\alpha}\frac{\hat{H}_m}{E_*}\right)\quad ,\quad \alpha=\frac{a}{L_*}\quad , \tag{2.3}$$

wherein $\hat{H}_m$ is the Hamiltonian of the matter in the Minkowski space and

$$L_*=\sqrt{\frac{\hbar G}{c^3}}\quad ,\quad E_*=M_*c^2\quad ,\quad M_*=\sqrt{\frac{\hbar c}{G}}\quad , \tag{2.4}$$

are the Planck-length, Planck-energy and the Planck-mass.

Note: In Dirac's theory, an analogue procedure leads to a semiclassical Hamiltonian that is similar to the above gravitative case [15]:

$$\hat{H}=\hat{\hat{H}}-m_0c^2=\sqrt{\eta^{\mu\nu}p_\mu p_\nu+m_0c^2}-m_0c^2\approx\frac{\hat{p}^2}{2m_0}-\frac{1}{8}\frac{\left(\hat{p}^2\right)^2}{m_0{}^3c^2}\ , \tag{2.5}$$

$$\hat{H}=\hat{H}_0\left(1-\frac{1}{2}\frac{\hat{H}_0}{E_0}\right)\ ,\quad \hat{H}_0=\frac{\hat{p}^2}{2m_0}\ ,\ E_0=m_0c^2\ . \tag{2.6}$$

The correction term in (2.6) has the same structure as that in (2.3), but with a negative sign.

3 Application to the photon gas

Let us now consider an ensemble of harmonic oscillators in the heating bath having a temperature T (Einstein's photon model), and determine the mean energy $\overline{\varepsilon}$ of a photon therein. Let us use for that the canonical partition with the one-particle partition sum Z and the Hamiltonian $\hat{H}$ in the flat Friedmann universe from (2.3)

$$Z=Tr\left(e^{-\beta\hat{H}}\right)\quad ,\quad \beta=\frac{1}{kT}\ , \tag{3.1}$$

$$\hat{H}=\hat{H}_m+\frac{\lambda}{2}\hat{H}_m^2\quad ,\quad \lambda=\frac{1}{\alpha E_*}\ . \tag{3.2}$$

As the λ term is a sufficiently small correction term only, we can readily note

$$Z = Tr\left\{e^{-\beta\hat{H}_m - \frac{\lambda}{2}\beta\hat{H}_m^2}\right\} = Tr\left\{e^{-\beta\hat{H}_m}\left(1-\tfrac{\lambda}{2}\beta\hat{H}_m^2\right)\right\} \quad , \tag{3.3}$$

$$Z = Tr\left(e^{-\beta\hat{H}_m}\right) - \frac{\lambda}{2}\beta\,Tr\left(\hat{H}_m^2\, e^{-\beta\hat{H}_m}\right) \quad , \tag{3.4}$$

$$Z = Z_0 - \frac{\lambda}{2}\beta\frac{\partial^2 Z_0}{\partial\beta^2} \qquad , \qquad Z_0 = Tr\left(e^{-\beta\hat{H}_m}\right) \quad . \tag{3.5}$$

The mean one-particle energy $\bar{\varepsilon}$ then, as we know, results from

$$\bar{\varepsilon} = -\frac{\partial \ln Z}{\partial\beta} \quad . \tag{3.6}$$

Z_0, however, is the one-particle partition sum of the Planck partition in the flat Minkowski space

$$Z_0 = \frac{1}{1-e^{-\beta\varepsilon}} \quad , \tag{3.7}$$

so that, from (3.5), (3.6) and (3.7), it follows that

$$\bar{\varepsilon} = -\frac{\partial}{\partial\beta}\ln\left(Z_0 - \frac{\lambda}{2}\beta\frac{\partial^2 Z_0}{\partial\beta^2}\right) \quad . \tag{3.8}$$

Our interest is focused on the correction of Planck´s radiation law at very high temperatures $kT \gg \hbar\omega$ or $\beta\varepsilon \ll 1$. For that case we obtain, with (3.7) and (3.8),

$$\lim_{T\to\infty} Z_0 = \frac{1}{\beta\varepsilon} = \frac{kT}{\varepsilon} \quad . \tag{3.9}$$

For $T \to \infty$, then, we get

$$\bar{\varepsilon}_\infty = \lim_{T\to\infty}\bar{\varepsilon} = -\frac{\partial}{\partial\beta}\ln\left\{\left(\frac{1}{\beta\varepsilon}\right) - \frac{\lambda}{2}\beta\frac{\partial^2}{\partial\beta^2}\left(\frac{1}{\beta\varepsilon}\right)\right\} \quad , \tag{3.10}$$

$$\bar{\varepsilon}_\infty = -\frac{\partial}{\partial\beta}\ln\left\{\frac{1}{\beta\varepsilon}\left(1-\frac{\lambda}{\beta}\right)\right\} \quad , \tag{3.11}$$

$$\bar{\varepsilon}_\infty = \frac{1}{\beta}\left(1-\frac{\lambda}{\beta}\right) \quad . \tag{3.12}$$

With (3.1) and (3.2), therefore, we obtain

$$\bar{\varepsilon}_\infty = kT\left(1-\frac{kT}{\alpha E_*}\right) \qquad , \qquad kT \ll E_* \quad . \tag{3.13}$$

The occurrence of the parameter $\alpha = a/L_*$ points to the influence of the Friedmann universe.

We recognize that the number of classical degrees of freedom per particle deviates from 2, although this correction, due to the magnitude of the Planck energy $E_* \approx 1{,}2\cdot 10^{28}\,\mathrm{eV}$, is extremely little.

4 A value for α

Can we in (2.3) set a value for α? In [5] we established an approach to a modified Planck's radiation law, which we developed in an imitation of Einstein's well-known laser model [16]. The modified Planck formula, which was based also on statistical considerations, reads as follows:

$$\bar{\varepsilon} = \frac{\varepsilon}{\exp\left(\frac{\varepsilon}{kT}\right)-\left(1-\frac{\varepsilon}{E_*}\right)} \quad . \tag{4.1}$$

For temperatures $T\to\infty$ this converts to

$$\bar{\varepsilon}_\infty = \lim_{T\to\infty}\bar{\varepsilon} = \lim_{T\to\infty}\left(\frac{kT}{1+(kT/E_*)}\right) = E_* \quad . \tag{4.2}$$

If, however, we remain at a temperature of $kT \ll E_*$, we obtain from (4.2) for this case:

$$\bar{\varepsilon}_\infty \approx kT\left(1-\frac{kT}{E_*}\right) \quad . \tag{4.3}$$

If we now compare (4.3) with our result (3.13) for the Friedmann universe, we can readily conclude that $\alpha = 1$ and also, therefore, from (2.3), that $a = L_*$. This is well in agreement with the ground state of a quantized universe. A quantization of the Friedmann cosmos provides namely a discretization of the cosmic scale factor and the energy content of the cosmos,

$$a \to a_n = (2n+1)L_* \qquad , \qquad E \to E_n = \sqrt{2n+1}\,E_* \quad . \tag{4.4}$$

how we can find in [4], [5] or [13]. With $n=1$ we read off, for ground state length and ground state energy, respectively, the values $a = a_1 = L_*$ and $E = E_1 = E_*$, such that $\alpha = 1$ seems to be a reasonable value.

According to formula (4.3), however, corrections to the classical Rayleigh-Jeans behaviour $\bar{\varepsilon}_\infty = kT$ and, thus, to the classical equipartition theorem $\bar{\varepsilon}_\infty = 2\cdot(kT/2)$ pro photon, will not appear before the temperatures come close to the Planck temperature $kT \sim E_* \sim 10^{28}\,\mathrm{eV}$. Such temperatures will at best appear at a cosmologically short time after the big bang.

5 Conclusion

We applied the semiclassical fundamental equation of quantum gravitation, derived from the Wheeler-DeWitt equation by C. Kiefer [1], to the electromagnetic radiation in a flat Friedmann universe. We used the method of canonical partition and the state function for the ideal photon gas to determine the mean energy $\bar{\varepsilon}$ of a photon. For very high temperatures (but still $kT \ll E_*$), there is a deviation from the classical behavior, which now took the form $\bar{\varepsilon}_\infty \approx kT\left(1 - kT/E_*\right)$, being influenced by the quantity of the Planck energy E_*.

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